LINEAR PARAMETER VARYING SYSTEMS: A GEOMETRIC THEORY AND APPLICATIONS

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Abstract: Linear Parameter Varying (LPV) systems appear in a form of LTI state space representations where the elements of the $A(\rho), B(\rho), C(\rho)$ matrices depend on an unknown but at any time instant measurable vector parameter $\rho \in \mathcal{P}$. This paper describes a geometric view of LPV systems. Geometric concepts and tools of invariant subspaces and algorithms for LPV systems affine in the parameters will be presented and proposed. Application of these results will be shown and referenced in solving various analysis (controllabilty/observability) problems, controller design and fault detection problems associated to LPV systems.

Keywords: geometric control, invariant subspaces, vector space distributions, dynamic inversion, decoupling, observer design

1. INTRODUCTION: MOTIVATIONS FOR LPV MODELING

Consider the following linear state space form of a dynamic system:

$$\dot{x} = A(\rho)x + B(\rho)u, \quad (1)$$
$$y = C(\rho)x + D(\rho)u, \quad (2)$$

where $\rho \in \mathcal{P}$ and $\mathcal{P}$ denotes the parameter set.

This representation can describe linear, time-varying (LTV) systems if $\rho = \rho(t)$, LPV systems if $\rho \in \mathcal{P}$ or also nonlinear systems if $\rho = \rho(x)$. If a nonlinear system is described by an input affine form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad y = h(x), \quad (3)$$

then choosing

$$f(x) = A(x)x, \quad B(x) = [g_1(x), \ldots, g_m(x)]$$

and $h(x) = C(x)x$ one obtains a formally linear representation with $\rho(x) = x$. This representation of the input affine systems is usually called as quasi LPV (qLPV) system. In general, the function ρ can be a more complex function of the state vector, it will however, be assumed that the ρ is available (measurable) for each time instant.

A more specific form of LPV systems are those, where the matrices are affine in the scheduling variables:

$$A(\rho(t)) = A_0 + \rho_1(t)A_1 + \ldots + \rho_N(t)A_N, \quad (4)$$
$$B(\rho(t)) = B_0 + \rho_1(t)B_1 + \ldots + \rho_N(t)B_N, \quad (5)$$
$$C(\rho(t)) = C_0 + \rho_1(t)C_1 + \ldots + \rho_N(t)C_N. \quad (6)$$

Most input affine systems (3) can be described in this form by proper choice of the scheduling variables. Moreover, in some applications either the sensor map C or the B are constant matrices.

It can be seen that the qLPV form of a system is non unique. The choice of the scheduling variables and associated matrices can influence the system properties like observability or controllability or can be more suitable to use in a formal (optimal) controller design than others. We will come back to these issues later.

The motivation for rewriting the original nonlinear system equations into LPV form originates from at least two sources. One is a geometric view on control systems. Analysis of LTI systems usually can be performed using finite dimensional vector space concepts like invariant subspaces, bases, linear transforms while analysis of nonlinear systems needs the calculations of distributions and co-distributions. While computations with vector spaces can be performed by using linear algebra tools, the corresponding operations with distributions can be made mostly symbolically. It will be shown later, that using affine LPV representations, various analysis and control problems can be solved using algorithms similar to those used for LTI systems. The additional requirement is that the scheduling variables have to satisfy certain conditions.

Another motivation comes from the optimal control aspects. The solution of the optimal control for nonlinear systems usually requires a solution to an associated Hamilton Jacobi Bellman partial differential equation. This is realistic if we are given a value function, or a good approximation to it, like the control Lyapunov function as it is assumed in deriving universal formulae like Sontag-formula, pointwise min-norm controller formula, Freeman (1995), Sontag (1998), Lu and Doyle (1995). The main objection concerning these results are that performance specifications can not be specified directly. The $\mathcal{L}_2$ analysis and controller synthesis elaborated by LPV and qLPV systems can address this goal via the solution of a set of LMI problems.

This paper will propose a geometric view on the LPV systems.

The geometric approach to dynamic systems appeared e.g. in Basile and Marro (1969), Wonham (1985) for LTI systems and in Isidori (1989) for input affine nonlinear systems where a central role is played by invariant subspaces like (A, B), (C, A) or unobservabilty subspaces and related algorithms like CAISA, UOSA or their corresponding nonlinear versions using vector space distributions and codistributions.

Geometric concepts and tools for parameter varying invariant subspaces, invariant subspace algorithms for a class of LPV systems in affine form (where the $A(\rho), B(\rho), C(\rho)$ matrices are affine in ρ) are presented.

Controllability, observability of LPV, qLPV systems and related problems will be studied in Section 3. It will be shown that a generalized Kalman-rank condition can be derived as a necessary condition (Balas *et al.*, 2003), but there are also conditions on the ρ functions indicating that their choice in describing nonlinear systems in LPV form can influence the above properties of the resulting model.

Prototype control problems like disturbance decoupling problem (DDP), DDP with stability are discussed in (Bokor *et al.*, 2002), dynamic decoupling, system inversion (Szabo *et al.*, 2003) and filter designs (Bokor and Balas, 2004) will be discussed for affine LPV systems in Section 4.

The advantage gained by using LPV formalism is that the solutions can be given in terms of linear algebraic manipulations like those presented for LTI systems in Basile and Marro (1969). This feature allows us to obtain solutions to some nonlinear problems rewritten in qLPV form that are difficult to compute in the original nonlinear form.

