Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Frequency Domain Approach to Computing Loop Phase Margins of Multivariable Systems ⋆

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Abstract: The loop phase margins of multivariable control systems are defined as the allowable individual loop phase perturbations within which stability of the closed-loop system is guaranteed. This paper presents a frequency domain approach to accurately computing these phase margins for multivariable systems. With the help of unitary mapping between two complex vector space, the MIMO phase margin problem is converted using the Nyquist stability analysis to the problem of some simple constrained optimization, which is then solved numerically with the Lagrange multiplier and Newton-Raphson iteration algorithm. The proposed approach can provide exact margins and thus improves the LMI results reported before, which could be conservative.

1. INTRODUCTION

Phase margin measures how far a system is away from instability if phase change is allowed only, which is important in control theory because it reflects the relative stability or robustness of the closed-loop system (Horowitz, 1963). For single-input-single-output (SISO) system, phase margin is well defined and can be easily determined by Nyquist plot or Bode diagram. However, it is not straightforward to be extended to a multi-input-multi-output (MIMO) system because of the coupling among loops as well as complexity of matrix perturbations of unity size with different directions (Wang, 2003). Although Gershgorin bands can be used to find the phase margin of MIMO systems (Ho et al., 1997), this method gives conservative results because it requires the diagonal dominance of the system. Bar-on and Jonckheere (1990) presented the definition for the phase margin of the MIMO system, where the stability of the closed-loop system is guaranteed if the phase perturbation of a unitary matrix in the feedback path is less than the phase margin of the system. Such a definition allows the perturbations to be in the entire set of unitary matrices, not necessarily to be diagonal. While this is a nice formulation, permissible perturbations in this class are simply too rich to imagine intuitively and connect to phase changes of individual loops, which practical control engineers have been used to. Thus, a more direct and useful class of phase perturbations is one of diagonal phase perturbations. This corresponds to a multivariable control system where each loop has some phase perturbation but no gain change. Even in this case, the problem is not simple as one cannot calculate phase margin from each loop separately due to loop interactions, that is, one-loop's phase margin depends on other-loop's one.

Recently, Wang et al. (2007) proposed a time-domain method to obtain the loop phase margin for multivariable systems. The basic idea is motivated by the link between the phase lag and the time delay. Consider an MIMO system under a decentralized delay feedback, the stabilizing ranges of all time delays is obtained by an LMI delay-dependent stability criterion with the help of the free-weighting-matrix method (He et al., 2004; Wu et al., 2004). Then, these stabilizing ranges of time delays is converted into the stabilizing ranges of phases by multiplying some fixed frequency based on a proposition in Bar-on and Jonckheere (1990), which are finally taken as the loop phase margins. Although the proposed method provides a systematic approach to calculate the loop phase margins for any given MIMO system, some conservativeness still exists: 1) LMI delay-dependent stability criterion is sufficient only and not necessary, which is common for all LMI techniques. 2) the fixed frequency determined by Bar-on and Jonckheere is under-estimated because their phase perturbation is not necessary to be diagonal.

This paper aims to reduce the above conservativeness in Wang et al. (2007). Rather than a time domain method, a frequency domain approach is proposed to accurately computing loop phase margins for multivariable systems. Based on the work of Bar-on and Jonckheere and with the help of unitary mapping between two complex vector space, the MIMO phase margin problem is converted to some simple constrained optimization problem, which is then solved numerically with the Lagrange multiplier method and Newton-Raphson iteration algorithm. New constraints are added in the optimization problem such that the diagonal structure of phase perturbations is main-

⋆ This work was sponsored by the Ministry of Education's AcRF Tier 1 funding, R-263-000-306-112, Singapore.
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tained. Accordingly, loop phase margin are well defined and easily determined.

