OPTIMAL CONTROLLER FOR STOCHASTIC SYSTEMS WITH ALGEBRAIC DEPENDENCIES

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Abstract: For systems with algebraic dependencies an extended design of an optimal controller is discussed. It relies on a new notion of a sequential controller proposed together with an efficient algorithm for determining a set of admissible controllers to a given system that contain the least number of delays. An extended optimizing recursion then follows naturally.

Keywords: Multivariable control systems, Optimal control, Optimal systems, Stochastic systems, Computational methods

1. INTRODUCTION

Control theory deals with mathematical models of real systems called the abstract control systems. A control system consists of a plant model Σ_1 and a controller model Σ_2. A system theoretic approach was given in (Žampa, 2004). If Σ_1 contains some algebraic dependencies, that is, if some inputs propagate to the outputs at the same time instant, a thorough determination of admissible controllers is a complex problem and its solution provides the richest possible class for choosing the optimal control strategy.

Well-posedness of the closed loop system implies that Σ must be *free of algebraic loops* (Žampa, 2004; Zhou *et al.*, 1996). Optimality, however, asks for a minimum number of delays in the control system to maximize information available for control. This leads to a new term of a sequential control strategy introduced together with optimal control problem formulation in Section 2. In Section 3, an efficient determination of the set of admissible strategies with minimal number of delays is proposed. An extension of the standard optimizing recursion (Bertsekas, 2000) is provided in Section 4 and an example in Section 5.

Notation. The field of real numbers is denoted by $\mathbb{R}$. Sets are denoted by calligraphic upper characters, e.g. $\mathcal{T}$. Vectors are written in bold, matrices as bold capitals. For a matrix $\boldsymbol{S}$, $\boldsymbol{S}[i,j]$ denotes its entry at position (i,j) and $\boldsymbol{S}[:,j]$ stands for the j-th column of $\boldsymbol{S}$. Furthermore, $\boldsymbol{S}'$ denotes transposition of $\boldsymbol{S}$ and the same applies to vectors. For $\boldsymbol{u} = (u_1, \ldots, u_m)'$ being a vector, $\tilde{\boldsymbol{u}}_\zeta = (u_{\zeta_1}, \ldots, u_{\zeta_m})'$ denotes a vector with permuted elements. To indicate time dependence, indexing $\boldsymbol{u}_k$, $u_{i,k}$ and $\tilde{\boldsymbol{u}}_{\zeta,k}$ is used and k is reserved for denoting time to avoid confusion.

2. PROBLEM FORMULATION

2.1 Discrete stochastic control system

The control system is defined on a finite set of time instants $\mathcal{T} := \{0, 1, \ldots, F\}$, where F is the

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control horizon. At the time instant k, $\boldsymbol{x}_k \in \mathbb{R}^n$, $\boldsymbol{y}_k \in \mathbb{R}^p$ denote the vector of non-measurable and observed (output) variables of Σ_1, respectively, and $\boldsymbol{u}_k \in \mathcal{U} \subset \mathbb{R}^m$ is the control. For $i, j \in \mathcal{T}$, $i < j$, $\boldsymbol{x}_i^j$ is a shorthand description of the sequence $(\boldsymbol{x}_i, \dots, \boldsymbol{x}_j)$. The same applies for other variables.

Behavior of the plant Σ_1 is for all $k \in \mathcal{T}$ completely described by a probability distribution function (pdf)

$$f_k(\boldsymbol{x}_k, \boldsymbol{y}_k \colon \boldsymbol{x}_{k-1}, \boldsymbol{y}_{k-1}, \boldsymbol{u}_{k-1}, \bar{\boldsymbol{u}}_k), \qquad (1)$$

where f_k is a pdf of $(\boldsymbol{x}_k, \boldsymbol{y}_k)$ parameterized by its complete and immediate cause

$$(\boldsymbol{x}_{k-1}, \boldsymbol{y}_{k-1}, \boldsymbol{u}_{k-1}, \bar{\boldsymbol{u}}_k).$$

Such pdf is called causal pdf (Žampa, 2004). Parameter $\bar{\boldsymbol{u}}_k$ represents control variables $u_{i,k} \in \boldsymbol{u}_k$ that propagate to the output at time k, i.e. without delay.