Applications of LPV modelling concepts and related results in aerospace control design can be found e.g. in Papageorgiou *et al.* (2000), Marcos and Balas (2001) and in road vehicle control systems design in the references Gaspar *et al.* (2003), Gaspar *et al.* (2004), Gaspar *et al.* (2005), Gaspar *et al.* (2005).

2. INVARIANCE CONCEPTS AND ALGORITHMS FOR LPV SYSTEMS

The concept of (A,B)-invariant and (C,A)-invariant subspaces is extended to the LPV systems by introducing parameter-varying (A,B)-invariant and parameter-varying (C,A)-invariant subspaces. The parameter dependence in the state matrix of these LPV systems is assumed to be affine in form.

Definition 1. A subspace $\mathcal{V}$ is called a *parameter-varying invariant subspace* for the family of the linear maps $A(\rho)$ (or shortly $\mathcal{A}$-invariant subspace) if

$$A(\rho)\mathcal{V} \subset \mathcal{V} \qquad \text{for all } \rho \in \mathcal{P}. \tag{7}$$

The extension of the concept of (A,B)-invariant subspace is as follows:

Definition 2. Let $\mathcal{B}(\rho)$ denote Im $B(\rho)$. Then a subspace $\mathcal{V}$ is called a *parameter-varying (A,B)-invariant subspace* (or shortly $(\mathcal{A}, \mathcal{B})$-invariant subspace) if any of the following equivalent conditions holds:

(1) there exists a mapping $F : [0, T] \to \mathbb{R}^{m\times n}$ such that for all $\rho \in \mathcal{P}$:

$$(A(\rho) + B(\rho)F(\rho))\mathcal{V} \subset \mathcal{V}; \tag{8}$$

(2) for all $\rho \in \mathcal{P}$:

$$A(\rho)\mathcal{V} \subset \mathcal{V} + \mathcal{B}(\rho). \qquad (9)$$

The dual notion of the previous definition is the following

Definition 3. Let $\mathcal{C}(\rho)$ denote Ker $C(\rho)$. Then a subspace $\mathcal{W}$ is called a *parameter-varying (C,A)-invariant subspace* (or shortly $(\mathcal{C},\mathcal{A})$-invariant subspace) if any of the following equivalent conditions holds:

(1) there exists a mapping $G : [0,T] \to \mathbb{R}^{n\times p}$ such that for all time instance $\rho \in \mathcal{P}$:

$$(A(\rho) + G(\rho)C(\rho))\mathcal{W} \subset \mathcal{W}; \qquad (10)$$

(2) for all $\rho \in \mathcal{P}$:

$$A(\rho)(\mathcal{W} \cap \mathcal{C}(\rho)) \subset \mathcal{W}. \qquad (11)$$

A similar concept was introduced in Basile and Marro (1987), called *robust controlled invariant* subspace and an algorithm was given in to determine this robust controlled invariant. Since the number of conditions is not finite, the algorithm, in general, is quite complex.

From a practical point of view it is an important question to characterize these parameter-varying subspaces by a finite number of conditions. Assuming the special structure (4) of the matrix $A(\rho)$ it is immediate that if the inclusions holds for all A_i, then they hold also for all $\rho \in \mathcal{P}$. It is not so straightforward to establish under which conditions is the reverse implication true.

In the remaining part of this paper we will introduce algorithms for the computation of $\mathcal{A}$, $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{A})$ invariant subspaces with constant $B(\rho) \equiv B$ matrices in the case of $\mathcal{A}$-, $(\mathcal{A},\mathcal{B})$-invariance and constant $C(\rho) \equiv C$ matrices in the context of $(\mathcal{C},\mathcal{A})$-invariance.

The $\mathcal{A}$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm over a given subspace $\mathcal{L}$ can be defined as:

$$\mathcal{AISAL}: \quad \mathcal{R}_0 = \mathcal{L}$$
$$\mathcal{R}_{k+1} = \mathcal{L} + \sum_{i=0}^{N} A_i \mathcal{R}_k, \qquad k \geq 0,$$
$$\mathcal{R}^* = \lim_{k\to\infty} \mathcal{R}_k. \qquad (12)$$

The algorithm will stop after a finite number of steps, i.e. $\mathcal{R}^* = \mathcal{R}_{n-1}$. It can be proved that $\mathcal{R}^*$ is such that $\mathcal{R} \subset \mathcal{R}^*$, $\mathcal{R}^*$ is $\mathcal{A}$-invariant and it is minimal with these properties. If $\mathcal{L} = \mathcal{B}$, this algorithm is referenced in the controllability analysis later.

The set of all $(\mathcal{A},\mathcal{B})$-invariant subspaces contained in a given subspace $\mathcal{K}$, is an upper semilattice with respect to subspace addition. This semilattice admits a maximum which can be computed from the $(\mathcal{A},\mathcal{B})$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm:

$$\mathcal{ABISA} \quad \mathcal{V}_0 = \mathcal{K} \qquad (13)$$
$$\mathcal{V}_{k+1} = \mathcal{K} \cap \bigcap_{i=0}^{N} A_i^{-1}(\mathcal{V}_k + \mathcal{B}). \qquad (14)$$

The limit of this algorithm will be denoted by $\mathcal{V}^*$ and its calculation needs at most n steps.