2. LOOP PHASE MARGINS

In Wang et al. (2007), the loop phase margins problem is formulated as follows:

Problem 1. For an $m \times m$ square open-loop, $G(s)$, under the decentralized phase perturbation, $\Delta = \text{diag}\{e^{j\phi_1}, \ldots, e^{j\phi_m}\}$, find the ranges, $(\underline{\phi_i}, \overline{\phi_i})$, $-\pi \leq \underline{\phi_i} < \overline{\phi_i} < \pi$, $i = 1, \ldots, m$, such that the closed-loop system is stable when $\phi_i \in (\underline{\phi_i}, \overline{\phi_i})$ for all i, but marginally stable when $\phi_i = \underline{\phi_i}$ or $\phi_i = \overline{\phi_i}$ for some i.

Definition 2. The solution to Problem 1, $\phi_i \in (\underline{\phi_i}, \overline{\phi_i})$, is called the phase margin of the i-th loop of $G(s)$ under other loops' phases of $\phi_j \in (\underline{\phi_j}, \overline{\phi_j})$, $j \neq i$, $i = 1, \cdots, m$. If $\underline{\phi_i} = \underline{\phi_j} = \underline{\phi}$ and $\overline{\phi_i} = \overline{\phi_j} = \overline{\phi}$, then $(\underline{\phi}, \overline{\phi})$ is called the common phase margin of $G(s)$.

To obtain the loop phase margins, a time domain approach is proposed with two steps. Firstly, consider the MIMO system under a decentralized delay feedback and obtain the stabilizing ranges of time delays for each loop based on LMI techniques with a delay-dependent stability criterion. Then, convert the stabilizing ranges of time delays into the stabilizing ranges of phases by multiplying some fixed frequency determined by a proposition in Bar-on and Jonckheere (1990). It should be pointed out that the loop phase margins obtained with this time domain approach are indeed stability margins but may not be exact or maximum margins available. This is due to conservativeness introduced in both steps. For the first step, LMI based delay-dependent stability criterion gives only sufficient but not necessary conditions for stability of the closed-loop system under loop delay perturbations. This conservativeness is common in all the LMI techniques. For the second step, the frequency is under-estimated because the phase perturbation in Bar-on and Jonckheere (1990) is not required to be diagonal. To reduce this conservativeness, a frequency domain approach is proposed, which is demonstrated in the following section.

3. THE PROPOSED APPROACH

The following lemma shows the key property of the stabilizing boundary.

Lemma 3. The stabilizing boundary is symmetric with respect to the origin.

Proof. Suppose that $(\phi_1, \ldots, \phi_m)$ is the point on the stabilizing boundary, then there exists some ω_c such that $\det[I+G(j\omega_c)\Delta] = \det[I+G(j\omega_c)\text{diag}\{e^{j\phi_1}, \ldots, e^{j\phi_m}\}] = 0$. Taking conjugate on both sides of the above equation yields

$$\begin{aligned}\det{}^*[I + G(j\omega_c)\Delta] &= \det[I + G^*(j\omega_c)\Delta^*]\\ &= \det[I + G(-j\omega_c)\text{diag}\{e^{-j\phi_1}, \ldots, e^{-j\phi_m}\}] = 0,\end{aligned}$$

which implies that for the pair $(-\phi_1, \ldots, -\phi_m)$, there exists $-\omega_c$ such that the closed-loop system is marginally stable. Hence, $(-\phi_1, \ldots, -\phi_m)$ is also the point on the stabilizing boundary.

It follows from Definition 2 that loop phase margins of a given MIMO system is the polytope in m-dimensional real vector space representing m independent loop phase perturbations. Consider the unity output feedback sys-

