MOTION PLANNING AND FEEDFORWARD CONTROL FOR DISTRIBUTED PARAMETER SYSTEMS UNDER INPUT CONSTRAINTS

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Abstract: The flatness–based feedforward tracking control design based on formal power series is extended to distributed parameter systems (DPS) with input constraints. Thereby, formal power series and suitable summation methods are utilized to derive a finite–dimensional approximation in generalized controller form coordinates. This representation serves as the basis for a novel feedforward control design approach, which treats the considered finite–time transition between equilibrium profiles as a two–point boundary value problem (BVP) with input constraints. Simulation results for a linear heat conduction system and a nonlinear diffusion–convection–reaction system (DCRS) illustrate the applicability of the approach.

Keywords: infinite dimensional system, finite dimensional system, flatness, nonlinear system inversion, boundary value problem, input constraints

1. INTRODUCTION AND PROBLEM FORMULATION

Formal power series have proven to provide a sophisticated tracking control design tool for boundary controlled DPS, which comprises motion planning to ensure uniform convergence of the feedforward control (Laroche *et al.*, 2000; Lynch and Rudolph, 2002). Recently, powerful summation methods for enhanced motion planning, convergence acceleration, and summation of divergent series have been introduced within this context (Wagner *et al.*, 2004). Moreover, an extension to feedback control with observer state estimation is available (Meurer and Zeitz, 2003; Meurer and Zeitz, 2005).

On the other hand, many applications require to fulfill certain input constraints due to physical limitations, e.g. limited heating power or inlet concentration. In (Graichen *et al.*, 2004), a design methodology is presented for finite–dimensional systems, which directly allows motion planning and feedforward control design for finite–time transitions between equilibrium points. This approach is extended in (Graichen and Zeitz, 2005) to account for input constraints. In the sequel, this approach is exemplarily applied to the feedforward control design for an infinite–dimensional scalar DCRS with state $x(z,t)$ defined on $(z,t) \in (0,1)\times \mathbb{R}^+$. The DCRS is described by the nonlinear parabolic partial differential equation

$$\frac{\partial x}{\partial t} = \lambda \frac{\partial^2 x}{\partial z^2} + \nu \frac{\partial x}{\partial z} + \beta x + \varphi(x) \qquad (1)$$

with boundary and initial conditions

$$\frac{\partial x}{\partial z}(0,t) = 0, \quad t > 0 \qquad (2)$$

$$x(1,t) = u(t) \in [u_{min}, u_{max}], \quad t > 0 \qquad (3)$$

$$x(z,0) = x_0(z), \quad z \in [0,1] \qquad (4)$$

and output equation

$$y(t) = x(0,t), \quad t \geq 0. \qquad (5)$$

Thereby, all model parameters and variables are assumed to be perfectly non–dimensionalized. From a physical point of view, it follows that $\lambda > 0$, $\nu \geq 0$, i.e. convection takes place in the direction of the negative z-axis. The input $u(t)$ acts at $z=1$ and is assumed to be constrained by u_{min} and u_{max}. The DCRS (1)–(5) might serve as a simple model of a tubular fixed–bed reactor or a bioreactor, where the nonlinear source function

$$\varphi(x) = \sum_{m=2}^{M<\infty} \mu_m x^m \tag{6}$$

could describe a higher order reaction rate (Meurer and Zeitz, 2004).

At first, the nonlinear DCRS (1)–(5) is parameterized by a formal power series in §2, which serves as a basis for the derivation of a finite–dimensional design model. This approximation is used for motion planning and feedforward control design under input constraints in §3 following the inversion–based approach. Simulation results in §4 illustrate the applicability of this approach in various scenarios.

2. SYSTEM APPROXIMATION VIA FORMAL POWER SERIES

Similar to (Lynch and Rudolph, 2002; Meurer and Zeitz, 2003), Eqns. (1)–(6) are completely parameterized by a formal power series for the state $x(z,t)$.