The set of all $(\mathcal{C},\mathcal{A})$-invariant subspaces containing a given subspace $\mathcal{L}$, is a lower semilattice with respect to subspace intersection. This semilattice admits a minimum which can be computed from the $(\mathcal{C},\mathcal{A})$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm (note that $\mathcal{C}$ = Ker C):

$$\mathcal{CAISA} \quad \mathcal{W}_0 = \mathcal{L} \qquad (15)$$
$$\mathcal{W}_{k+1} = \mathcal{L} + \sum_{i=0}^{N} A_i(\mathcal{W}_k \cap \mathcal{C}). \qquad (16)$$

The limit of this algorithm will be denoted by $\mathcal{W}^*$ and its calculation needs at most n steps.

In the so called "geometrical approach" to fault detection, a central role is played by certain unobservability subspaces, (Massoumnia, 1986; Massoumnia *et al.*, 1989) or observability codistributions, (Persis and Isidori, 2000a). An unobservability subspace $\mathcal{S}$ is a subspace such that there exist matrices G and H with the property that $(A+GC)\mathcal{S} \subset \mathcal{S}$, i.e., $\mathcal{S}$ is (C,A)-invariant, and $\mathcal{S} \subset KerHC$. The family of unobservability subspaces containing a given set $\mathcal{L}$ has a minimal element $\mathcal{S}^*$. In what follows, the parameter varying versions of these notions are introduced.

Definition 4. A subspace $\mathcal{S}$ is called parameter-varying unobservability subspace if there exists a constant matrix H and a parameter varying matrix $G : [0,T] \to \mathbb{R}^{n\times p}$ such that

$$\mathcal{S} = \langle \text{Ker } HC | \mathcal{A} + \mathcal{G}C \rangle, \qquad (17)$$

where $\mathcal{A} + \mathcal{G}C$ denotes the system $A(\rho) + G(\rho)C$.

It can be shown that the family of parameter-varying unobservability subspaces containing a given subspace $\mathcal{L}$ is closed under subspace intersection. The minimal element of this family is the result of the parameter-varying $\mathcal{U}$nobservability $\mathcal{S}$ubspace $\mathcal{A}$lgorithm:

$$\mathcal{USA}: \quad \mathcal{S}_0 = \mathcal{X} \qquad (18)$$
$$\mathcal{S}_{k+1} = \mathcal{W}^* + \left(\bigcap_{i=0}^{N} A_i^{-1}\mathcal{S}_k \cap \mathcal{C} \right) \qquad (19)$$
$$\mathcal{S}^* = \lim_{k\to\infty} \mathcal{S}_k \qquad (20)$$

where $\mathcal{W}^*$ is computed by $\mathcal{CAISA}$. Exactly the same algorithms are obtained in the context of bilinear systems, see (Hammouri *et al.*, 1999).

Proposition 1. The subspace $\mathcal{S}^*$ is the smallest parameter–varying unobservability subspace containing the subspace $\mathcal{L}$.

These concepts and algorithms play important roles in solving the analysis and design problems in the oncoming paragraphs.

3. CONTROLLABILITY AND OBSERVABILITY

Although most of the existing approaches to formal controller synthesis impose condtions like controllabilty (stabilizabilty), observabilty (detectabilty) of the systems, apart from LTI case, it is still very involved to verify these properties. There are various approaches to analyze the above system properties. The Silverman–Meadows (Silverman and Medows, 1969), and general nonlinear approaches are based on the concept of the local controllability distributions and directly use $A(t), B(t)$. These result in a rank condition for some distribution in order to test controllability. The rank condition fails if some differential algebraic conditions are fulfilled in terms of the $A(t), B(t)$ parameters of the problem. The advantage of these approaches is that one can obtain this differential algebraic relation explicitly. The disadvantage is that full state controllability might be the only useful information obtained this way. In order to go further and obtain a controllability decomposition, one has to solve involved partial differential equations in a general nonlinear setting, see e.g. Isidori (1989).

This section investigates these issues for LTV, LPV and qLPV systems with an attempt to derive conditions similar to the Kalman rank condition derived for LTI systems. We refer mainly to the controllability problem, since results about observability can be obtained using duality.

Definition 5. A state x_0 is said to be *controllable at time* t_0 if there exist a control function $u(t)$ depending on x_0 and t_0 and defined over some finite closed interval $[t_0, T]$ such that for the corresponding solution one has $x(T) = \Phi_{x_0}(T, t_0) = 0$, where $\Phi_{x_0}(t, t_0)$ denotes the solution of the system starting from $x(t_0) = x_0$. If this is true for every state x and every t_0 then the system will be called (completely) controllable.

Let the dynamic system be given in state space form as:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t). \tag{21}$$

Denote by $X(t)$ the fundamental matrix associated with the above system satisfying the matrix differential equation $\dot{X}(t) = A(t)X(t), \quad X(t_0) = I$, and $X(t) \in \mathbb{R}^{n\times n}$. Let us denote by $\Phi(t, t_0)$ the state transition matrix that is nonsingular for any t, and $\Phi(t, t_0) = X(t)X(t_0)^{-1}$.

The fundamental solution $\Psi(t, t_0)$ associated with the adjoint equation $\dot{P}(t) = -A(t)^*P(t), \quad P(t_0) = I$, is denoted by $\Psi(t, t_0) = \Phi(t_0, t)^* = X(t)^{-*}X(t_0)^*$. One has that $P(t) = X(t)^{-*}$, i.e., $\Phi(t, t_0) = X(t)P(t_0)^*$.

