Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Constrained Controllability of Second Order Semilinear Systems

J. Klamka*

*Institute of Control Engineering, Silesian University of Technology,
street Akademicka 16, 44-100 Gliwice, Poland,
(e-mail: jerzy.klamka@polsl.pl)

Abstract: In the paper finite-dimensional dynamical control systems described by second order semilinear stationary ordinary differential state equations are considered. Using a generalized open mapping theorem, sufficient conditions for constrained local controllability in a given time interval are formulated and proved. These conditions require verification of constrained global controllability of the associated linear first-order dynamical control system. It is generally assumed, that the values of admissible controls are in a convex and closed cone with vertex at zero. Moreover, several remarks and comments on the existing results for controllability of semilinear dynamical control systems are also presented. Finally, simple numerical example, which illustrates theoretical considerations is also given. It should be pointed out, that the results given in the paper extend for the case of semilinear second-order dynamical systems constrained controllability conditions, which were previously known only for linear second-order systems.

1. INTRODUCTION.

Controllability is one of the fundamental concept in mathematical control theory (Brammer, 1972, Klamka, 1991, Naito, 1987, Seidman, 1987). This is a qualitative property of dynamical control systems and is of particular importance in control theory. Systematic study of controllability was started at the beginning of sixties, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out. Roughly speaking, controllability generally means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability, which strongly depend on class of dynamical control systems and on the set of admissible controls (Brammer, 1972, Klamka, 1991, 1993, 1996, Naito, 1987, Seidman, 1987, Son, 1990, Zhou, 1984).

In recent years various controllability problems for different types of nonlinear dynamical systems have been considered in many publications and monographs. The extensive list of these publications can be found for example in the monograph (Klamka, 1991) or in the survey paper (Klamka, 1993). However, it should be stressed, that the most literature in this direction has been mainly concerned with controllability problems for finite-dimensional linear and nonlinear dynamical systems with unconstrained controls and without delays (Brammer, 1972, Klamka, 1996, Naito, 1987,) or for linear infinite-dimensional dynamical systems with constrained controls and without delays (Fuji and Sakawa, 1974, Klamka, 1991, Seidman, 1987).

Let us recall, that semilinear dynamical control systems contain both linear and pure nonlinear parts in the differential state equations (Klamka, 1993, Naito, 1987, Zhou, 1984). More precisely, we shall formulate and prove sufficient conditions for constrained local controllability in a prescribed time interval for semilinear second-order stationary dynamical systems which nonlinear term is continuously differentiable near the origin. It is generally assumed that the values of admissible controls are in a given convex and closed cone with vertex at zero, or in a cone with nonempty interior. Proof of the main result is based on the generalized open mapping theorem presented in the paper (Robinson, 1976).

Roughly speaking, in the paper it will be proved that under suitable assumptions constrained global controllability of a linear first-order associated approximated dynamical system implies constrained local controllability near the origin of the original semilinear second-order dynamical system. This is a direct generalization to constrained controllability case some well-known previous results concerning controllability of linear dynamical control systems with unconstrained controls (Chukwu and Lenhart, 1991, Klamka, 1991, 1993, 1996).

Finally, is should be mentioned, that other different controllability problems both for linear dynamical control systems and nonlinear dynamical control systems have been also considered in the papers (Chukwu and Lenhart, 1991, Fuju and Sakawa, 1974, Peichl and Schappacher, 1986, Son, 1990)

978-1-1234-7890-2/08/$20.00 © 2008 IFAC

10.3182/20080706-5-KR-1001.0061

2. SYSTEM DESCRIPTION

In this paper, we shall consider constrained local controllability problems for second-order finite-dimensional semilinear stationary dynamical control systems, described by ordinary differential state equations.

$$w''(t) = Gw(t) + f(w(t),u(t)) + Hu(t) \qquad t\in[0,T] \quad (2.1)$$

where vector $w(t)\in R^n = W$ and the control $u(t)\in R^m = U$, G is n×n dimensional constant matrix, H is n×m dimensional constant matrix. Moreover, let us assume that the nonlinear mapping f: W×U→W is continuously differentiable near the origin and such that f(0,0)=0.