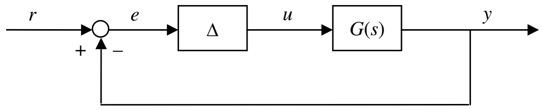

Fig. 1. Diagram of a MIMO control system

tem depicted in Fig. 1, where $G(s)$ represents the open-loop transfer function matrix of size of $m \times m$, and $\Delta(s) = \text{diag}\{e^{j\phi_i}\}$, $i = 1, 2, \cdots, m$, is the diagonal phase perturbation matrix. Note that unlike a common robust stability analysis where the nominal case means $\Delta(s) = 0$, our nominal case means no phase perturbations, i.e., $\phi_i = 0, i = 1, 2, \cdots, m$, and thus $\Delta(s) = I_m$, the identity matrix. Except the above difference, we follow the typical robust stability analysis framework. In particular, we assume, throughout this paper, nominal stabilization of the closed-loop system, that is, the closed-loop system is stable when $\Delta(s) = I_m$. By the assumed nominal stabilization, the system can be de-stabilized if and only if there is a phase perturbation Δ such that

$$\det(I + G(j\omega)\Delta) = 0, \tag{1}$$

which is equivalent to the existence of some unit vector $\mathbf{z} \in \mathbb{C}^m$ such that

$$\mathbf{z} = \Delta\mathbf{v} = -\Delta G\mathbf{z}, \tag{2}$$

where "$-$" denotes the negative feedback configuration. Thus, Δ is a unitary matrix which maps the unit vector $\mathbf{v}$ into $\mathbf{z}$. If all solutions to (2), $\mathbf{z}$ and $\mathbf{v}$, can be found, boundary points, ϕ_i, $i = 1, 2, \cdots, m$, are simply the phase angle of divisions by the corresponding elements from $\mathbf{z}$ and $\mathbf{v}$. However, solutions to (2) do not always exist for $\forall\omega \in (-\infty, +\infty)$ since solutions to (1) are frequency-dependent. Hence, the basic idea of the proposed method is composed of two parts. (i) find the frequency range, Ω, within which the existence of all solutions to (2) is guaranteed; (ii) find numerical solutions to (2) within a framework of the constrained optimization.

The following proposition proposed by Bar-on and Jonckheere (1990) can be used to over-estimate Ω.

Proposition 4. (Bar-on and Jonckheere, 1990). There exists a unitary perturbation Δ destabilizing $G(j\omega)$ if and only if there exists an ω such that $0 \leq \underline{\sigma}(G(j\omega)) \leq 1 \leq \overline{\sigma}(G(j\omega))$.

Let $\hat{\Omega} = \{\omega | 0 \leq \underline{\sigma}(G(j\omega)) \leq 1 \leq \overline{\sigma}(G(j\omega))\}$, then $\Omega \subseteq \hat{\Omega}$ because Δ in Proposition 4 does not limit to be diagonal, which implies that Ω is over-estimated by $\hat{\Omega}$. Note that singular values are the square roots of eigenvalues of the cascade system $G^*(s)G(s)$ and suppose the state-space representation of $G(s)$ to be

$$\begin{aligned}\dot{x}_1 &= Ax_1 + Bu,\\ y_1 &= Cx_1,\end{aligned}$$

then, $G(s) = C(sI - A)^{-1}B$ and $G^*(s) = G^T(-s) = [C(-sI - A)^{-1}B]^T = -B^T(sI + A^T)^{-1}C^T$. The state-space representation of $G(s)^*$ is

$$\dot{x}_2 = -A^T x_2 + C^T y_1,$$
$$y_2 = -B^T x_2.$$

The system $G(s)^*G(s)$ is then written as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}A & 0\\ C^T C & -A^T\end{bmatrix}}_{\tilde{A}}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}B\\ 0\end{bmatrix}}_{\tilde{B}} u,$$
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & -B^T\end{bmatrix}}_{\tilde{C}}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

