CONTROL OF WEAKLY DAMPED FINITE AND INFINITE DIMENSIONAL EULER-LAGRANGE SYSTEMS

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Abstract: This contribution presents the control of a certain kind of mechanical systems based on energy considerations. The plants under investigation are underactuated lumped and distributed parameter systems, which consist of two masses and an elastic element. After a brief introduction of the used mathematical objects, the proposed controller design method is applied to a simple introductory example - the Mass-Spring-Mass system. After that, an infinite dimensional system - the Mass-Beam-Mass system - is under consideration. In this case, the elastic element is given by a Bernoulli-Euler beam. Finally some simulation results demonstrate the improvements gained by the introduced control structure.

Keywords: infinite dimensional systems, energy considerations, Bernoulli-Euler beam

1. INTRODUCTION

The tendency to use lightweight constructions in mechanical engineering opens up a huge field of applications for control engineers. Especially economic reasons force the use of lightweight constructions, because the acquisition and operating costs of such plants are reduced dramatically. Another consequence of the reduction of masses in a construction is normally an increased elasticity. Thus rigid body approximations do not apply anymore for such structures and appropriate control strategies are needed.

In the following a control structure designed for rigid body systems is assumed. This structure consists of a trajectory planning unit, which supplies a planned position w^1_{plan} and a planned force F_{plan}. The deviation of the actual and the planned trajectory is fed into a standard PD-controller, whose output F_e is also applied to the plant. This configuration is depicted in Fig. 1. In this

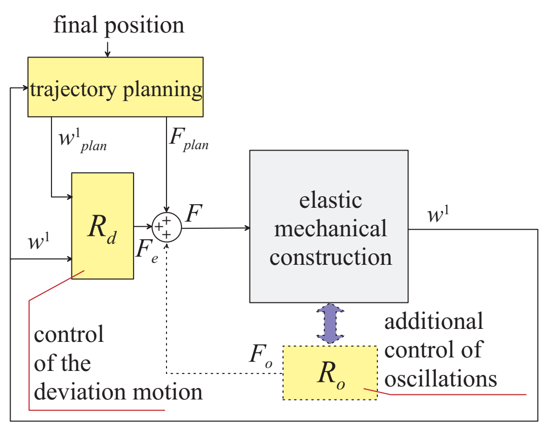

Fig. 1. Assumed control structure

contribution a design of an add-on controller R_o is presented, whereby the oscillations of the plant are absorbed and no additional modification of the control structure is necessary.

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The treated plants are two mechanical structures with significant elasticity. It will be shown, that the rigid body motion and the oscillations of the plant can be decoupled by means of a coordinate transformation. The decoupled representation allows to introduce a controller, which acts purely on the oscillation part of motion. In a second design step, the stability of a desired position of the overall system is achieved by an extension of the control law.

In the first chapter a brief definition of some mathematical objects, which will be used throughout the upcoming analysis, is given. After that, the lumped parameter Mass-Spring-Mass system is under investigation. This simple plant serves as an illustrative example for the proposed controller design. In the following chapter the distributed parameter Mass-Beam-Mass system is considered. Here the decoupling of the motion is extended to the infinite dimensional case. Consequently it is again possible to construct a controller, which acts only on the unmeant oscillations of the structure. Some additional investigations concerning the stability of the original system close this chapter. The proposed controller design is tested in several simulations and the achieved improvements are visualized. Finally this contribution closes with some concluding remarks on the controller implementation.

2. SOME NOTATIONS

The modelling of the investigated mechanical systems will be done by the use of Hamilton's principle. The description will be carried out on a smooth, trivial bundle $\mathcal{E} = (\mathcal{E}, \mathrm{pr}_1, \mathcal{B})$ with the reference manifold $\mathcal{E} = \mathcal{B} \times \mathcal{M}_C$, the configuration manifold $\mathcal{B} = \mathcal{Z} \times \mathcal{M}_R$ and the projection pr_1. The local coordinate of $\mathcal{Z}$ represents the system time and is denoted by $X^1 = t$. The local coordinates corresponding to $\mathcal{M}_R$ are given by X^i, $i = 2, \ldots, p$. These coordinates will represent the independent coordinates of a physical system. The local coordinates u^α, $\alpha = 1, \ldots, q$ correspond to the manifold $\mathcal{M}_C$ and equal the dependent coordinates of the system. Derivatives of the dependent coordinates with respect to the independent coordinates are the local coordinates of the corresponding jet bundles $J^n\mathcal{E}$ (Saunders, 1989) denoted by $u^\alpha_{[J]}$. Here, the partial derivatives are written in the multiindex notation, where the k^{th} order partial derivatives of the function γ^α are denoted by