2.1 Formal power series parameterization

For the formal power series parameterization – see (Wagner *et al.*, 2004; Meurer and Zeitz, 2005) for a rigorous definition – the state $x(z,t)$ is assumed to follow an ansatz of the form

$$\hat{x}(z,t) = \sum_{n=0}^{\infty} \hat{x}_n(t) z^n, \quad z \in [0,1],\ t \geq 0 \tag{7}$$

with yet unknown time–varying coefficients $\hat{x}_n(t)$. Substituting (7) into (1), (2), (5), (6) and sorting terms of equal order in z yields

$$\begin{aligned}\frac{d\hat{x}_n(t)}{dt} &= \lambda(n+2)(n+1)\hat{x}_{n+2}(t) + \nu(n+1)\hat{x}_{n+1}(t)\\ &\quad +\beta\hat{x}_n(t) + \hat{\varphi}_n(\hat{\mathbf{x}}_n^{[1+]}(t)),\ n \in \mathbb{N}^0, \qquad (8)\\ \hat{x}_1(t) &= 0, \qquad (9)\\ \hat{x}_0(t) &= y(t), \qquad (10)\end{aligned}$$

where $\hat{\varphi}_n(\hat{\mathbf{x}}_n(t))$ denotes the coefficient of z^n when evaluating (6) with (7) using Cauchy's product formula, and $\hat{\mathbf{x}}_n^{[j+]} := [\hat{x}_0, \hat{x}_j, \ldots, \hat{x}_{nj}]^T$ with $n, j \in \mathbb{N}^0$.

A differential recursion for $\hat{x}_n(t), n\geq 2$ is obtained by solving (8)–(10) for $\hat{x}_{n+2}(t)$. This recursion can be evaluated in terms of the output $y(t)$ and its time–derivatives $y^{(j)}(t), j\in\mathbb{N}$ up to infinite order. As a result, it can be verified that

$$\hat{x}_n(t) = \psi_n(t) = \begin{cases} \bar{\psi}_n(\mathbf{y}_{\frac{n}{2}}(t)) & n \text{ even} \\ \bar{\psi}_n(\mathbf{y}_{\frac{n-1}{2}}(t)) & n \text{ odd} \end{cases} \tag{11}$$

with $\mathbf{y}_m := [y, \dot{y}, \ldots, y^{(m)}]^T$ and $\psi_0(t) = y(t)$, $\psi_1(t) = 0$. Furthermore, for any $n \in \mathbb{N}^0$ it is possible to express the coefficients $\hat{x}_{2n+1}(t)$ algebraically in terms of $\hat{\mathbf{x}}_n^{[2+]}(t)$, simply by solving (8) for $\hat{x}_{2n+1}(t)$ and subsequential substitution of the appearing derivatives and coefficients using (8) and (9), i.e.

$$\hat{x}_{2n+1} = \vartheta_{2n+1}(\hat{\mathbf{x}}_n^{[2+]}), \quad n \in \mathbb{N}^0. \tag{12}$$

Thus, substitution of (7) with (11) into (3) provides a parameterization of the boundary input

$$u(t) = \sum_{n=0}^{\infty} \psi_n(t) \tag{13}$$

in terms of the output $y(t)$ and its time–derivatives up to infinite order. This in particular relates formal power series parameterizability (Wagner *et al.*, 2004) to the notion of differential flatness (Fliess *et al.*, 1995).

A standard control problem concerns the startup of a tubular reactor, which is characterized by a finite–time transition between the initial stationary profile $x_S^0(z) = x_0(z)$ and the operating stationary profile $x_S^T(z) = x(z, t\geq T)$. For the DCRS (1)–(5), stationary profiles $x_S^j(z),\ j \in \{0, T\}$ are governed by the BVP

$$\begin{aligned}&\lambda\frac{d^2x_S^j}{dz^2} + \nu\frac{dx_S^j}{dz} + \beta x_S^j + \varphi(x_S^j) = 0,\\ &\frac{dx_S^j}{dz}(0) = 0, \quad x_S^j(1) = y_j^* = const.\end{aligned} \tag{14}$$