To obtain the set $\hat{\Omega}$, we may begin with finding the ω_i such that $\sigma(G(j\omega_i)) = 1$, which implies that $\det[I - G^*(j\omega_i)G(j\omega_i)] = \det[I - \tilde{C}(j\omega_i I - \tilde{A})^{-1}\tilde{B}] = \det[I - \tilde{B}\tilde{C}(j\omega_i I - \tilde{A})^{-1}] = \det[j\omega_i I - (\tilde{A} + \tilde{B}\tilde{C})]\det(j\omega_i I - \tilde{A})^{-1} = 0$. Suppose that $G(j\omega_i)$ has no poles on the imaginary axis, which is the case for most MIMO plants in practice, $\det(j\omega_i I - \tilde{A}) \neq 0$ for $\forall\omega$. Thus, $\det[j\omega_i I - (\tilde{A} + \tilde{B}\tilde{C})] = 0$ yields that ω_i are pure imaginary eigenvalues of $(\tilde{A} + \tilde{B}\tilde{C})$. Note that $\sigma(G(j\omega))$ is a continuous function of ω, and between the interval of two consecutive ω_i and ω_{i+1}, no other $\omega \in (\omega_i, \omega_{i+1})$ exists such that $\sigma(G(j\omega)) = 1$, otherwise, ω_i and ω_{i+1} are not consecutive any more. This implies that $\sigma(G(j\omega))$ is always greater or less than 1 for $\forall\omega \in (\omega_i, \omega_{i+1})$. Hence, by calculating $\overline{\sigma}(G(j\omega))$ and $\underline{\sigma}(G(j\omega))$ for one $\omega \in (\omega_i, \omega_{i+1})$, we know whether $(\omega_i, \omega_{i+1}) \subseteq \hat{\Omega}$. By Lemma 3, ω is symmetric with respect to the origin, only positive ω need to be checked, which can simplify the process of calculation.

For every $\omega \in \hat{\Omega}$, $\mathbf{z}$ can be found from (2) by solving an equivalent constrained optimization problem. Since $\hat{\Omega}$ is over-estimated for Ω, some $\omega \in \hat{\Omega}$ may cause the Newton-Raphson algorithm divergent, which implies that no diagonal phase perturbation exists to destabilize the closed-loop system at that frequency. In the following, we show how to find $\mathbf{z}$ in the framework of constrained optimization.

Let $\mathbf{z} = [z_1, z_2, \cdots, z_m]^T$ and $\mathbf{v} = [v_1, v_2, \cdots, v_m]^T$. A diagonal unitary mapping via $\mathbf{z} = \Delta\mathbf{v}$ yields $|z_k| = |v_k|$, i.e., $z_k^* z_k = v_k^* v_k$, $k = 1, 2, \cdots, m$. One can write $z_k^* z_k = \mathbf{z}^* H_k \mathbf{z}$, where $H_k = [h_{i,j}] \in \mathbb{R}^{m\times m}$ is given by

$$h_{i,j} = \begin{cases} 1, & i = j = k;\\ 0, & \text{otherswise,}\end{cases}$$

and $v_k^* v_k = \mathbf{v}^* H_k \mathbf{v} = \mathbf{z}^* G^* H_k G \mathbf{z}$ since $\mathbf{v} = -G\mathbf{z}$. Thus, $z_k^* z_k = v_k^* v_k$ yields $\mathbf{z}^*(H_k - G^* H_k G)\mathbf{z} = 0$. Unit $\mathbf{z}$ and $\mathbf{v}$ yield $\mathbf{z}^*\mathbf{z} = 1$ and $\mathbf{v}^*\mathbf{v} = \mathbf{z}G^*G\mathbf{z} = 1$. Due to the diagonal nature of Δ, $\mathbf{v}^*\mathbf{v} = \sum_{k=1}^m v_k^* v_k = \sum_{k=1}^m z_k^* z_k = \mathbf{z}^*\mathbf{z} = 1$, which implies only $m+1$ independent constraints as follows:

$$\begin{cases}\mathbf{z}^*\mathbf{z} = 1,\\ \mathbf{z}^*(H_k - G^* H_k G)\mathbf{z} = 0,\ k = 1, 2, \cdots, m.\end{cases} \tag{3}$$

Once z_k and v_k which meet the above constraints can be obtained and, $z_k/v_k = \mathrm{e}^{j\phi_k}$, where ϕ_k is the phase change from v_k to z_k. However, solutions to (3) is not unique because $\phi_k \pm 2k\pi$, $k \in \mathbb{N}$, is also a solution. Here, we limit $\phi_k \in [-\pi, \pi)$ since the nominal system ($\phi_i = 0, i = 1, 2, \cdots, m$) is stable according to our assumption.