For simplicity of considerations we assume zero initial conditions, i.e.

$$w(0) = 0 \text{ and } \quad w'(0) = 0$$

Using standard substitutions

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} w(t) \\ w'(t) \end{bmatrix}$$

we may transform second-order dynamical system (2.1) into equivalent first-order semilinear stationary 2n-dimensional control system described by the following ordinary differential state equation

$$x'(t) = Ax(t) + F(x(t),u(t)) + Bu(t) \quad \text{for } t\in[0,T], \quad T>0 \qquad (2.2)$$

with zero initial conditions: $\qquad x(0) = 0$

where state vector $x(t)\in R^{2n} = X$ and the control $u(t)\in R^m = U$, A is 2n×2n dimensional constant matrix, B is 2n×m dimensional constant matrix.

$$A = \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ H \end{bmatrix} \quad F(x(t),u(t)) = \begin{bmatrix} 0 \\ f(x(t),u(t)) \end{bmatrix}$$

Moreover, from the previous assumptions concerning nonlinear term f(x(t),u(t)) it follows, that the nonlinear mapping F: X×U→X is also continuously differentiable near the origin and such that F(0,0)=0.

In practice admissible controls are always required to satisfy certain additional constraints. Generally, for arbitrary control constraints it is rather very difficult to give easily computable criteria for constrained controllability even in the linear case. However, for some special cases of the constraints it is possible to formulate and prove simple algebraic constrained controllability conditions. Therefore, we assume that the set of values of admissible controls $U_c\subset U$ is a given closed and convex cone with nonempty interior and vertex at zero. Then, the set of admissible controls for the dynamical control systems (2.1) and (2.2) has the following form $U_{ad}=L_\infty([0,T],U_c)$.

Then for a given admissible control u(t) there exists a unique solutions $w(t;u)\in R^n$ of the second-order differential equation (2.1) and similarly, unique solution $x(t;u)\in R^{2n}$ of the first-order ordinary differential state equation (2.2) and with zero initial condition described by the integral formula [6], [11].

$$x(t;u) = \int_0^t S(t-s)(F(x(s;u(s)),u(s)) + Bu(s))ds \qquad (2.3)$$

where the matrix semigroup S(t) = exp(At) for t≥0 is 2n×2n dimensional exponential transition matrix for the linear part of the semilinear first-order control system (2.2).

For the semilinear stationary finite-dimensional second-order dynamical system (2.1) or equivalently for first-order dynamical system (2.2), it is possible to define many different concepts of controllability. However, in the sequel we shall focus our attention on the constrained controllability in a given time interval [0,T].

In order to do that, first of all let us introduce the notion of the so called attainable or reachable set at time T>0 from zero initial conditions, denoted shortly by $K_T(U_c)$ and defined as follows (Klamka, 1991, Seidman, 1987).

$$K_T(U_c) = \{x\in X : x = x(T,u), \ u(t)\in U_c \ \text{ for a.e. } t\in[0,T]\} \qquad (2.4)$$

where x(t,u), t > 0 is the unique solution of the differential first-order state equation (2.2) with zero initial conditions and a given admissible control $u\in U_{ad}=L_\infty([0,T],U_c)$.

Taking into account the form of the semilinear state equation (2.2) it should be pointed out that under the assumptions stated on the nonlinear term F such solution always exists and is unique [6], [11].

Now, using the concept of the attainable set $K_T(U_c)$, let us recall the well known (see e.g. Klamka, 1991, 1993, 1996) definitions of local and global constrained controllability in [0,T] for semilinear second-order dynamical system (2.1).

Definition 2.1 The dynamical system (2.1) is said to be U_c-locally controllable in [0,T] if the attainable set $K_T(U_c)$ contains a neighborhood of zero in the space X.

Definition 2.2 The dynamical system (2.1) is said to be U_c-globally controllable in [0,T] if $K_T(U_c) = X$.