Hence variations in y_j^* result in different stationary profiles $x_S^j(z)$. For the exact realization of finite–time transitions, a desired path $t \mapsto y^*(t)$ of class $\mathcal{C}^\infty$ with

$$\begin{aligned}&y^*(0) = y_0^* = x_S^0(1),\ y^*(T) = y_T^* = x_S^T(1),\\ &y^{*(i)}\Big|_{0,T} = 0,\ i \geq 1\end{aligned} \tag{15}$$

has to be specified ensuring uniform convergence of the series (7) with coefficients given by (11) – see e.g. (Laroche *et al.*, 2000). For the considered DCRS (1)–(6), convergence requires to impose restrictions on the parameters λ, ν, β, μ (Lynch and Rudolph, 2002; Meurer and Zeitz, 2004), which are usually too restrictive for many physical situations. In order to overcome these limitations, in (Wagner *et al.*, 2004; Meurer and Zeitz, 2005) the notion of an (N,ξ)–approximate k–sum is introduced for enhanced motion planning, convergence acceleration, and summation of divergent series.

Definition 1. The (N,ξ)–approximate k–sum $\mathcal{S}_{N,\xi,k}$ of a formal power series (7) is defined as

$$\mathcal{S}_{N,\xi,k} := \frac{\sum_{n=0}^{N}\left(\sum_{j=0}^{n}\hat{x}_j(t)z^j\right)\frac{\xi^n}{\Gamma(1+nk)}}{\sum_{n=0}^{N}\frac{\xi^n}{\Gamma(1+nk)}}. \tag{16}$$

This summation approach modifies the widely studied k–summation (Balser, 2000) and directly accounts for the fact that in general only a finite number of series coefficients $\hat{x}_n(t),\ n = 0, 1, \ldots, N$ can be computed from the obtained differential recursion. Furthermore this technique provides a powerful method for convergence acceleration and summation of divergent series (Wagner *et al.*, 2004; Meurer and Zeitz, 2005).

Following the ideas of (Meurer and Zeitz, 2003; Meurer and Zeitz, 2005), the derived formal power series parameterization can be utilized to derive a finite–dimensional approximation of the given infinite–dimensional DCRS (1)–(6).

2.2 Design model based on formal power series

As shown in §2.1, solving Eqn. (8) for $\hat{x}_{n+2}(t)$ provides a differential recursion for any coefficient $\hat{x}_n(t),\ n \geq 2$ of the formal power series ansatz (7). On the other hand, Eqn. (8) can be interpreted as a set of ordinary differential equations (ODEs) for the states $\hat{x}_{2n}(t),\ n \geq 0$. Since any $\hat{x}_{2n+1}(t)$ can be substituted using (12), only the coefficients $\hat{x}_{2n}(t),\ n \in \mathbb{N}^0$ with even index have to be considered as states. Thus, the infinite set of ODEs

$$\frac{d\hat{x}_{2n}}{dt} = \lambda(2n+2)(2n+1)\hat{x}_{2n+2} + \beta\hat{x}_{2n} + \theta_{2n}(\hat{\mathbf{x}}_n^{[2+]}) \quad (17)$$

with

$$\theta_{2n}(\hat{\mathbf{x}}_n^{[2+]}) = \nu(2n+1)\vartheta_{2n+1}(\hat{\mathbf{x}}_n^{[2+]}) + \hat{\varphi}_{2n}(\hat{\mathbf{x}}_n^{[2+]}) \quad (18)$$

is obtained for $n \in \mathbb{N}^0$. It is shown in (Meurer and Zeitz, 2003), that a flat finite–dimensional design model can be obtained under the formal assumption of an at least unit radius of convergence of (7). Therefore, the set (17) has to be truncated at some $n = N - 1,\ N \in \mathbb{N}$. The remaining unknown coefficients $\hat{x}_{2N}(t)$ can be substituted by introducing the input $u(t)$ via the truncated formal ansatz

$$u \approx \sum_{n=0}^{2N} \hat{x}_n. \quad (19)$$

In order to weaken the rather restrictive convergence requirement, the less restrictive assumption of k–summability (Balser, 2000) is introduced, such that instead of applying (19), its $(2N, \xi)$–approximate k–sum