A seminal result, see Kalman (1960), concerning controllability of LTI, LTV systems can be stated as the equivalence of the following statements.

Proposition 2. The system (21) is

(1) controllable on $[\sigma, \tau]$;
(2) the controllability Grammian
$$W(\sigma, \tau) = \int_\sigma^\tau \Phi(\sigma, s)B(s)B(s)^*\Phi(\sigma, s)^*ds$$
is positive definite;
(3) There is no nonzero solution $p(t)$ of the adjoint equation such that $\langle p(t), b_i(t)\rangle = 0$, for almost all $t \in [\sigma, \tau]$, and $i = 1, \cdots, m$, where b_i is the ith column of B and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\mathbb{R}^n$,
(4) There is no nonzero vector $p \in \mathcal{X}$ such that $\langle p, \Phi(\sigma, s)b_i(s)\rangle = 0$, for almost all $s \in [\sigma, \tau]$, and $i = 1, \cdots, m$.

Proposition 2 assumes that the fundamental matrix for a particular system can analytically be derived. For LTI systems this is straight forward (assuming $t_0 = 0$ for simplicity):

$$\Phi(t) = e^{At} = \sum_{i=1}^{n} \psi_i(t)A^{i-1},$$

and the test of controllability using controllability grammians can be decided by consulting the dimension of the reachability subspace

$$\mathcal{R} = \sum_{k=0}^{n-1} \mathrm{Im}A^kB,$$

The proof of this, see e.g. Kailath (1980), includes that it is possible to generate linearly independent functions $\psi_i, i = 1, \ldots, n$ if the Kalman-rank condition

$$rank[B, AB, \ldots, A^{n-1}B] = n$$

is satisfied.

To extend these results to affine LPV and qLPV systems, it will be necessary to use that the fundamental matrix can be found, at least locally, as exponential function of the "coordinates of second kind" associated with the equation

$$\dot{x} = \sum_{i=0}^{N} \rho_i(t)A_ix,$$

i.e., the solutions of the Wei–Norman equation

$$\dot{g}(t) = (\sum_{i=1}^{K} e^{\Gamma_1 g_1} \cdots e^{\Gamma_{i-1} g_{i-1}} E_{ii})^{-1} a(t), \quad g(0) = 0, \tag{22}$$

where $a(t) = [\rho_0(t), \rho_1(t) \cdots \rho_N(t)]^T$, $\{\hat{A}_1, \ldots, \hat{A}_K\}$ is a basis of the Lie-algebra $\mathcal{L}(A_1, \ldots, A_N)$,

$$[\hat{A}_i, \hat{A}_j] = \sum_{l=1}^{K} \Gamma_{i,j}^l \hat{A}_l, \quad \Gamma_i = [\Gamma_{i,j}^l]_{j,l=1}^K,$$

and E_{ii} is the matrix with a single 1 entry at the i-th diagonal element. The fundamental matrix is given locally by the expression:

$$\Phi(t) = e^{g_1(t)\hat{A}_1} e^{g_2(t)\hat{A}_2} \cdots e^{g_n(t)\hat{A}_n}.$$

Lemma 1. For systems (21) the points attainable from the origin are those from the subspace spanned by the vectors

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} := span\{\prod_{j=1}^{K} A_{l_j}^{i_j} B_k\} \tag{23}$$

where $K \geq 0$, $l_j, k \in \{0, \cdots, N\}$, $i_j \in \{0, \cdots, n-1\}$, i.e., $\mathcal{R} \subset \mathcal{R}_{(\mathcal{A},\mathcal{B})}$.

Denote by $\mathcal{L}(A_0, \ldots, A_N)$ the finitely generated Lie-algebra containing the matrices $A_0, \ldots, A_N$, and let $\hat{A}_1, \ldots, \hat{A}_K$ be a basis of this algebra, then

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} = \sum_{l=0}^{N} \sum_{n_1=0}^{n-1} \cdots \sum_{n_K=0}^{n-1} \text{Im } (\hat{A}_1^{n_1} \ldots \hat{A}_K^{n_K} B_l).$$

A direct consequence of this fact is that if the inclusion $\mathcal{R}_{\mathcal{A},\mathcal{B}} \subset \mathbb{R}^n$ is strict, i.e, if $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is a proper subspace, then the system (21) cannot be completely controllable. The equality can be decided by a rank test similar to the LTI case, called the *generalized Kalman rank condition* in Szigeti (1992).

The main question is that under what condition is the reachability set of the original system (21) equals to the Lie algebra, i.e., when we have $\mathcal{R} = \mathcal{R}_{\mathcal{A},\mathcal{B}}$? This will lead to further conditions on the choice of the scheduling variables ρ both in LPV and qLPV case.