2.2 Batch control strategy

Let $\mathcal{G}$ denote the set of admissible stochastic control strategies γ for the system Σ_1. The aim of optimal control system synthesis is to find a control strategy $\gamma^* \in \mathcal{G}$ in a form of a conditional pdf that for the given plant Σ_1 guarantees the best quality behavior of Σ. Such quality is usually evaluated by a criterion $J(\gamma) : \mathcal{G} \mapsto \mathbb{R}$. Considering the usual case of a criterion with an additive loss function and noting that the value of loss function is a random variable, it follows that

$$J(\gamma) = E\Big\{ \sum_{k=0}^{F} L_k(\boldsymbol{x}_0^k, \boldsymbol{y}_0^k, \boldsymbol{u}_0^k); f, \gamma \Big\} \qquad (2)$$

$$\gamma^* = \operatorname*{argmin}_{\gamma \in \mathcal{G}} J(\gamma), \qquad (3)$$

where $E\{\cdot; f, \gamma\}$ stands for the operator of expected value parameterized by the behavior of the controlled system Σ_1 and controller Σ_2 realizing control strategy γ.

Problem formulation 1. For a given controlled subsystem Σ_1, determine a set of admissible strategies $\mathcal{G}$. Then, find a control strategy $\gamma^* \in \mathcal{G}$.

Determination of $\mathcal{G}$ is trivial in SISO systems, because there exist only two boundary cases, $\bar{\boldsymbol{u}} = \emptyset$ and $\bar{\boldsymbol{u}} = u$. Their respective $\mathcal{G}$ contains either all strategies $\gamma_k(u_k | u_0^{k-1}, y_0^k)$ or is restricted to $\gamma_k(u_k | u_0^{k-1}, y_0^{k-1})$. MIMO systems require much more care and thus it became a routine to constrain the optimization to the set of control strategies $\gamma(\boldsymbol{u}_k | \boldsymbol{u}_0^{k-1}, \boldsymbol{y}_0^{k-1})$ as soon as $\bar{\boldsymbol{u}}$ in (1) is not an empty set. The result, however, may not exhibit the best possible behavior since the control strategy does not use all information available at time k. It may thus be desirable to design a system with the least number of delays (Willems, 1991).

Definition 1. Let $f_{i,k}(y_{i,k} : \boldsymbol{x}_{k-1}, \boldsymbol{y}_{k-1}, \boldsymbol{u}_{k-1}, \bar{\boldsymbol{u}}_k)$, $k \in \mathcal{T}$, denote the pdf of $y_{i,k}$, derived from (1). The output variable $y_{i,k}$ is said to *algebraically depend* on input $u_{j,k}$ if for all k holds that

$$\begin{aligned} f_{i,k}(y_{i,k} &: \boldsymbol{x}_{k-1}, \boldsymbol{y}_{k-1}, \boldsymbol{u}_{k-1}, \\ &u_{1,k}, \dots, \hat{u}_{j,k}, \dots, u_{m,k}) \\ &\neq f_{i,k}(y_{i,k} : \boldsymbol{x}_{k-1}, \boldsymbol{y}_{k-1}, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k), \end{aligned}$$

where $\hat{u}_{j,k}$ denotes that $u_{j,k}$ is deleted.

Definition 2. The *adjacency matrix* of Σ_1 at time $k \in \mathcal{T}$ is a $(p \times m)$ matrix $\boldsymbol{S}_k$, such that $\boldsymbol{S}_k[i,j] = 1$ if $y_{i,k}$ algebraically depends on $u_{j,k}$ and $\boldsymbol{S}_k[i,j] = 0$ if it does not. The matrix $\boldsymbol{S}_k$ determines the structure of algebraic input-output dependencies (SAIOD) of Σ_1 at time k.

Remark 1. SAIOD of Σ_1 can generally vary with time as apparent from Definition 1 and Definition 2. Thorough this paper, however, only control systems with time-invariant SAIOD are considered. Thus $\boldsymbol{S}_k = \boldsymbol{S}$. Control at time k is standardly generated at once, as a vector $\boldsymbol{u}_k$. Such control strategy is said to be a *batch control strategy.* Maximum information (Data) available for control at time k will be denoted D_k.

Lemma 1. D_k available for generating the control vector $\boldsymbol{u}_k$ consists of all $\boldsymbol{u}_0^{k-1}$, $\boldsymbol{y}_0^{k-1}$ and all $y_{i,k}$ not algebraically dependent on any $u_{j,k}$, $j = 1, 2, \dots, m$, that is, of all such $y_{i,k}$ that $\boldsymbol{S}[i,j] = 0$, $\forall j$.