Now, in the last part of this section we shall discuss the relationships between controllability of the first-order system (2.2) for $F(x(t),u(t)) = 0$, and associated with second-order system (2.1) the first-order system for the case when $f(w(t),u(t)) = 0$. Therefore, we shall consider the following two first order linear dynamical systems:

$$w'(t) = Gw(t) + Hu(t) \qquad t\in[0,T] \qquad (2.5)$$

$$x'(t) = Ax(t) + Bu(t) \qquad t\in[0,T] \qquad (2.6)$$

First of all, taking into account the form and dimensionality of the matrices A, B, G and H simple calculations show that the following of the equality holds:

$$rank\left[B \;\vdots\; AB \;\vdots\; A^2B \;\vdots\; \dots \;\vdots\; A^{2n-1}B\right] =$$

$$= rank\begin{bmatrix} H & 0 & GH & 0 & G^2H & 0 & \dots & G^{n-1}H & 0 \\ 0 & H & 0 & GH & 0 & G^2H & \dots & 0 & G^{n-1}H \end{bmatrix}$$

Hence, we have the following rather well-known in the literature corollary (see e.g., monograph (Klamka, 1991) concerning different relationships between unconstrained controllability of dynamical systems (2.5) and (2.6).

Corollary 2.1 Second-order linear dynamical system (2.1) or equivalently first-order 2n-dimensional dynamical system (2.6) is controllable in any time interval without control constraints if and only if the linear first-order n-dimensional dynamical system (2.5) is controllable in any time interval without control constraints.

Remark 2.1 However, it should be pointed out, that for the case of constrained controllability problem the above Corollary 2.1 does not hold and there are no general relationships between constrained controllability of first-order and second-order linear dynamical systems.

3. PRELIMINARIES

In this section, we shall introduce certain notations and present some important facts from the general theory of nonlinear operators.

Let U and X be given spaces and $g(u):U\to X$ be a mapping continuously differentiable near the origin 0 of U. Let us suppose for convenience that $g(0)=0$. It is well known from the implicit-function theorem (see e.g., [8]) that, if the derivative $Dg(0)$: $U\to X$ maps the space U onto the whole space X, then the nonlinear map g transforms a neighborhood of zero in the space U onto some neighborhood of zero in the space X.

Now, let us consider the more general case when the domain of the nonlinear operator g is Ω, an open subset of U containing 0. Let U_c denote a closed and convex cone in U with vertex at 0.

In the sequel, we shall use for controllability investigations some property of the nonlinear mapping g which is a consequence of a generalized open-mapping theorem (Robinson, 1976). This result seems to be widely known, but for the sake of completeness we shall present it here, though without proof and in a slightly less general form sufficient for our purpose.

Lemma 3.1. (Robinson, 1976). Let X, U, U_c, and Ω be as described above. Let $g:\Omega\to X$ be a nonlinear mapping and suppose that on Ω nonlinear mapping g has derivative Dg, which is continuous at 0. Moreover, suppose that $g(0) = 0$ and assume that linear map $Dg(0)$ maps U_c onto the whole space X. Then there exist neighborhoods $N_0 \subset X$ about $0\in X$ and $M_0\subset\Omega$ about $0\in U$ such that the nonlinear equation $x=g(u)$ has, for each $x\in N_0$, at least one solution $u\in M_0\cap U_c$, where $M_0\cap U_c$ is a so called conical neighborhood of zero in the space U.

Lemma 3.2 Let D_ux denotes derivative of x with respect to u. Moreover, if $x(t;u)$ is continuously differentiable with respect to its u argument, we have for $v\in L_\infty([0,T],U)$

$$D_ux(t;u)(v)_{x=0,u=0} = z(t,u,v)$$

where the mapping $t\to z(t,u,v)$ is the solution of the linear ordinary equation

$$z'(t) = Az(t) + D_xF(0,0)z(t) + Bv(t) + D_uF(0,0)v(t) \qquad (3.1)$$

with zero initial conditions $z(0,u,v) = 0$.