In the following paragraphs we consider affine LPV systems with constant B. The fundamental matrix can be written in exponential form as:

$$\Phi(t) = \sum_{n_1=0}^{n-1} \cdots \sum_{n_K=0}^{n-1} \hat{A}_1^{n_1} \ldots \hat{A}_K^{n_K} \psi_{n_1,\cdots,n_K}(t). \tag{24}$$

Since $\Phi(t_0, t) = X(t_0)P(t)^*$, we are interested in the matrix $Q(t) = P(t)^*$, that satisfies the equation

$$\dot{Q}(t) = -Q(t)A(t)), \quad Q(t_0) = \mathbb{I}. \tag{25}$$

Denote by $\mathbf{K} := \{0, 1, \cdots n-1\}^K$ and by $\mathbf{i} := (i_1, \cdots, i_K)$. Introducing the notation $\hat{A}^{\mathbf{i}} := \hat{A}_1^{i_1} \ldots \hat{A}_K^{i_K}$, let us choose a linearly independent set of matrices from the set $\{\hat{A}^{\mathbf{i}} \,|\, \mathbf{i} \in \mathbf{K}\}$, say $\{\hat{A}^{\mathbf{j}} \,|\, \mathbf{j} \in \mathbf{J}, \mathbf{J} \subset \mathbf{K}\}$ and construct the associated vector of functions $[\varphi_{\mathbf{j}}(0)]_{\mathbf{j}\in\mathbf{J}}$ from the function coefficients $\psi_{n_1,\cdots,n_K}(t)$.

Unfortunately the dependence of $[\varphi_{\mathbf{j}}(0)]_{\mathbf{j}\in\mathbf{J}}$ on the original ρ parameters can have complicated form. For the given notation, one has

$$\Phi(t) = \sum_{\mathbf{j}\in\mathbf{J}} \hat{A}^{\mathbf{j}} \varphi_{\mathbf{j}}(t). \tag{26}$$

Note, that the system $\{\varphi_{\mathbf{j}}(\sigma) \,|\, \mathbf{j} \in \mathbf{J}\}$ is neither necessarily linearly independent, nor unique.

The subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is exactly the image space of the matrix

$$R_{\mathcal{A},\mathcal{B}} := [\, \hat{A}_{\mathbf{j}} B \,]_{\mathbf{j}\in\mathbf{J}}. \tag{27}$$

Since

$$X(\sigma)^{-1} W(\sigma, \tau) X(\sigma)^{-*} = \int_\sigma^\tau [\, \hat{A}_{\mathbf{j}} B \,]_{\mathbf{j}\in\mathbf{J}} [\, \varphi_{\mathbf{j}}(s) \,]_{\mathbf{j}\in\mathbf{J}} [\, \varphi_{\mathbf{j}}(s) \,]_{\mathbf{j}\in\mathbf{J}}^* [\, \hat{A}_{\mathbf{j}} B \,]_{\mathbf{j}\in\mathbf{J}}^* ds, \tag{28}$$

The controllability Grammian is given as

$$W(\sigma, \tau) = R_{\mathcal{A},\mathcal{B}} (\int_\sigma^\tau [\, \varphi_{\mathbf{j}}(s) \,]_{\mathbf{j}\in\mathbf{J}} [\, \varphi_{\mathbf{j}}(s) \,]_{\mathbf{j}\in\mathbf{J}}^* ds) R_{\mathcal{A},\mathcal{B}}^*.$$

Assume that $dim \mathcal{R}_{\mathcal{A},\mathcal{B}} = m$, then one can deduce that $rank W(\sigma, \tau) = rank R_{\mathcal{A},\mathcal{B}}$ if there are m linearly independent functions in $\{\varphi_{\mathbf{j}}(\sigma) \,|\, \mathbf{j} \in \mathbf{J}\}$. The Kalman - controllability rank condition in this case is also sufficient.

Definition 6. The time varying system (21) is c-excited if there are linearly independent functions $\{\tilde{\varphi}_1, \cdots, \tilde{\varphi}_m\}$ in $\{\varphi_{\mathbf{j}}(\sigma) \,|\, \mathbf{j} \in \mathbf{J}\}$, where $m = rank R_{\mathcal{A},\mathcal{B}}$.

This property depends on the choice of the functions ρ although indirectly, via the use of Wei - Norman equations.

Theorem 1. The LPV system is controllable, iff the generalized Kalman - rank condition

$$rank R_{\mathcal{A},\mathcal{B}} = n$$

is satisfied and the set of functions $\{\varphi_{\mathbf{j}}(\sigma) \,|\, \mathbf{j} \in \mathbf{J}\}$ contains n linearly independent functions.

To test the generalized Kalman - rank conditions on affine LPV, qLPV systems the $\mathcal{A}$-invariant subspace algorithm (12) can be applied, see also Balas *et al.* (2003) for details.

Summarizing the above discussions, the approach presented here is based on Kalman's original formulation of controllability conditions (Kalman, 1960) that exploits the properties of the fundamental matrix. The disadvantage of this approach is that explicit computation of the fundamental matrix is complicated for practical problems. In some special cases, e.g., affine LPV systems, the Wei–Normann approach might lead to useful conclusions using matrix Lie algebra techniques, obtaining a generalized Kalman rank condition for controllability. This condition is only necessary and for sufficiency a second condition on a set of functions given in terms of the scheduling ρ_i functions, called c-exited property, has to be satisfied, too.

4. DYNAMIC INVERSION AND TRACKING FOR LPV SYSTEMS

The solution of the dynamic inversion of systems was investigated in the classical paper Silverman (1969), where he considered the properties and calculation of the inverse of LTI systems but guaranteeing neither minimality (observability, detectability) nor stability properties of the resulting inverse system. The problem was also considered by Hirschorn (1981) and Fliess (1986) for nonlinear input-output systems. For certain classes of nonlinear state space representations Isidori and Nijmeijer provided algorithms and also sufficient or necessary conditions of invertibility in Isidori (1989), Nijmeijer (1991a).