$$u \approx \frac{\sum_{n=0}^{2N}\left(\sum_{j=0}^{n}\hat{x}_j\right)\frac{\xi^n}{\Gamma(1+nk)}}{\sum_{n=0}^{2N-1}\frac{\xi^n}{\Gamma(1+nk)}}, \quad (20)$$

is used. Utilizing (12), it follows after some intermediate calculations that

$$\begin{aligned}\hat{x}_{2N} = &-\sum_{n=0}^{N-1} A_{2n}(\xi,k)\hat{x}_{2n} + b_N(\xi,k)u + \\ &-\sum_{n=0}^{N-1} A_{2n+1}(\xi,k)\vartheta_{2n+1}(\hat{\mathbf{x}}_n^{[2+]}),\end{aligned} \quad (21)$$

where

$$A_n(\xi,k) = 1 + \sum_{j=n}^{2N-1}\sigma_j, \quad b_N(\xi,k) = \sum_{n=0}^{2N}\sigma_j \quad (22)$$

with $\sigma_j = \Gamma(1+2Nk)/\Gamma(1+jk)/\xi^{2N-j}$, $j, N \in \mathbb{N}$. This allows the insertion of the input $u(t)$ into the ODEs (17) for $n = 0, 1, \ldots, N-1$. Summarizing these results, the state–space representation (23), (24) as depicted in Fig. 1 is obtained for the states $\hat{\mathbf{x}}_{N-1}^{[2+]} = [\hat{x}_0, \hat{x}_2, \ldots, \hat{x}_{2N-2}]^T$. This input affine SISO system of finite dimension N allows an interpretation as a generalized nonlinear controller normal form due to the band structure of matrix $\hat{\mathbf{A}}_{\xi,k}$ with one side diagonal and the triangular structure of the nonlinear function $\hat{\boldsymbol{\theta}}(\hat{\mathbf{x}}_{N-1}^{[2+]})$. It is hence easy to verify, that the output (24) is a flat output which parameterizes the state $\hat{\mathbf{x}}_{N-1}^{[2+]}(t)$ and the input $u(t)$ of (23).

As illustrated in (Meurer and Zeitz, 2005) for an appropriate choice of the summation parameters ξ and k, the derived system of ODEs yields a sufficient approximation of the original dynamics governed by the DCRS (1)–(6). Based on this approximation, flatness–based feedback tracking control with observer can be designed (Meurer and Zeitz, 2003; Meurer and Zeitz, 2005). In the following, the design model (23)–(24) is used to determine a feedforward control $u(t)$ which satisfies the input constraints (3) by applying the ideas of (Graichen and Zeitz, 2005) to the infinite–dimensional case.

3. MOTION PLANNING AND FEEDFORWARD CONTROL DESIGN

In (Graichen and Zeitz, 2005), an approach for feedforward control design is presented based on the inverse of the given nonlinear system. Since the derived N–dimensional design model (23), (24) is structurally flat, a change of coordinates $[y, \dot{y}, \ldots, y^{(N-1)}]^T = \Lambda(\hat{\mathbf{x}}_{N-1}^{[2+]})$ allows to transform the system (23), (24) into the controller normal form (Isidori, 1995)

$$y^{(N)} = \alpha(y, \dot{y}, \ldots, y^{(N-1)}, u), \quad (25)$$

which represents a chain of integrators of length N. The particular structure of the form (25) enables an algebraic solution of the feedforward control

$$u^*(t) = \alpha^{-1}(y^*(t), \dot{y}^*(t), \ldots, y^{*(N)}(t)). \quad (26)$$

in dependence of the desired output trajectory $y^*(t)$ and its first N time derivatives. The relation (26) represents the inverse of system (25) or (23), (24), respectively. In view of (3), it additionally follows that the feedforward control

$$u^*(t) \overset{!}{\in} [u_{min}, u_{max}]. \quad (27)$$

has to satisfy the constraints. In order to realize finite–time transitions for the DCRS (1)–(6) based on the design model (23)–(24), a trajectory $y^*(t) \in \mathcal{C}^N$ has to be specified, which satisfies the BCs (15). In particular, the BCs $y^{*(N)}\big|_{0,T}$ for the highest order time derivative guarantee that the