Proof. Data from the time instants $0, \dots, k-1$ as well as observed variables at time k, which are not algebraically dependent on any $u_{j,k}$, can not cause an algebraic loop within the control system Σ. Hence they belong to D_k. To show that all algebraically dependent output variables must be excluded, assume $y_{i,k}$ belongs to D_k and simultaneously $y_{i,k}$ algebraically depends on some $u_{j,k}$. Algebraic dependence implies that $u_{j,k}$ must have already been generated. This contradicts the definition of D_k. □

Corollary 1. Assume that the set $\mathcal{G}$ contains only batch control strategies and that all batch deterministic strategies are also contained in $\mathcal{G}$. Then, from (Žampa *et al.*, 2004), the solution of optimal control synthesis is given by the following optimizing recursion

$$\begin{aligned} V_k(D_k) &= \min_{\boldsymbol{u}_k \in \mathcal{U}} E\{V_{k+1}(D_{k+1}) + L_k(\cdot) | D_k), \boldsymbol{u}_k\} \\ \boldsymbol{u}_k^* &= \operatorname*{argmin}_{\boldsymbol{u}_k \in \mathcal{U}} E\{V_{k+1}(D_{k+1}) + L_k(\cdot) | D_k, \boldsymbol{u}_k)\} \\ J^* &= E\{V_0(D_0)\}, \end{aligned}$$

where $V_k(D_k)$ is a Bellman function, $V_{F+1}(\cdot) = 0$ and $k = F, \dots, 1, 0$.

2.3 Sequential control strategy

The optimizing recursion in Corollary 1 solves the optimal synthesis problem if only batch control strategies are considered. Nevertheless, relaxing the assumption on a batch generation of $\boldsymbol{u}_k$ may allow utilization of more information than D_k for control. On the other hand, it will also require careful analysis of the admissible structures.

Definition 3. Let

$$(\boldsymbol{v}_\zeta^{(1)\prime}, \dots, \boldsymbol{v}_\zeta^{(q)\prime})' = \tilde{\boldsymbol{u}}_\zeta$$

be a disjoint grouping of a ζ-th permutation of the control variables $\boldsymbol{u}$, where $\dim(\boldsymbol{v}_\zeta^{(i)}) \neq 0$. Furthermore, let

$$(\boldsymbol{z}_\eta^{(0)\prime}, \boldsymbol{z}_\eta^{(1)\prime}, \dots, \boldsymbol{z}_\eta^{(q)\prime})' = \tilde{\boldsymbol{y}}_\eta$$

be a disjoint grouping of an η-th permutation of the observed variables $\boldsymbol{y}$. A control strategy $\gamma_\zeta = (\gamma_\zeta^{(1)}, \dots, \gamma_\zeta^{(q)})$ that generates $\boldsymbol{u}_k$ sequentially at time k, such that

$$\gamma_{\zeta,k}^{(i)}(\boldsymbol{v}_{\zeta,k}^{(i)} | D_{\zeta,k}^{(i)}),$$

where $D_{\zeta,k}^{(i)} = D_{\zeta,k}^{(i-1)} \cup \{\boldsymbol{v}_{\zeta,k}^{(i-1)}, \boldsymbol{z}_{\eta,k}^{(i)}\}$, $i = 1, \dots, q$, and $D_{\zeta,k}^{(0)} = \{\boldsymbol{u}_0^{k-1}, \boldsymbol{y}_0^{k-1}\}$, is called *sequential control strategy*. Moreover, if $\dim(\boldsymbol{v}_\zeta^{(i)}) = 1$ for all $i = 1, \dots, m$, γ_ζ is said to be *strictly sequential*.

Sequential control strategy realizes the vectors $\boldsymbol{v}_{\zeta,k}^{(i)}$ sequentially at the time instant k. Each $\gamma_{\zeta,k}^{(i)}$ uses all information available for $\gamma_{\zeta,k}^{(j)}$, $j < i$, plus data that may be contained in $\boldsymbol{z}_{\eta,k}^{(i)}$. As there are no restrictions on the dimension of $\boldsymbol{z}_{\eta,k}^{(i)}$, some $\boldsymbol{z}_{\eta,k}^{(i)}$ may be empty. The group $\boldsymbol{z}_{\eta,k}^{(0)}$ thus contains observed variables $y_{\eta_i,k} \in \tilde{\boldsymbol{y}}_{\eta,k}$ that are not used for control at time k.

Note that any batch control strategy is a special case of sequential control strategy, consider, for example, the case $q = 1$. Hence allowing the use of sequential strategies can not decrease the control quality. This reasoning leads to an alternative problem formulation.

Problem formulation 2. For a given controlled subsystem Σ_1, find a set of admissible sequential control strategies $\mathcal{G}$. Then, find a control strategy $\gamma^* \in \mathcal{G}$.

The only condition for a strategy γ to belong to $\mathcal{G}$ is that the control system composed of Σ_1 and of Σ_2 does not contain algebraic loops. Only the SAIODs of the plant and of the control strategy are needed for a check for algebraic loops within the control system.