The tracking problem for nonlinear systems has been investigated e.g. in Di Benedetto (1994), Grizzle *et al.* (1994), Huang *et al.* (1990). Dynamic inversion and state feedback have been applied in design of tracking controllers for multivariable systems and applied in some flight control systems, see Bennani (1998), Costa *et al.* (2001), Looye (2001), Morton *et al.* (1996), Meyer and Cicilani (1975), Bugajski *et al.* (1992), Bugajski and Enns (1992). The advantage of this approach is that it can be extended to nonlinear systems since it is closely related to feedback linearization. The simplest form (called also pseudo inverse control) as it is formulated for LTI systems can be characterized as follows.

Given a system

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

the goal is to generate a control input such that the state or output y follows the reference or desired trajectory generated by

$$\dot{x} = A_d x_d + B_d r, \quad y_d = C_d x_d.$$

Assume that B is of full column rank, then the input can be obtained from the equation $\dot{y} = C\dot{x}$. Assuming that $CB \neq 0$, the input is given by

$$u = (CB)^+(\dot{y} - CAx),$$

where $(.)^+$ denotes the pseudo inverse. The controller is obtained by replacing $\dot{y}$ by $\dot{y}_d$ and applying an external input v:

$$u_{dinv} = (CB)^+(\dot{y}_r - CAx + v).$$

Define the tracking error as $= y - y_d$, and let $v = \Lambda e$, Λ has negative eigenvalues. Applying this control (using full state measurement), one arrives at a stable linear dynamics of the tracking error:

$$\dot{e} = \Lambda e.$$

It can be seen that the assumptions on deriving the above controller are that B is monic and that all input components can be computed from the first order derivatives of the outputs (the latter means that all the relative degrees of the LTI system were equal to 1, i.e. $r_i = 1, i = 1, \ldots, p$, where p is the output dimension).

Since the above concepts and system properties can be defined for LTV, LPV and nonlinear systems as well, the dynamic inversion controller approach can be applied to these systems, too.

It will be shown that the geometric concepts and algorithms described in Section 2 can be used to design dynamic inversion controllers for LPV and qLPV systems.

Consider the class of linear parameter-varying (LPV) systems of m inputs and p outputs that can be described as:

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \quad (29)$$

$$y(t) = Cx(t) \quad (30)$$

where

$$A(\rho(t)) = A_0 + \rho_1(t)A_1 + \ldots + \rho_N(t)A_N, \quad (31)$$

$$B(\rho(t)) = B_0 + \rho_1(t)B_1 + \ldots + \rho_N(t)B_N, \quad (32)$$

$$(33)$$

The dimension of the state space is supposed to be n.

While the LPV systems are usually time varying, the qLPV systems are nonlinear. It is known for nonlinear input affine system described as

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \quad (34)$$

$$(y_j)_{j=1,p} = (h_j(x))_{j=1,p}.$$

that the (left)invertibility condition can be stated as:

$$dim\, span\{\, g_i(x_0)\,|\, i = 1, m\} = m, \quad (35)$$
$$span\{\, g_i(x)\,|\, i = 1, m\} \cap T_x Z^* = 0, \quad (36)$$

where $T_x Z^*$ is the tangent of the locally maximal output zeroing sub manifold. The zero dynamics algorithm for computing Z^* can be found in Isidori (1989), Nijmeijer and van der Schaft (1991). However, in some cases Z^* can be determined relative easily relating it to the maximal controlled invariant distribution Δ^* contained in $kerdh$.

If the ρ functions are exciting, then it can be proved that $T_x Z^* = \mathcal{V}^*$, where $\mathcal{V}^*$ is the maximal $(\mathcal{A}, \mathcal{B})$-invariant subspace contained in $kerC$.

For this situation the invertibility conditions reduce to:

$$dim\, ImB = m, \quad \mathcal{V}^* \cap ImB = 0,$$

where $\mathcal{V}^*$ can be computed using the ($\mathcal{A}$,B)-invariant subspace algorithm. It is known that $dim\mathcal{V}^* = n - r, \quad r = r_1 + \dots r_p$ where r is called the vector relative degree.

Recall that for the pseudo inverse control it was assumed that all states were available for feedback implying that all inputs could be computed from first output derivatives. This leads to $r = n$, i.e. $\mathcal{V}^*$ is the zero subspace. This implies that under these assumptions the second invertibility condition was automatically satisfied. For the general case the inversion is a bit more involved leading to a controller whose dynamics are defined by the zero dynamics of the system.

If the invertibility conditions are satisfied, one can always choose a coordinate transform of the form $z = Tx$, where $T^{-1} = \left[\, \Lambda \;\, ImB \;\, \mathcal{V}^* \,\right], \quad \Lambda \subset \mathcal{V}^{*\perp}$.