Suppose that $\overline{\phi} = \max\{|\phi_k|\}$ and $\underline{\phi} = \min\{|\phi_k|\}$, the inner product of $\mathbf{v}$ and $\mathbf{z}$ is

$$\begin{aligned}\langle\mathbf{v}, \mathbf{z}\rangle = \mathbf{v}^*\mathbf{z} &= \sum_{k=1}^m v_k^* z_k = \sum_{k=1}^m \mathrm{e}^{j\phi_k} v_k^* v_k\\ &= \sum_{k=1}^m |v_k|^2 \cos\phi_k + j\sum_{k=1}^m |v_k|^2 \sin\phi_k,\end{aligned}$$

where

$$\begin{aligned}\sum_{k=1}^m |v_k|^2\cos\phi_k &= \sum_{k=1}^m |v_k|^2 \cos|\phi_k|\\ &\geq \cos\overline{\phi}\sum_{k=1}^m |v_k|^2 = \cos\overline{\phi}.\end{aligned}$$

To ensure $\overline{\phi} = \max\{|\phi_k|\}$ really hold, $\cos\overline{\phi}$ has to be minimized, which can be achieved by minimizing its upper bound $\sum_{k=1}^m |v_k|^2\cos\phi_k$. Likewise,

$$\begin{aligned}\sum_{k=1}^m |v_k|^2\cos\phi_k &= \sum_{k=1}^m |v_k|^2 \cos|\phi_k|\\ &\leq \cos\underline{\phi}\sum_{k=1}^m |v_k|^2 = \cos\underline{\phi},\end{aligned}$$

and $\cos\underline{\phi}$ has to be maximized to ensure $\underline{\phi} = \min\{|\phi_k|\}$ really hold, which can be achieved by maximizing its lower bound $\sum_{k=1}^m |v_k|^2\cos\phi_k$. Clearly,

$$2\sum_{k=1}^m |v_k|^2\cos\phi_k = \mathbf{v}^*\mathbf{z} + \mathbf{z}^*\mathbf{v} = -[\mathbf{z}^*(G^* + G)\mathbf{z}].$$

Maximizing $\sum_{k=1}^m |v_k|^2\cos\phi_k$ is equivalent to minimizing $\mathbf{z}^*(G^* + G)\mathbf{z}$ with the constraints (3). Thus, finding the stabilizing boundary of loop phase perturbation is then equivalently converted to the constrained minimization problem as follows:

$$\begin{aligned}&\min[\mathbf{z}^*(G^* + G)\mathbf{z}]\\ &\text{s.t.}\begin{cases}\mathbf{z}^*\mathbf{z} = 1,\\ \mathbf{z}^*(H_k - G^* H_k G)\mathbf{z} = 0,\ k = 1, 2, \cdots, m.\end{cases}\end{aligned} \tag{4}$$

On the contrary, if $\sum_{k=1}^m |v_k|^2\cos\phi_k$ needs to be maximized, an equivalent constrained maximization framework can be constructed in a similar way. Here, we focus on the constrained minimization problem (4) only and omitted its counter part since the numerical algorithm proposed to solve both of them are completely the same.