Definition 4. Let $\gamma_{i,k}(u_{i,k} | \boldsymbol{u}_0^{k-1}, \boldsymbol{y}_0^{k-1}, \bar{\boldsymbol{y}}_k)$ be a control strategy for $u_{i,k}$. The control variable $u_{i,k}$ is said to *algebraically depend* on $y_{j,k}$ at time $k \in \mathcal{T}$ if

$$\gamma_{i,k}(u_{i,k} | \boldsymbol{u}_0^{k-1}, \boldsymbol{y}_0^{k-1}, y_{1,k}, \dots, \hat{y}_{j,k}, \dots, y_{p,k}) \neq \gamma_{i,k}(u_{i,k} | \boldsymbol{u}_0^{k-1}, \boldsymbol{y}_0^{k-1}, \boldsymbol{y}_k),$$

where $\hat{y}_{j,k}$ denotes that $y_{j,k}$ is deleted.
Structure of algebraic input output dependencies of the control strategy γ is described by an *adjacency matrix* $\boldsymbol{R}$ of the control strategy. Matrix $\boldsymbol{R}$ of dimension $(m \times p)$ is such that $\boldsymbol{R}[i,j] = 1$ if u_i algebraically depends on y_j and $\boldsymbol{R}[i,j] = 0$ if it does not.

There is a class of control strategies with a particular SAIOD that is of interest during the optimal control synthesis. This follows from the fact that in order to use all information available at each stage of the sequential control strategy, its SAIOD should be as rich as possible.

Definition 5. Consider the SAIOD of an admissible control strategy described by $\boldsymbol{R}$. The structure is said to be *maximal*, if the closed loop loses well-posedness by a change of its arbitrary zero entry to one. With an abuse of notation, the corresponding control strategy will be called maximal as well. The set of all maximal strategies is denoted $\mathcal{G}_M$.

Clearly, since a maximal strategy is defined as admissible, so is any other strategy with a SAIOD that is contained in the maximal one. This leads to the final problem formulation.

Problem formulation 3. For a given controlled subsystem Σ_1, determine a set of control strategies with maximal structure $\mathcal{G}_M$. Then, find a control strategy $\gamma^* \in \mathcal{G}_M$.

Retrieval of all maximal structures from the set of all possible structures is not a trivial problem. One option is to use graph theory. Then, one analyzes whether the directed graph resulting from feedback connection of SAIOD of the plant and of the control strategy is acyclic, e.g. by the use of Floyd's algorithm (Gibbons, 1999). Acyclic directed graphs need to be checked for maximality. Complexity of such a solution is obviously exponential and thus unsuitable for systems with higher numbers of inputs and outputs. The solution presented in the next section takes advantage of special features of the strategy synthesis and significantly reduces the computational complexity.

3. DETERMINATION OF MAXIMAL STRUCTURES

3.1 Basic algorithm

Recall that any sequential (and hence also batch) control strategy can be realized as a strictly sequential control strategy. This implies that there is no need to check all admissible SAIOD for maximality. Indeed, there are only at most $m!$ possible permutations $\tilde{\boldsymbol{u}}_\zeta$ to be analyzed.

Let $\tilde{\boldsymbol{u}}_\zeta$ be a permutation of $\boldsymbol{u}$. Let γ_ζ be an admissible strictly sequential control strategy realizing $\tilde{\boldsymbol{u}}_\zeta$ such that changing any zero entry of its corresponding $\boldsymbol{R}_\zeta$ to one either validates admissibility of γ_ζ or requires a change in $\tilde{\boldsymbol{u}}_\zeta$, then γ_ζ is said to have maximal SAIOD *conditioned by permutation* ζ. Such control strategy is called conditioned maximal strategy.

Proposition 1. Let $\tilde{\boldsymbol{u}}_\zeta$ be the ζ-th permutation of the elements of $\boldsymbol{u}$ and let γ_ζ be the *conditioned strictly sequential control strategy* realizing $\tilde{\boldsymbol{u}}_\zeta$ as

$$u_{\zeta_1} \to \cdots \to u_{\zeta_i} \to \cdots \to u_{\zeta_m}$$

Then $\boldsymbol{R}_\zeta$ describing SAIOD of γ_ζ can be computed by the following algorithm.

$$
\begin{aligned}
&\boldsymbol{R}_\zeta := \neg \boldsymbol{S}' \\
&\texttt{for } i = m-1, \ldots, 1 \\
&\qquad \boldsymbol{R}_\zeta[\zeta_i, :] := \boldsymbol{R}_\zeta[\zeta_i, :] \wedge \boldsymbol{R}_\zeta[\zeta_{i+1}, :]; \\
&\texttt{end}
\end{aligned}
$$

where the symbols $\neg$ and $\wedge$ denote logical operators of negation and AND, respectively.