where $\quad D_xF(0,0) = D_xF(x(t;u),u)_{x=0,u=0}$

$$D_uF(0,0) = D_u(F(x(t;u),u)_{x=0,u=0}$$

Proof. Using formula (2.3) and the standard well known differentiability results for composite function we have

$$D_u x(t;u) = \int_0^t D_u(S(t-s)(F(x(s;u(s)),u(s)) + Bu(s)))ds =$$
$$= \int_0^t S(t-s)(D_u(F(x(s;u(s)),u(s)) + Bu(s)))ds =$$
$$= \int_0^t S(t-s)(D_xF(x(s;u(s)),u(s))D_ux(s;u(s))ds +$$
$$+ \int_0^t S(t-s)(D_uF(x(s;u(s)),u(s)) + B)ds \qquad (3.2)$$

Differentiating equality (3.2) with respect to the variable t, we have

$$(d/dt)D_u x(t;u) = D_x F(x(t;u),u)D_u x(t;u) + \\ + D_u F(x(t;u),u) + B + \\ + \int_0^t (d/dt)S(t-s)(D_x F(x(s;u(s)),u(s))D_u x(s;u(s))ds + \\ + \int_0^t (d/dt)S(t-s)(D_u F(x(s;u(s)),u(s)) + B)ds \quad (3.3)$$

Therefore, since S(t) is for t>0 a differentiable semigroup of n×n dimensional matrices then (d/dt)S(t-s) = AS(t-s) and we have

$$(d/dt)D_u x(t;u) = D_x F(x(t;u),u)D_u x(t;u) + \\ + D_u F(x(t;u),u) + B + \\ + \int_0^t AS(t-s)(D_x F(x(s;u(s),u(s))D_u x(s;u(s))ds + \\ + \int_0^t AS(t-s)(D_u F(x(s;u(s)),u(s))ds + B)ds \quad (3.4)$$

Hence,

$$(d/dt)D_u x(t;u)_{x=0,u=0} = \\ = D_x F(0,0)D_u x(t;u(t)) + D_u F(0,0) + B + \\ + \int_0^t AS(t-s)(D_x F(0,0)D_u x(t;u(s)) + D_u F(0,0) + B)ds = \\ = D_x F(0,0)D_u x(t;u(t)) + D_u F(0,0) + B + \\ + A\int_0^t S(t-s)(D_x F(0,0)D_u x(t;u(t)) + D_u F(0,0) + B)ds \quad (3.5)$$

On the other hand it is well known, that the solution of the linear ordinary differential equation (3.1) with constant coefficients and zero initial condition has the following integral form

$$z(t) = \int_0^t S(t-s)(D_x F(0,0)z(s) + (B + D_u F(0,0))v(s))ds \quad (3.6)$$

Hence, substituting (3.6) into (3.5) multiplying both sides by v(t) and denoting

$$z(t) = D_u x(t;u)_{x=0,u=0} v(t)$$

we have

$$\dot{z}(t) = D_x F(0,0)z(t) + (D_u F(0,0) + B)v(t) + \\ + A\int_0^t S(t-s)(D_x F(0,0)z(s) + (D_u F(0,0) + B)v(t))dsv = \\ = Az(t) + D_x F(0,0)z(t) + (D_u F(0,0) + B)v(t) \quad (3.7)$$

Therefore, differential equation (3.6) can be expressed as follows

$$\dot{z}(t) = Az(t) + D_x F(0,0)z(t) + Bv(t) + D_u F(0,0)v(t) \quad (3.8)$$

Hence Lemma 3.2 follows.

4. CONTROLLABILITY CONDITIONS.

In this section we shall study constrained local relative controllability in [0,T] for semilinear dynamical system (2.1) using the associated linear 2n-dimensional control dynamical system

$$z'(t) = Cz(t) + Du(t) \qquad t \in [0,T] \qquad (4.1)$$

with zero initial condition z(0)=0, where

$$C = A + D_x F(0,0) \qquad D = B + D_u F(0,0) \qquad (4.2)$$

The main result is the following sufficient condition for constrained local controllability of the semilinear second order dynamical system (2.1).

Theorem 4.1 Suppose that
(i) f(0,0) = 0,
(ii) $U_c \subset U$ is a closed and convex cone with vertex at zero,
(iii) The associated linear control system (4.1) is U_c-globally controllable in [0,T].

Then the semilinear second order stationary dynamical control system (2.1) is U_c-locally controllable in [0,T].