$$
\frac{d\hat{\mathbf{x}}_{N-1}^{[2+]}}{dt} = \underbrace{\begin{pmatrix} \beta & 2\lambda & 0 & \dots & 0 & 0 \\ 0 & \beta & 12\lambda & \dots & 0 & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta & (2N-2)(2N-3)\lambda \\ \hat{A}_0 & \hat{A}_2 & \hat{A}_4 & \dots & \hat{A}_{2N-4} & \beta + \hat{A}_{2N-2} \end{pmatrix}}_{=:\ \hat{\mathbf{A}}_{\xi,k}} \hat{\mathbf{x}}_{N-1}^{[2+]} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hat{b}_N(\xi,k) \end{pmatrix}}_{=:\ \hat{\mathbf{b}}_{\xi,k}} u + \underbrace{\begin{pmatrix} \theta_0(\hat{\mathbf{x}}_0^{[2+]}) \\ \theta_2(\hat{\mathbf{x}}_1^{[2+]}) \\ \vdots \\ \theta_{2N-4}(\hat{\mathbf{x}}_{N-2}^{[2+]}) \\ \hat{\theta}_{2N-2}(\hat{\mathbf{x}}_{N-1}^{[2+]}) \end{pmatrix}}_{=:\ \hat{\boldsymbol{\theta}}(\hat{\mathbf{x}}_{N-1}^{[2+]})} \tag{23}
$$

$$
y = [1, 0, \dots, 0]^T \hat{\mathbf{x}}_n^{[2+]} =: \hat{\mathbf{c}}^T \hat{\mathbf{x}}_n^{[2+]}. \tag{24}
$$

Fig. 1. Schematic state–space representation of the design model in generalized controller form from (17) truncated at $n = N-1$ and (21), (22) substituted. Here $\hat{A}_{2j} = 2N(2N-1)\lambda A_{2j}(\xi,k),\ j = 0,1,\dots,N-1,\ \hat{b}_N(\xi,k) = 2N(2N-1)\lambda b_N(\xi,k),\ \hat{\theta}_{2N-2}(\hat{\mathbf{x}}_{N-1}^{[2+]}) = \theta_{2N-2}(\hat{\mathbf{x}}_{N-1}^{[2+]}) + 2N(2N-1)\lambda \sum_{n=0}^{N-1} A_{2n+1}(\xi,k)\vartheta_{2n+1}(\hat{\mathbf{x}}_n^{[2+]})$.

feedforward control $u^*(t)$ in (26) is $\mathcal{C}^0$–continuous at the initial and terminal points $t = 0, T$.

Note that the consideration of the finite–dimensional approximation can be interpreted in the sense of approximate motion planning for the infinite–dimensional DCRS (1)–(6). Nevertheless, this motion planning approach directly allows to account for input constraints as outlined below.

3.1 Flatness-based design for unconstrained input

In case of an unconstrained input, the flatness–based feedforward design can be applied to the finite–dimensional design model in flat coordinates (25) and the transition problem is simply solved by planning a sufficiently smooth flat output trajectory $y^*(t) \in \mathcal{C}^N$. With respect to the stationary initial and terminal setpoints y_0^* and y_T^*, e.g. a polynomial trajectory $y^*(t)$ of order $2N+1$ has the particular structure

$$
y^*(t) = y_0^* + (y_T^* - y_0^*) \sum_{i=N+1}^{2N+1} a_i \left(\frac{t}{T}\right)^i \tag{28}
$$

for $t \in [0,T]$. The coefficients a_i, $i = N+1, \dots, 2N+1$ have to be determined to meet the $(N+1)$ BCs in (15) for $t = T$. By means of the output trajectory $y^*(t)$, the feedforward trajectory $u^*(t)$ can easily be calculated from (26).
The flatness–based approach allows a purely algebraic solution of the transition problem but input constraints (27) can only be considered heuristically, e.g. by changing the transition time T.