Accordingly, the system will be decomposed to:

$$\dot{x_1} = A_{11}(t)x_1 + A_{12}(t)x_2 + B_1 v \quad (37)$$
$$\dot{x_2} = A_{21}(t)x_1 + A_{22}(t)x_2 \quad (38)$$
$$y = C_1 x_1. \quad (39)$$

It follows, that applying the feedback

$$u = F_1(\rho(t))x_1 + F_2(\rho(t))x_2 + v, \quad (40)$$

such that $\mathcal{V}^*$ is $(\mathcal{A} + \mathcal{B}F, \mathcal{B})$ invariant, one can obtain the system:

$$\dot{x}_1 = A_{11}(t)x_1 + B_1 v \quad (41)$$
$$y = C_1 x_1. \quad (42)$$

A basis for $\mathcal{V}^*$ can be selected from

$$\{c_1, \cdots, S_1^{\gamma_1}(t), \cdots, c_p, \cdots, S_p^{\gamma_p}(t)\}, \quad (43)$$

where $S_i^l(t)B = 0$, for $l < \gamma_i$, and

$$S_i^0(t) = c_i, \quad (44)$$
$$S_i^{k+1}(t) = \dot{S}_i^k(t) + S_i^k(t)A_{11}(t), \quad (45)$$

see Silverman and Medows (1969), Bestle and Zeitz (1983), then one can define a coordinate transform $\mathcal{S}(t)$ that maps x_1 to $\tilde{y}$, where

$$\tilde{y} = \left[y_1, \cdots, y_1^{(\gamma_1)}, \cdots, y_p, \cdots, y_p^{(\gamma_p)}\right]^T. \quad (46)$$

Since one can chose $F_1(\rho) = 0$, the inverse of an LPV system is given by:

$$\dot{\eta} = A_{22}(t)\eta + A_{21}(t)S^{-1}(t)\tilde{y}, \quad (47)$$
$$u = F_2(\rho(t))\eta + \tilde{B}^{-r}S^{-1}(t)(\dot{\tilde{y}} -$$
$$- (S(t)A_{11}(t)S^{-1}(t) + \dot{S}(t)S^{-1}(t))\tilde{y}),$$

where $\tilde{B}^{-r}$ is the right inverse of $\tilde{B}$.

Remark 1. One can observe that to compute the matrix $S(t)$ one needs certain derivatives of the parameter functions ρ_i, i.e., certain derivatives of the output y, but the order of these derivatives are bounded by $\max_i \gamma_i$.

Remark 2. The method presented above can also be applied to quasi LPV systems.

Using formulae (47) for the inverse, one can form the (q)LPV version of the tracking controller by replacing $\dot{\tilde{y}}$ by the reference signal y_d and its derivatives. This will result in a linear closed loop error system and the stability can be guaranteed by forming an outer error feedback loop resulting in the following dynamic inversion controller structure:

$$\dot{\eta}_{dinv} = A_{22}\eta_{dinv} + A_{21}S^{-1}\tilde{y}_d + \Lambda(\tilde{y} - \tilde{y}_d) \quad (48)$$
$$u_{dinv} = F_2\eta_{dinv} + \lambda(y_d), \quad (49)$$

where

$$\lambda(y_d) = \tilde{B}^{-r}S^{-1}(t)(\dot{\tilde{y}}_d - \quad (50)$$
$$-(S(t)A_{11}(t)S^{-1}(t) + \dot{S}(t)S^{-1}(t))\tilde{y}_d),$$

and Λ is a gain matrix playing similar role in the stability of the error system like in the LTI case. The details of obtaining the error systems can be found in Balas *et al.* (2004).

5. FAULT DETECTION AND ISOLATION IN LPV SYSTEMS

An interesting family of control problems where geometric theory plays a key role is the Disturbance Decoupling (with stability) and dynamic decoupling. The dual problem to the latter is used in the development of filters for fault detection and isolation (FDI) in dynamic systems. The DDP for the LPV case has been discussed in Bokor *et al.* (2002). Here we summarize results on FDI filter design as elaborated for (q)LPV systems.

Consider the following system:

$$\dot{x}(t) = A(\rho)x(t) + B(\rho)u(t) + L_1(\rho)m_1(t) + L_2(\rho)m_2(t)$$
$$y(t) = Cx(t), \tag{51}$$

where $A(\rho), B(\rho)$ have affine structure, $L_1(\rho), L_2(\rho)$ represent known fault directions and are supposed to be affine in ρ, too, i.e. $L_i(\rho) = \sum_{j=1}^{N} \rho_j L_j, i = 1, 2$.

The signals $m_1(t), m_2(t)$ are called *fault signatures* and they are zero if there is no fault but are arbitrary and unknown otherwise.

The goal is to design a residual generator with output denoted by r_1, such that if $m_1 \neq 0$ then $r_1 \neq 0$ and if $m_1 = 0$ then $\lim_{t\to\infty} ||r_1(t)|| = 0$, i.e., there is a stability condition requirement on the residual generator. In addition, the effect of the other fault has to be completely isolated from the time evolution of r_1. This concept can be extended to the situation where more than two faults can occur simultaneously. The filters that satisfy these requirements are usually called FDI filters.

In the solution of this problem for LTI systems, a key role is played by the (C,A)-invariant subspaces and certain unobservability subspaces, (Massoumnia, 1986; Massoumnia *et al.*, 1989), or observability codistributions (Persis and Isidori, 2000a; Persis and Isidori, 2000b) in the nonlinear version of this problem.

The design of LPV FDI filters can be performed using the following result Bokor and Balas (2004).