Proof. Let us define for the nonlinear dynamical system (3.1) a nonlinear map $g: L_\infty([0,T],U_c) \rightarrow X$ by g(u) = x(T,u).

Similarly, for the associated linear dynamical system (4.1), we define a linear map $H: L_\infty([0,T],U_c) \rightarrow X$ by Hv = z(T,v).

By the assumption (iii) the linear dynamical system (4.1) is U_c-globally relative controllable in [0,T]. Therefore, by the Definition 2.2 the linear operator H is surjective i.e., it maps the cone U_{ad} onto the whole space X. Furthermore, by Lemma 3.2 we have that Dg(0)=H.

Since U_c is a closed and convex cone, then the set of admissible controls $U_{ad}=L_\infty([0,T],U_c)$ is also a closed and convex cone in the function space $L_\infty([0,T],U)$. Therefore, the nonlinear map g satisfies all the assumptions of the generalized open mapping theorem stated in the Lemma 3.1. Hence, the nonlinear map g transforms a conical neighborhood of zero in the set of admissible controls U_{ad} onto some neighborhood of zero in the state space X. This is by Definition 2.1 equivalent to the U_c-local relative controllability in [0,T] of the semilinear dynamical control system (2.1). Hence, our theorem follows.

In practical applications of the Theorem 4.1, the most difficult problem is to verify the assumption (iii) about constrained global controllability of the linear stationary dynamical system (4.1) (Klamka, 1991, 1996, Son, 1990). In order to avoid this disadvantage, we may use the following Theorem.

Theorem 4.2. (Klamka, 1991, 1993, 1996, Son, 1990). Suppose, that the set U_c is a given convex cone with vertex at zero and a nonempty interior in the space of control values R^m.

Then the associated linear dynamical control system (4.1) is U_c-globally controllable in [0,T] if and only if

(1) it is controllable without any constraints, i.e.

$\mathrm{rank}[D,CD,C^2D,...,C^{n-1}D] = 2n$,

(2) there is no real eigenvector $v\in R^n$ of the matrix C^{tr} satisfying inequalities

$v^{tr}Du\leq 0$, for all $u\in U_c$.

It should be pointed out that for the single input semilinear second order dynamical systems (2.1), associated linear dynamical control system (4.1), i.e. for the case of scalar controls and m=1, Theorem 4.2 reduces to the following Corollary.

Corollary 4.1 (Brammer, 1978, Klamka, 1991, Son, 1990). Suppose, that the dynamical system (2.1) has single input, i.e. m=1 and $U_c=R^+$.
Then the associated linear dynamical control system (4.1) is U_c-globally controllable in [0,T] if and only if it is controllable without any constraints, i.e.

$\mathrm{rank}[D,CD,C^2D,...,C^{n-1}D]=2n$,

and matrix C has only complex eigenvalues.

Remark 4.1 It should be stressed that the important advantage of the Corollary 4.1 is that instead rather difficult condition 2 given in Theorem 4.2 it is enough to verify only eigenvalues of the matrix C.

5. EXAMPLE

Finally, let us consider constrained controllability of the simple illustrative example. Let the semilinear second-order finite-dimensional dynamical control system defined on a given time interval [0,T], has the following form

$$\begin{aligned} w_1''(t) &= w_2(t) + \sin u(t) \\ w_2''(t) &= -\sin w_1(t) + u(t) \end{aligned} \qquad (5.1)$$

Therefore, n=2, m=1, $w(t)=(w_1(t),w_2(t))^{tr}\in R^2=W$, $u(t)\in U_c=R^+$, and using the notations given in the previous sections matrices G and H and the nonlinear mapping f have the following form

$$G=\begin{bmatrix}0&1\\0&0\end{bmatrix} \qquad H=\begin{bmatrix}0\\1\end{bmatrix} \quad f(w(t),u(t))=\begin{bmatrix}\sin u(t)\\ \sin w_1(t)\end{bmatrix}$$