Proof. Idea of the proof was suggested by Pešek (1997). From Definition 3, a strictly sequential control strategy at time k is given by

$$\gamma_{\zeta,k} = \left(\gamma^{(1)}_{\zeta,k}(v^{(1)}_{\zeta,k}|D^{(1)}_{\zeta,k}), \ldots, \gamma^{(m)}_{\zeta,k}(v^{(m)}_{\zeta,k}|D^{(m)}_{\zeta,k})\right).$$

(1) The admissibility condition implies that $D^{(i)}_{\zeta,k}$ in $\gamma^{(i)}_{\zeta,k}(v^{(i)}_{\zeta,k}|D^{(i)}_{\zeta,k})$ *must not contain* any element of $\boldsymbol{y}_k$ that will acquire their values after application of at least one $v^{(j)}_{\zeta,k}$, $j \geq i$ on Σ_1.
(2) The maximality condition, on the other hand, requires $D^{(i)}_{\zeta,k}$ to *contain all* elements of $\boldsymbol{y}_k$ that have already acquired their values after realization of all $v^{(j)}_{\zeta,k}$, $j < i$.

It is advantageous to start the algorithm by determining $y_{r,k} \in \boldsymbol{y}_k$ that belong to $D^{(m)}_{\zeta,k}$. By Definition 2, information on such observed variables is contained in $\boldsymbol{S}'[\zeta_m, :]$. In the next step, $D^{(m-1)}_{\zeta,k}$ can be to found readily from $\boldsymbol{S}'[\zeta_m, :]$ and $\boldsymbol{S}'[\zeta_{m-1}, :]$. The calculation proceeds towards $D^{(1)}_{\zeta,k}$.
The actual computational algorithm follows from the fact that since both $\boldsymbol{S}'$ and $\boldsymbol{R}$ are defined over $\{0,1\}^{m \times p}$, logical operations can be applied. □

Note that only such $y_{r,k}$ that $\boldsymbol{S}'[\eta_r, i] = 0$, $\forall i = 1, \ldots, m$ may belong to $D^{(1)}_{\zeta,k}$, which complies with Lemma 1. Proposition 1 assigns to each permutation $\tilde{\boldsymbol{u}}_\zeta$ a maximal conditioned control strategy. To obtain a set of maximal control strategies, all strategies whose SAIOD is a strict subset of SAIOD of another control strategy should be eliminated. In particular, consider maximal conditioned control strategies γ_η and γ_ζ. If

$$\boldsymbol{R}_\eta[i,j] = 1 \Rightarrow \boldsymbol{R}_\zeta[i,j] = 1, \ \forall i, j,$$

then $\boldsymbol{R}_\eta$ is said to be a subset of $\boldsymbol{R}_\zeta$ and denoted $\boldsymbol{R}_\eta \subset \boldsymbol{R}_\zeta$. Moreover, if also $\boldsymbol{R}_\eta \supset \boldsymbol{R}_\zeta$, which implies $\boldsymbol{R}_\eta = \boldsymbol{R}_\zeta$. Then, γ_η and γ_ζ are called *equivalent*. Otherwise, $\boldsymbol{R}_\eta \not\supset \boldsymbol{R}_\zeta$ and γ_η is not maximal. It follows that if $\boldsymbol{R}_\eta \subset \boldsymbol{R}_\zeta$ holds for a pair of maximal conditioned strategies, then γ_η can be omitted from construction of $\mathcal{G}_M$.

3.2 Reducing computational complexity

At most $m!$ strictly sequential control strategies need to be constructed. Computational complexity can be reduced by analyzing the plant adjacency matrix $\boldsymbol{S}$ before generating $\tilde{\boldsymbol{u}}_i$.

(1) Assume there exist r elements of $\boldsymbol{u}$ such that $\boldsymbol{S}[:, i_1] = \boldsymbol{S}[:, i_2] = \cdots = \boldsymbol{S}[:, i_r]$ for some $i_1, i_2, \ldots, i_r$. Then, without loss of generality, $\{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}$ can be grouped together and considered as one control variable when searching for control strategies with maximal SAIOD.
(2) By the proof of Proposition 1, if there are i, j such that $\boldsymbol{S}[:, i] \subset \boldsymbol{S}[:, j]$, then any strictly sequential control strategy where u_i precedes u_j is either not maximal or there exists an equivalent maximal control strategy that differs only by the order of u_i and u_j. Hence, no control strategy is constructed for such permutation.
(3) If there are s elements of $\boldsymbol{u}$ such that $\boldsymbol{S}[j, i_1] = \cdots = \boldsymbol{S}[j, i_s] = 0$, $\forall j$, then, by the maximality condition, these variables can be aggregated and applied as a batch after all other control variables have already been calculated and applied.