With the approach of Lagrange multiplier Bertsekas (1982), let

$$\begin{aligned}F(\kappa) = &\ \mathbf{z}^*(G^* + G)\mathbf{z} + \lambda_1(\mathbf{z}^*\mathbf{z} - 1)\\ &+ \sum_{k=1}^m \lambda_{k+1}\mathbf{z}^*(H_k - G^* H_k G)\mathbf{z},\end{aligned}$$

where $\kappa = [z_1, \cdots, z_m, \lambda_1, \lambda_2, \cdots, \lambda_{m+1}]^T$. The constrained optimization problem (4) can be solved by finding zeros of the following function

$$f(\kappa) = \frac{\partial F(\kappa)}{\partial \kappa} = \begin{bmatrix} (G^* + G)\mathbf{z} + \lambda_1 \mathbf{z} \\ + \sum_{k=1}^{m} \lambda_{k+1}(H_k - G^* H_k G)\mathbf{z} \\ \mathbf{z}^*\mathbf{z} - 1 \\ \mathbf{z}^*(H_1 - G^* H_1 G)\mathbf{z} \\ \vdots \\ \mathbf{z}^*(H_m - G^* H_m G)\mathbf{z} \end{bmatrix}, \quad (5)$$

Numerical solutions to (5) are obtained by the Newton-Raphson algorithm:

$$\kappa_{n+1} = \kappa_n - J^{-1}[f(\kappa_n)]f(\kappa_n), \quad (6)$$

where J is the Jacobian matrix of $f(\kappa)$. If J is singular, then a Moore-Penrose inverse is used Lancaster and Tismenetsky (1985). Once the iteration routine converges to a zero of $f(\kappa)$, the eigenvalues of the Hessian matrix

$$H = \frac{\partial^2 F}{\partial \mathbf{z}^2} = \left[(G^* + G) + \lambda_1 I_m + \sum_{k=1}^{m} \lambda_{k+1}(H_k - G^* H_k G) \right]$$

is calculated to see whether it is a local minimum or maximum. For the local minimum (or maximum), a new initial search direction is chosen as the negative of the eigenvector of H corresponding to the most positive (or negative) eigenvalue to achieve the local maximum (or minimum). Since the cost function and the constraints in (4) are quadratic form of $\mathbf{z}$, the local minimum (or maximum) is also the global minimum (or maximum).

It should be pointed out here that $\mathbf{z}, \mathbf{v} \in \mathbb{C}^m$ will lead to the failure of solving the optimization problem (4) because neither the cost function nor the constraints are holomorphic functions of $\mathbf{z}$ or ω, refer to Grasse and Bar-on (1997). Fortunately, the standard technique of converting (4) to an equivalent real constrained optimization problem is applicable by the process of decomplexification, which makes use of a canonical isomorphism between $\mathbb{C}^m$ and $\mathbb{R}^{2m}$. Let $z_k = x_k + jy_k$, $x_k, y_k \in \mathbb{R}$, $k = 1, 2, \cdots, m$; $\mathbf{z_c} = [x_1, y_1, \cdots, x_m, y_m]^T \in \mathbb{R}^{2m}$; $G_{i,j} = x_{i,j} + jy_{i,j}$, $x_{i,j}, y_{i,j} \in \mathbb{R}$, $i, j = 1, 2, \cdots, m$; and

$$G_c = \begin{bmatrix} x_{1,1} & -y_{1,1} & \cdots & x_{1,m} & -y_{1,m} \\ y_{1,1} & x_{1,1} & \cdots & y_{1,m} & x_{1,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m,1} & -y_{m,1} & \cdots & x_{m,m} & -y_{m,m} \\ y_{m,1} & x_{m,1} & \cdots & y_{m,m} & x_{m,m} \end{bmatrix} \in \mathbb{R}^{2m \times 2m},$$

there holds $\mathbf{z}^*(G^* + G)\mathbf{z} = \mathbf{z_c}^T(G_c^T + G_c)\mathbf{z_c}$; $\mathbf{z}^*\mathbf{z} = \mathbf{z_c}^T\mathbf{z_c}$; $\mathbf{z}^*(H_k - G^* H_k G)\mathbf{z} = \mathbf{z_c}^T(H_k^c - G_c^T H_k^c G_c)\mathbf{z_c}$, $k = 1, 2, \cdots, m$, where $H_k^c = [h_{i,j}] \in \mathbb{R}^{2m \times 2m}$ with

$$h_{i,j} = \begin{cases} 1, \ i = j = 2k \text{ or } 2k - 1; \\ 0, \ \text{otherswise.} \end{cases}$$