Moreover, let the cone of values of controls be a cone of positive numbers i.e., $U_c=R^+$, and therefore, the set of admissible controls has the following form $U_{ad}=L_\infty([0,T],R^+)$. Hence, we have

$$A=\begin{bmatrix}0&I\\G&0\end{bmatrix}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\0&0&0&0\end{bmatrix} \qquad B=\begin{bmatrix}0\\H\end{bmatrix}=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$F(0,0)=\begin{bmatrix}0\\f(0,0)\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\mathrm{D_x}F(x(t),u(t))=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-\cos w_1(t)&0&0&0\end{bmatrix}$$

$$\mathrm{D_x}F(0,0)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{bmatrix}$$

$$D_uF(x(t),u(t))=\begin{bmatrix}0\\0\\\cos u(t)\\0\end{bmatrix} \qquad D_uF(0,0)=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

$$C = A + D_x F(0,0,0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$D = B + D_u F(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, the characteristic equation of the matrix C det(sI-C)=s^4+1=0 and hence, the matrix C has only complex eigenvalues. Moreover, we have

$$rank\left[D, CD, C^2 D, C^3 D\right] = rank \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} = 4 = 2n$$

Hence, both assumptions of the Theorem 4.2 are satisfied and therefore, the associated linear 2n-dimensional dynamical control system (4.1) with above matrices C and D is R^+-globally controllable in a given time interval [0,T]. Moreover, all the assumptions stated in the Theorem 4.1 are also satisfied and thus the second-order semilinear dynamical control systems (5.1) is R^+-locally controllable in [0,T].

6. CONCLUSIONS

In the paper sufficient conditions for constrained local controllability near the origin for semilinear second-order stationary finite-dimensional dynamical control systems have been formulated and proved. It was generally assumed, that control values are in a given convex cone with vertex at zero and nonempty interior. In the proof of the main result generalized open mapping theorem (Robinson) has been used. These conditions extend to the case of constrained controllability of second-order finite-dimensional semilinear dynamical control systems the results published in (Klamka, 1991, 1993, 1996) for unconstrained nonlinear systems.

The method presented in the paper is quite general and covers wide class of semilinear dynamical control systems. Therefore, similar constrained controllability results may be derived for more general class of semilinear dynamical control systems. For example, it seams, that it is possible to extend sufficient constrained controllability conditions given in the previous sections for more general class of semilinear dynamical control systems with single point delay in the control or with multiple point delays in the controls or in the state variables and for the discrete-time semilinear control systems.

REFERENCES.

Brammer R.F. (1972). Controllability in linear autonomous systems with positive controllers. *SIAM Journal on Control*, **10**, 339-353.

Chukwu E.N., and Lenhart S.M. (1991). Controllability questions for nonlinear systems in abstract spaces. *Journal of Optimization Theory and Applications*, **68**, 437-462.

Fujii N., and Sakawa Y. (1974). Controllability for nonlinear differential equations in Banach space. *Automatic Control Theory and Applications*, **2**, 44-46.

Klamka J. (1991). *Controllability of Dynamical Systems*. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Klamka J. (1993). Controllability of dynamical systems - a survey. *Archives of Control Sciences*, **2**, 281-307.

Klamka J. (1996) Constrained controllability of nonlinear systems. *Journal of Mathematical Analysis and Applications*, **201**, 365-374.

Naito K. (1987). Controllability of semilinear control systems dominated by the linear part. *SIAM Journal on Control and Optimization*, **25**, 715-722.

Peichl G., and Schappacher W. (1986). Constrained controllability in Banach spaces. *SIAM Journal on Control and Optimization*, **24**, 1261-1275.

Robinson S.M. (1976). Stability theory for systems of inequalities. Part II. Differentiable nonlinear systems. *SIAM Journal on Numerical Analysis*, **13**, 497-513.

Seidman T.I. (1987). Invariance of the reachable set under nonlinear perturbations. *SIAM Journal on Control and Optimization*, **25**, 1173-1191.

Son N.K. (1990). A unified approach to constrained approximate controllability for the heat equations and the retarded equations. *Journal of Mathematical Analysis and Applications*, **150**, 1-19.

Zhou H.X. (1984). Controllability properties of linear and semilinear abstract control systems. *SIAM Journal on Control and Optimization*, **22**, 405-422.