4. OPTIMIZING RECURSION

At this stage, maximal control strategies that constitute $\mathcal{G}_M$ have been determined in the form of strictly sequential strategies. While the strict sequential strategies have proved useful in constructing $\mathcal{G}_M$, they may unnecessarily complicate the optimizing recursion.

Agregation of elements of $\boldsymbol{u}$ using analysis of $\boldsymbol{S}$, was discussed in previous section. Another grouping relies on analysis of $\boldsymbol{R}_\zeta$ and may be done after $\mathcal{G}_M$ had been constructed. Namely, if

$$\boldsymbol{R}_\zeta[\zeta_r,:] = \boldsymbol{R}_\zeta[\zeta_s,:], \quad r < s,$$

then $\boldsymbol{R}_\zeta[\zeta_r,:] = \boldsymbol{R}_\zeta[\zeta_i,:]$, $i = r,\ldots,s$. Assume, moreover, that the relation is not true for $r-1$ nor for $s+1$. By construction, $u_{\zeta_r}, u_{\zeta_{r+1}},\ldots,u_{\zeta_s}$ are in the given permutation ζ ordered subsequently and $D^{(r)}_{\zeta,k} = D^{(r+1)}_{\zeta,k} = \cdots = D^{(s)}_{\zeta,k}$. From Definition 3, it follows that

$$\boldsymbol{v}^{(i)}_\zeta = (u_{\zeta_r},\ldots,u_{\zeta_s})'$$

for some i. Subsequential regrouping the elements yields $\tilde{\boldsymbol{u}}_\zeta = \big(\boldsymbol{v}^{(1)\prime}_\zeta,\ldots,\boldsymbol{v}^{(q)\prime}_\zeta\big)$, $q \le m$, which is the desired input form for optimizing recursion.

Proposition 2. Let $\mathcal{G}_M$ contain (possibly non-strictly) sequential control strategies with maximal SAIOD together with their deterministic versions. Let $V_k(D_k)$ be the Bellman function at time k defined by

$$V_k(D_k) = \min_{\gamma^F_k\in(\mathcal{G}_M)^F_k} E\Big\{\sum_{i=k}^{F} L_i(\cdot)|D_k; f, \gamma^F_k\Big\},$$

where γ^F_k is the control strategy applied from the time instant k until the final time F, $(\mathcal{G}_M)^F_k = \mathcal{G}_M\times\cdots\times\mathcal{G}_M$ and $V_{F+1}(\boldsymbol{x}^F_0,\boldsymbol{y}^F_0,\boldsymbol{u}^F_0) = 0$. Furthermore, let $V^{(i)}_{\zeta,k}$ denote the Bellman function corresponding to application of $\boldsymbol{v}^{(i)}_{\zeta,k}$. To obtain the optimal control strategy minimizing (2), the following optimizing recursion must be calculated for each $\gamma_\zeta\in\mathcal{G}_M$ at time k:

$$V^{(i)}_{\zeta,k}(D^{(i)}_{\zeta,k}) = \min_{\boldsymbol{v}^{(i)}_{\zeta,k}\in\mathcal{U}} E\big\{V^{(i+1)}_{\zeta,k}(\cdot)|D^{(i)}_{\zeta,k},\boldsymbol{v}^{(i)}_{\zeta,k}\big\} \quad (4)$$

$$D^{(i)}_{\zeta,k} = D_k \cup \big(\cup^{i}_{j=1}\boldsymbol{z}^{(j)}_{\eta,k}\big) \cup \big(\cup^{i-1}_{j=1}\boldsymbol{v}^{(j)}_{\zeta,k}\big) \quad (5)$$

$$\boldsymbol{v}^{(i)*}_{\zeta,k} = \underset{\boldsymbol{v}^{(i)}_{\zeta,k}\in\mathcal{U}}{\operatorname{argmin}}\ E\big\{V^{(i+1)}_{\zeta,k}(\cdot)|D^{(i)}_{\zeta,k},\boldsymbol{v}^{(i)}_{\zeta,k}\big\}, \quad (6)$$

where $i = q, q-1,\ldots,1$, the initial condition at time k is given by

$$V^{(q+1)}_{\zeta,k}(D_{k+1}) = V_{k+1}(D_{k+1}) + L_k\big(\boldsymbol{y}^k_0,\boldsymbol{x}^k_0,\boldsymbol{u}^k_0\big)$$
$$D^{(q+1)}_{\zeta,k} = D_{k+1}$$

and $D^{(1)}_{\zeta,k} = D_k$. The optimal control strategy at time k is given by

$$V_k(D_k) = \min_\zeta \big\{V^{(1)}_{\zeta,k}(D_k)\big\} \quad (7)$$

$$\zeta^*(D_k) = \underset{\zeta}{\operatorname{argmin}}\ \big\{V^{(1)}_{\zeta,k}(D_k)\big\}. \quad (8)$$