Thus, the constrained optimization (4) in $\mathbb{C}^m$ is equivalent to the optimization problem in $\mathbb{R}^{2m}$ as follows:

$$\min[\mathbf{z_c}^T(G_c^T + G)\mathbf{z_c}] \quad (7)$$
$$\text{s.t.} \begin{cases} \mathbf{z_c}^T\mathbf{z_c} = 1, \\ \mathbf{z_c}^T(H_k^c - G_c^T H_k^c G_c)\mathbf{z_c} = 0, \ k = 1, 2, \cdots, m. \end{cases}$$

Newton-Raphson iteration algorithm is then used to calculate the stabilizing boundary of the diagonal phase perturbation. Once the boundary is obtained, hypercubes are ready to be prescribed and the loop phase margin can be easily determined according to Definition 2.

The algorithm to find loop phase margins is summarized as follows:

Step 1. Determine the frequency range Ω such that the solutions to (1) or (2) exist;
Step 2. Construct the framework of the constrained optimization (4), which is then converted equivalently to its isomorphism in real space as (7);
Step 3. For every $\omega \in \Omega$, solve (7) with Lagrange multiplier and find $\mathbf{z}$ by Newton-Raphson iteration (6);
Step 4. Use the similar procedures in *Step 3* to solve maximum of (7) with different initial values;
Step 5. The points on the stabilizing boundary of loop phase margins are given by $\phi_i = \arg\{z_i/v_i\}$, $i = 1, 2, \cdots, m$, and loop phase margins are hypercubes prescribed in the stabilizing region.

4. ILLUSTRATION EXAMPLE

Example 5. (Bar-on and Jonckheere, 1990). Consider the system (A, B, C) as follows

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -0.75 & 1 & 0.25 \\ 0 & 0 & 0 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0.25 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Its transfer function matrix is

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^4 + 1.75s^3 + 7.5s^2 + 4s + 8} \times \begin{bmatrix} 0.0625s + 0.25 & s^2 + s + 4 \\ 0.25s^2 + 0.1875s + 0.75 & s + 4 \end{bmatrix}.$$

It follows from Bar-on and Jonckheere (1990) that $\hat{\Omega} = (0.643, 1.613)$. For every $\omega \in \hat{\Omega}$, obtain the solution $\mathbf{z}$ to the constrained optimization (7) with the Newton-Raphson iteration (6), where initial values are arbitrarily chosen. The frequency range for the convergence of (6) yields $\Omega = (0.764, 0.884) \cup (1.533, 1.572)$, shown as Fig. 2, where the solid and dashed lines represent the minimum and maximum loci of the cost function, respectively, who constitute two closed contours, denoted by A and B. As the cost function moves along contour A (or B), the pair (ϕ_1, ϕ_2) moves along their stabilizing boundary A (or B) correspondingly, shown as Fig. 3, where the dotted lines are determined by the symmetry with respect to $(0, 0)$ from Lemma 3.

It needs to be clarified that two (or more) boundaries may be obtained in the limited range $[-\pi, \pi)$ like this example shows. In such a case, the stabilizing region of ϕ_i is the polytope encompassed by the nearest boundary (Boundary B for this example). To show the stabilizing region of (ϕ_1, ϕ_2) for this example more clearly, zoom Boundary B in Fig. 4, where $\phi_1 \in [-\pi, \pi)$ and $\phi_2 \in [-0.2423, 0.2423]$ for $\forall \omega \in (1.533, 1.572)$. With the help of Lemma 3, $\phi_1 \in (-\pi, \pi)$ and $\phi_2 \in (-0.2423, 0.2423)$ are one of the phase margins for loop 1 and 2, respectively. Since there is no boundary for ϕ_1, the common phase margin for this example can be determined by letting $\phi_1 = \phi_2$, which is $(-0.3108, 0.3108)$, or $(-17.808°, 17.808°)$.