3.2 Solution of BVPs for constrained input

In order to directly incorporate the input constraints (27) within the feedforward control design as proposed in (Graichen and Zeitz, 2005), a new set–up function $\hat{\alpha} = y^{*(N)}$ is introduced to parameterize the highest order derivative $y^{*(N)}$ of the output, which yields the chain of N integrators

$$
y^{*(N)} = \hat{\alpha} \tag{29}
$$

subject to the $2\,N$ BCs

$$
\begin{aligned} &y^*(0) = y_0^*, \quad y^*(T) = y_T^*, \\ &y^{*(i)}\Big|_{0,T} = 0,\ i = 1, \dots, N-1\,. \end{aligned} \tag{30}
$$

From a mathematical point of view, the N ODEs (29) and the $2\,N$ BCs (30) form a two–point BVP for the flat output $y^*(t)$. The solution $y^*(t)$, $t \in [0,T]$ depends on the set–up of the function $\hat{\alpha} = y^{*(N)}$ with respect to the following objectives:

(i) $\mathcal{C}^0$–continuity of the feedforward trajectory $u^*(t)$ at the bounds $t = 0, T$ implies that the output trajectory $y^*(t)$ must meet the two additional BCs

$$
y^{*(N)}(0) = 0, \quad y^{*(N)}(T) = 0. \tag{31}
$$

(ii) The solvability of the BVP (29), (30) defined by N ODEs and $2\,N$ BCs, requires at least N free parameters. Similar to (Graichen *et al.*, 2004), the parameters $\boldsymbol{p}^* = (p_1^*, \dots, p_N^*)$ are provided in a function $\Phi(t, \boldsymbol{p}^*)$, which is used for the set–up $\hat{\alpha} = \Phi(t, \boldsymbol{p}^*)$ if the resulting input

$$
u_\Phi = \alpha^{-1}(y^*, \dot{y}^*, \dots, \Phi(t, \boldsymbol{p}^*)) \tag{32}
$$

following from (26) lies within the specified bounds (u_{min}, u_{max}). The function $\Phi(t, \boldsymbol{p}^*)$ can e.g. be constructed by a polynomial in the following way:

$$
\Phi(t, \boldsymbol{p}^*) = b_0(\boldsymbol{p}^*) + b_1(\boldsymbol{p}^*)\frac{t}{T} + \sum_{i=1}^{N} p_i^* \left(\frac{t}{T}\right)^{i+1}. \tag{33}
$$

The free parameters p_i^*, $i = 1, \dots, N$ are the coefficients of the highest order terms. In order to satisfy the two BCs (31), the first two coefficients are derived by $b_0(\boldsymbol{p}^*) = 0$ and $b_1(\boldsymbol{p}^*) = -\sum_{i=1}^{N} p_i^*$.

(iii) The consideration of the input constraints (27) requires to check if $u_\Phi \in (u_{min}, u_{max})$ is satisfied by (32). If u_Φ is outside the bounds, $\hat{\alpha} = y^{*,N}$ must be “re–planned” in (29), such that the bounds u_{min} and u_{max} are met. This is accomplished by the following case–dependent definition of the function