Proof. The proposition follows from the result in (Žampa *et al.*, 2004) adapted to the notion of a sequential control strategy. At a time k, the optimal control strategy γ^F_{k+1} for the time segment $k+1,\ldots,F$ is assumed to be known. Since each $\gamma_{\zeta,k}\in\mathcal{G}_M$ comes with a different SAIOD, control values for each structure must be found by recursion (4)–(6). Clearly, $\cup^i_{j=1}\boldsymbol{z}^{(j)}_{\eta,k}$ is determined by nonzero entries of $\boldsymbol{R}_\zeta[\zeta_i,:]$. At $i=1$, the recursion generates $V^{(1)}_{\zeta,k}(D_k)$, which is the value of Bellman function at time k for the strategy $\big(\gamma_{\zeta,k},\gamma^F_{k+1}\big)$. Selecting ζ that provides the smalles value of $V^{(1)}_{\zeta,k}(D_k)$ in (7), (8) concludes the control strategy design at time k. □

5. ILLUSTRATING EXAMPLE

Let the plant Σ_1 be described by

$$\begin{aligned}\Sigma_1: y_1(k) &= u_1(k) + 2x(k)\\ y_2(k) &= u_2(k) + x(k),\\ y_3(k) &= u_1(k) + u_3(k) + 3x(k),\\ x(k+1) &= \alpha x(k) + \xi(k)\end{aligned}$$

where ξ is a white noise with $\mathcal{N}(0,\sigma^2)$ and the usual notation is used for time dependence. The given quality criterion is

$$J = E\Big\{\sum_{k=0}^{F} L_k\big(\boldsymbol{y}(k)\big)\Big\} = E\Big\{\sum_{k=0}^{F}\sum_{i=1}^{3} y_i(k)^2\Big\}. \quad (9)$$

All $y_i(k)$ are algebraically dependent on at least one $u_j(k)$ and hence $D_k = \big(\boldsymbol{u}^{k-1}_0,\boldsymbol{y}^{k-1}_0\big)$. Moreover, thanks to simplicity of Σ_1 estimation of $x(k)$ can be solved trivially as

$$\begin{aligned}&E\big\{x(k)|D_k\big\} = \alpha(y_1(k-1) - u_1(k-1))/2\\ &\operatorname{var}\big\{x(k)|D_k\big\} = \sigma^2, &(10)\\ &E\big\{x(k)|D_k,u_1(k),y_1(k)\big\} = (y_1(k) - u_1(k))/2\\ &\operatorname{var}\big\{x(k)|D_k,u_1(k),y_1(k)\big\} = 0, &(11)\\ &E\big\{x(k)|D_k,u_2(k),y_2(k)\big\} = y_2(k) - u_2(k)\\ &\operatorname{var}\big\{x(k)|D_k,u_2(k),y_2(k)\big\} = 0, &(12)\end{aligned}$$

depending on information available at the given stage of control generation. Note also that the solution (10) of $E\big\{x(k)|D_k\big\}$ is not unique.

A batch control strategy minimizing (9) follows from Corollary 1 and its realization is

$$\begin{aligned}\gamma_0: u^*_1(k) &= -\alpha\big(y_1(k-1) - u_1(k-1)\big)\\ u^*_2(k) &= -\alpha\big(y_2(k-1) - u_2(k-1)\big)\\ u^*_3(k) &= -\alpha\big(y_3(k-1) - u_3(k-1) - y_1(k-1)\big).\end{aligned}$$

The corresponding value of (9) is $J(\gamma_0) = 14F\sigma^2$.

Considering sequential control strategies, it follows from

$\boldsymbol{S}$	u_1	u_2	u_3
y_1	1	0	0
y_2	0	1	0
y_3	1	0	1

that no control strategies need to be constructed for permutations, where u_3 precedes u_1. Thus only 3 strategies are to be determined, defined by permutations $\tilde{\boldsymbol{u}}_\zeta$ as $\tilde{\boldsymbol{u}}_1 = (u_1, u_2, u_3)'$, $\tilde{\boldsymbol{u}}_2 = (u_1, u_3, u_2)'$, $\tilde{\boldsymbol{u}}_3 = (u_2, u_1, u_3)'$ and their respective adjacency matrices

$$\begin{array}{c|ccc} \boldsymbol{R}_1 & y_1 & y_2 & y_3 \\ \hline u_1 & 0 & 0 & 0 \\ u_2 & 1 & 0 & 0 \\ u_3 & 1 & 1 & 0 \end{array}, \quad \begin{array}{c|ccc} \boldsymbol{R}_2 & y_1 & y_2 & y_3 \\ \hline u_1 & 0 & 0 & 0 \\ u_2 & 1 & 0 & 1 \\ u_3 & 1 & 0 & 0 \end{array}, \quad \begin{array}{c|ccc} \boldsymbol{R}_3 & y_1 & y_2 & y_3 \\ \hline u_1 & 0 & 1 & 0 \\ u_2 & 0 & 0 & 0 \\ u_3 & 1 & 1 & 0 \end{array}.$$