Table 1. Comparison with other methods

Method	Loop Phase Margin		Common Margin
	ϕ_1	ϕ_2	
The Proposed	$(-\pi, \pi)$	$(-0.2423, 0.2423)$	$(-0.3108, 0.3108)$
Bar-on and Jonckheere (1990)	—	—	$(-0.2701, 0.2701)$
Wang et al. (2007)	$[0, 0.1878]$	$[0, 0.1231]$	$[0, 0.1265]$

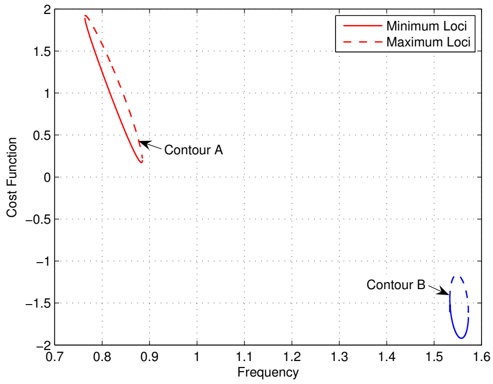

Fig. 2. Solving the constrained optimization for $\omega \in \Omega$

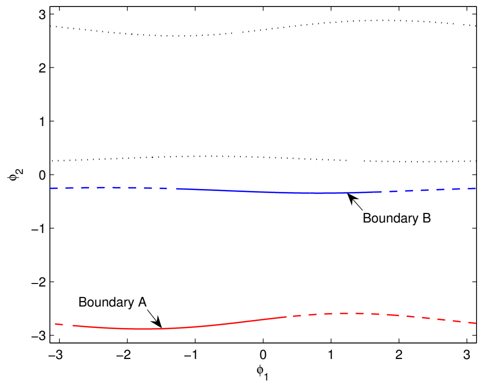

Fig. 3. Stabilization boundaries for (ϕ_1, ϕ_2)

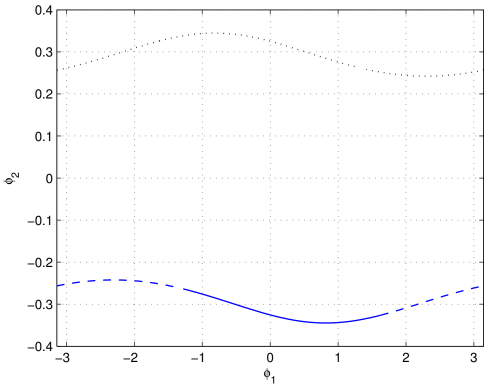

Fig. 4. Stabilization Region of (ϕ_1, ϕ_2)

Table 1 listed some comparison of the proposed method with the existing frequency domain and time domain methods of Bar-on and Jonckheere (1990) and Wang et al. (2007). The method of Bar-on and Jonckheere (1990) actually gives the common phase margin only, which is smaller than the result given by the proposed method. This is because for the decentralized control, the phase perturbations are not necessarily ergodic in the entire set of unitary matrices. The proposed method also gives a larger loop and common phase margins than the method of Wang et al. (2007) does, which implies the evident improvements to the time domain approaches.

5. CONCLUSION

In this paper, the loop phase margins of multivariable control systems are defined as the allowable individual loop phase perturbations within which stability of the closed-loop system is guaranteed. A frequency domain approach is presented to accurately computing these phase margins, which is converted using the Nyquist stability analysis to the problem of some simple constrained optimization with the help of unitary mapping between two complex vector space. Numerical solutions is then found with the Lagrange multiplier and Newton-Raphson iteration algorithm. The proposed approach can provide exact loop phase margins and thus improves the results by time domain approaches which are based on LMI techniques.

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