Proposition 3. For LPV systems given in equation (51) one can design a – not necessarily stable – residual generator of type

$$\dot{w}(t) = N(\rho)w(t) - G(\rho)y(t) + F(\rho)u(t) \tag{52}$$
$$r(t) = Mw(t) - Hy(t), \tag{53}$$

if for the smallest (parameter varying) unobservability subspace $\mathcal{S}^*$ containing $\mathcal{L}_2$ one has $\mathcal{S}^* \cap \mathcal{L}_1 = 0$, where $\mathcal{L}_i = \cup_{j=0}^{N} Im L_{i,j}$.

The first step in the design is to consider a common unobservability subspace for the matrices A_i in equation (51), i.e., a subspace $\mathcal{W}$ such that $\mathcal{W}$ is (C, A_i) invariant and $L_2(\rho) \in \mathcal{L}_2 \subset \mathcal{W}$ for all x, and an output mixing map H such that Ker $HC = \mathcal{W} +$ Ker C. Moreover, let us suppose, that $\mathcal{L}_1 \cap \mathcal{W} = 0$, where $\mathcal{L}_i = \cup_{j=0}^{N} Im L_{i,j}$.

Denote by G_i the gain matrices determined such that $(A_i + G_i C)\mathcal{W} \subset \mathcal{W}$, and let us consider the filter

$$\dot{\xi} = (A(\rho) + G(\rho)C)\xi + B(\rho)u - G(\rho)y,$$

where $G(\rho) = G_0 + \rho_1 G_1 + \cdots + \rho_N G_N$. Then for $e_1 = P(x - \xi)$, where P is the matrix of the orthogonal projection on $\mathcal{W}^{\perp}$, and denoting by $N(\rho) = P(A(\rho) + G(\rho)C)P$ one has the following error equations:

$$\dot{e}_1 = N(\rho)e_1 + L_1(\rho)m_1. \tag{54}$$

By setting $r_1 = Me_1$, where $MP = HC$, and supposing that there exist a gain $G(\rho)$ such that equation (54) is stable, then one has $\lim_{t\to\infty} ||r_1(t)|| = 0$, if $m_1 = 0$. Moreover, from $\mathcal{L}_1 \cap \mathcal{W} = 0$ one has $\mathcal{L}_1 \cap$Ker $M = 0$, i.e., r_1 can not vanish identically for a nonzero m_1.

Summarizing, a sufficient condition for the construction of a residual generator is the following:

Proposition 4. Let us consider the smallest (parameter varying) unobservability subspace $\mathcal{S}^*$ containing $\mathcal{L}_2$, and suppose that $\mathcal{S}^* \cap \mathcal{L}_1 = 0$. If, in addition, the system (54) is stable, then the filter (52-53) is a solution to the FPRG problem. The map $N(\rho)$ should satisfy $N(\rho)(\mathcal{X}/\mathcal{S}^*) \subset \mathcal{X}/\mathcal{S}^*$ for all $\rho \in \mathcal{P}$, H satisfy Ker $HC =$ Ker $C + \mathcal{S}^*$ and M is the unique solution of $MP = HC$, where P is the projection $P : \mathcal{X} \to \mathcal{X}/\mathcal{S}^*$, and $F = PB$.

The stability concept associated with the above LPV FDI filters is quadratic stability. LPV systems discussed so far are called quadratically stable if there exist a matrix $Q = Q^T > 0$ such that

$$A(\rho)^T Q + QA(\rho) < 0 \tag{55}$$

for all the parameters $\rho \in \mathcal{P}$. A necessary and sufficient condition for a system to be quadratically stable is that the condition in equation (55) holds for all the corner points of the parameter space, i.e., one can obtain a finite system of LMI's that has to be fulfilled for $A(\rho)$ with a suitable positive definite matrix Q, see Gahinet *et al.* (1996), Becker and Packard (1993), Fen *et al.* (1996), Packard and Becker (1992). Results using affine parameter dependent matrix $Q(\rho)$ are also available in the literature.

In order to obtain a quadratically stable residual generator one can set $N(\rho) = A_0(\rho) + G(\rho)M$ in equation (52), where $G(\rho) = G_0 + \rho_1 G_1 + \cdots \rho_N G_N$ is determined such that the LMI defined in equation (55), i.e.,

$$(A_0(\rho) + G(\rho)M)^T Q + Q(A_0(\rho) + G(\rho)M) < 0$$

holds for suitable $G(\rho)$ and $Q = Q^T > 0$. By introducing the auxiliary variable $K(\rho) = G(\rho)Q$, one has to solve the following set of LMIs on the corner points of the parameter space:

$$A_0(\rho)^T Q + QA_0(\rho) + M^T K(\rho)^T + K(\rho)M < 0.$$

Remark 3. If Ker $C \subset \mathcal{U}^*$ then one can choose $G(\rho)$ such that the matrix $N(\rho)$ is parameter independent with arbitrary eigenvalues, since the equation $G(\rho)CU = UT - A(\rho)U$ has a solution for arbitrary T, where U is the insertion map of $\mathcal{X}/\mathcal{U}^*$.

The proofs and the computation of an acceptable $G_(\rho)$ with example can be found in Bokor and Balas (2004).

6. CONCLUSIONS

This paper proposed a geometric view on some analysis and design problems related to dynamical systems that can be described by LPV or qLPV models. If these models are affine in the scheduling parameter functions, then it is possible to give solutions to these problems by applying simple algorithms based on linear algebra. The use of LPV modelling is also promising from optimal control point of view since powerful numeric tools (based on LMI solvers) are now available for design. These features can make them attractive when solving control problems related to time varying and general nonlinear systems.

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