Let $k = F$, then, since $V_{F+1}(x_0^F, \boldsymbol{y}_0^F, \boldsymbol{u}_0^F) = 0$, the optimal control for γ_1, defined by $\tilde{\boldsymbol{u}}_1$ and $\boldsymbol{R}_1$, is obtained by

$$\begin{aligned} V_{1,F}^{(3)} &= \min_{u_3(F)\in\mathcal{U}} E\{L_F(\cdot)|D_F, \boldsymbol{u}(F), y_1(F), y_2(F)\} \\ &= y_1(F)^2 + y_2(F)^2 \\ u_3^*(F) &= 3(u_2(F) - y_2(F)) - u_1(F) \end{aligned}$$

$$\begin{aligned} V_{1,F}^{(2)} &= \min_{u_2(F)\in\mathcal{U}} E\{V_{1,F}^{(3)}|D_F, u_1(F), u_2(F), y_1(F)\} \\ &= y_1(F)^2 \\ u_2^*(F) &= (u_1(F) - y_1(F))/2 \end{aligned}$$

$$\begin{aligned} V_{1,F}^{(1)} &= \min_{u_1(F)\in\mathcal{U}} E\{V_{1,F}^{(2)}|D_F, u_1(F)\} \\ u_1^*(F) &= \alpha(u_1(F-1) - y_1(F-1)) \end{aligned}$$

and $V_{1,F}^{(1)} = V_{1,F} = 4\sigma^2$. Formulae (12), (11) and (10) were used for estimation of x_k. The remaining optimal conditioned control strategies $\gamma_{2,F}^*$, $\gamma_{3,F}^*$, obtained by similar calculations, are given by

$$\begin{aligned} \gamma_{2,F} :& u_1^*(F) = \alpha(u_1(F-1) - y_1(F-1)) \\ & u_3^*(F) = (3y_1(F) - u_1(F))/2 \\ & u_2^*(F) = (u_1(F) - y_1(F))/2, \end{aligned}$$

$$\begin{aligned} \gamma_{3,F} :& u_2^*(F) = \alpha(u_2(F-1) - y_2(F-1)) \\ & u_1^*(F) = 2(u_2(F) - y_2(F)) \\ & u_3^*(F) = 3(u_2(F) - y_2(F)) - u_1(F) \end{aligned}$$

and the corresponding values of $V_{\zeta,F}$ are

$$V_{2,F} = 4\sigma^2, \qquad V_{3,F} = \sigma^2.$$

Since

$$V_F(D_F) = \min_{\zeta\in\{1,2,3\}} V_{\zeta,F} = V_{3,F} = \sigma^2$$

is a constant, the optimal strategy for all time instants is γ_3 and that the value of (9) becomes $J(\gamma_3) = F\sigma^2$. In this example, a sequential control strategy illustrated its higher control quality over a batch strategy.

6. CONCLUSION

Optimal control problem was a motivation for considering connection of systems with algebraic dependencies. Instead of the usual (and tedious) search for admissible maximal control strategies within the set of all possible structures of the strategies, the causality property was invoked. Namely, the notion of a sequential strategy was defined and then utilized in an efficient algorithm that provides maximal structures of admissible strategies. This may be seen as the main result of the paper. As its consequence, the standard optimizing recursion was extended.

The permutation number ζ^* may be thought of as a switching parameter determining which of the possible structures of a controller ought to be used at time k. This falls into the framework of hybrid or, more precisely, switched systems (Hespanha and Morse, 2002).

The problem can be extended to systems that may switch between control strategies $\gamma_{\zeta,k}$ during generation of the control sequence $\tilde{\boldsymbol{u}}_{\zeta,k}$ at each time k. This poses a nontrivial problem, where a suboptimal solution is possible using an open-loop feedback. Finally, a more practical modification of the proposed optimal control design consists in fixing the control strategy for the whole control horizon, i.e., ζ^* is constant for all $k \in \mathcal{T}$. The choice of optimal ζ^* by (7) and (8) is then carried out only once, at $k = -1$.

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