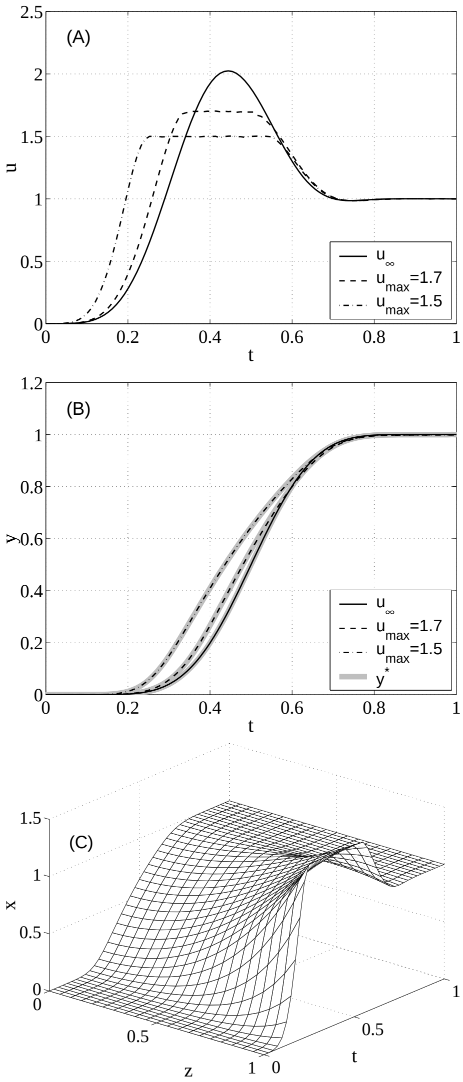

Fig. 2. Simulation results for linear heat equation: (A) Feedforward control $u(t) = x(1,t)$ for u_{max} varied; (B) comparison of output $y(t) = x(0,t)$ and desired output $y^*(t)$ for respective u_{max}; (C) evolution of the profile $x(z,t)$ in (z,t)–domain for feedforward control with $u_{max} = 1.5$.

$$\hat{\alpha} = \begin{cases} \Phi(t, \boldsymbol{p}^*) & \text{if } u_\Phi \in (u_{min}, u_{max}) \\ \alpha(y^*, \dot{y}^*, \ldots, u_{min}) & \text{if } u_\Phi \leq u_{min} \\ \alpha(y^*, \dot{y}^*, \ldots, u_{max}) & \text{if } u_\Phi \geq u_{max}. \end{cases} \quad (34)$$

The calculation of the feedforward control $u^*(t)$, $t \in [0,T]$ in (26) requires the solution of the BVP (29)–(30) with (33) and (34) in dependence of the free parameters $\boldsymbol{p}^*$.[1] Note that the two additional BCs in (31) are already satisfied by $\Phi(t, \boldsymbol{p}^*)$ in (33), and $\hat{\alpha} = \Phi(t, \boldsymbol{p}^*)$, $t = 0, T$ in (34) holds because $u_\Phi(0, \boldsymbol{p}^*) = u_0^*$ and $u_\Phi(T, \boldsymbol{p}^*) = u_T^*$ must lie within the constraints $[u_{min}, u_{max}]$.

In the special case that the feedforward control $u^*(t)$ lies within the bounds for the whole transition interval $t \in [0,T]$, the output trajectory

[1] Note that the transition time T must be reasonably chosen with respect to the constraints and system dynamics.

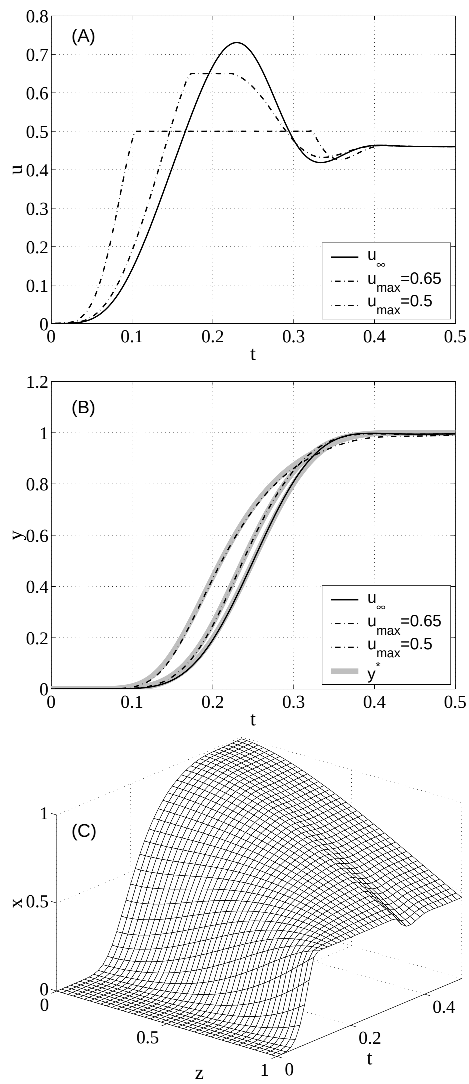

Fig. 3. Simulation results for nonlinear DCRS: (A) Feedforward control $u(t) = x(1,t)$ for u_{max} varied; (B) comparison of output $y(t) = x(0,t)$ and desired output $y^*(t)$ for respective u_{max}; (C) evolution of the profile $x(z,t)$ in (z,t)–domain for feedforward control with $u_{max} = 0.5$.

$y^*(t)$ is simply determined by the N–times integration of the polynomial set–up $y^{*(N)} = \Phi(t, \boldsymbol{p}^*)$. Thereby, the free parameters $\boldsymbol{p}^* = (p_1^*, \ldots, p_N^*)$ are calculated such that the BCs in (30) are satisfied, which results in an output trajectory $y^*(t)$ with the same polynomial structure (28) as in the flatness–based design.

4. SIMULATION RESULTS

The nonlinear BVP (29)–(30) with (33)–(34) and the N unknown parameters $\boldsymbol{p}^*$ can be solved with the standard MATLAB function bvp4c[2]. The bvp4c–function is a finite–difference code and determines a numerical solution by solving a set of

[2] ftp://ftp.mathworks.com/pub/doc/papers/bvp/

algebraic equations resulting from the difference approximation. Moreover, bvp4c estimates the error of the numerical solution on each subinterval and adapts the mesh points. The user has to provide an initial mesh as well as a guess of the solution at the mesh points. Furthermore, an initial guess of the free parameters of the BVP is needed.

In order to illustrate the achieved tracking performance, the determined feedforward control is applied to a high–order 'method–of–lines' discretized version of the governing DCRS (1)–(6). As a first example, the DCRS (1)–(6) with $\lambda = 1$, $\nu = \beta = \varphi(x) = 0$, i.e. the linear heat equation with input constraints $u_{min} = 0$ and $u_{max} > 0$ varied is considered. The respective design model (23)–(24) is of dimension $N = 8$ with $\xi = 25$, $k = 1$. It is desired to transfer the system from the initial state $x_S^0(z) = 0$ to the terminal state $x_S^T(z) = 1$ in finite time $T = 1$. Figure 2 shows simulation results for feedforward control with u_{max} varied within the unconstrained case u_∞ and $u_{max} \in \{1.5, 1.7\}$. In any case, almost no deviation appears between desired $y^*(t)$ and actual output trajectory $y(t) = x(0,t)$ due to the re–planning of the trajectory (34) when reaching the input constraint u_{max}. Obviously with decreasing constraints, the feedforward control has to start with a greater slope and follows the upper constraint u_{max} for a larger time–interval in order to perform the finite–time transition.

Secondly, consider the nonlinear DCRS (1)–(6) with $\lambda = 1$, $\nu = 5$, $\beta = 10$, $\varphi(x) = -7x^2$. For these parameters, the initial stationary profile $x_S^0(z) = 0$ and the terminal stationary profile determined by $y_T^* = 1$ in (14) remain stable. The transition time is chosen as $T = 0.5$. The respective design model (23), (24) is of dimension $N = 8$ with $\xi = 6.5$, $k = 1$ appropriately determined from the unconstrained case. Simulation results for this scenario are depicted in Figure 3 with input constraints $u_{min} = 0$ and $u_{max} > 0$ varied within the unconstrained case u_∞ and $u_{max} \in \{0.5, 0.65\}$. Similar to the linear heat equation, only negligible deviations occur between desired $y^*(t)$ and actual output trajectory $y(t) = x(0,t)$. This is in particular remarkable for the constraint $u_{max} = 0.5$ where the input constraint almost corresponds to the stationary input value needed to achieve the final stationary profile.

5. CONCLUSIONS

The presented approach for feedforward control of nonlinear DCRS with input constraints is based on an appropriate formal power series approximation of the infinite–dimensional system combined with the inversion–based feedforward control design technique proposed in (Graichen and Zeitz, 2005). Formal power series are used to determine a differentially flat finite–dimensional design model which is re–formulated as a two–point boundary value problem for approximate motion planning and feedforward control design with input constraints. Simulation results for the linear heat equation and a nonlinear tubular reactor model with quadratic reaction rate clearly confirm the applicability of the proposed approach.

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