$$\gamma^\alpha_{[J]} = \frac{\partial^k}{\partial\, X^{1\, j_1} \cdots \partial\, X^{p\, j_p}} \gamma^\alpha \ ,$$

$$[J] = [j_1, \ldots, j_p] \ , \quad k = \#J = \sum_{i=1}^{p} j_i .$$

A computer algebra support for Hamilton's principle in this framework is provided by the Maple9© package "JetVariationalCalculus" (H. Ennsbrunner, 2003).

3. THE MASS-SPRING-MASS SYSTEM

This system consists of two masses m_1, m_2 linked by a spring c_{sys} and a damper d_{sys}, as shown on the left hand side of Fig. 2. The lower mass m_1 is moved by a horizontally acting force F. The right hand side of Fig. 2 presents the rigid body model, which is taken into account by the trajectory planning unit and the deviation controller in the control structure sketched in Fig. 1.

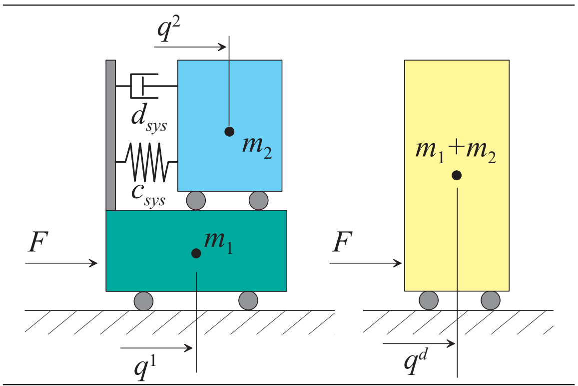

Fig. 2. Mass-Spring-Mass System

3.1 Modelling

Throughout the modelling we will use the dependent coordinates $u^1 = q^1$, $u^2 = q^2$ and the independent coordinate $X^1 = t$. The equations of motion can be derived by means of Hamilton's principle from the functional

$$\begin{aligned} \mathcal{L} &= \int_{t_1}^{t_2} l \ \mathrm{d}t \qquad (1)\\ &= \int_{t_1}^{t_2} \left(\frac{1}{2} m_1 \left(q^1_{[1]}\right)^2 + m_2 \left(q^2_{[1]}\right)^2 \right.\\ &\quad \left. - c_{sys} \left(q^2 - q^1\right)^2 + F_d q^1 - F_d q^2 + F q^1 \right) \ \mathrm{d}t \end{aligned}$$

resulting in

$$\begin{aligned} m_1 q^1_{[2]} - c_{sys} \left(q^2 - q^1\right) - d_{sys} \left(q^2_{[1]} - q^1_{[1]}\right) &= F \qquad (2)\\ m_2 q^2_{[2]} + c_{sys} \left(q^2 - q^1\right) + d_{sys} \left(q^2_{[1]} - q^1_{[1]}\right) &= 0, \end{aligned}$$

where $F_d = d_{sys} \left(q^2_{[1]} - q^1_{[1]}\right)$ is used. The equation of motion for the rigid body system is given by $(m_1 + m_2)\, q^d_{[2]} = F$.

3.2 System Decoupling

To enable the design of an add-on controller, as requested in the introduction, the decoupling of the rigid body motion and oscillations is needed. This splitting can be achieved by the introduction of a coordinate map $f : \mathcal{M}_C \rightarrow \bar{\mathcal{M}}_C$ defined by the two maps

$$\phi_d : q^d = \frac{m_1 q^1 + m_2 q^2}{m_1 + m_2}, \tag{3}$$

and $\phi_o : q^o = q^2 - q^1$. Here the new local coordinates q^d and q^o according to $\bar{\mathcal{M}}_C$ are introduced. The physical interpretation of the coordinate q^d is the center of gravity of the original system.

Consequently the linear transformation f allows us to transform the function l, defined in (1), to $\bar{l} = \left(j^1 f^{-1}\right)^* l$, where the first prolongation of the inverse mapping f^{-1} is used. Finally this procedure results in the functional

$$\begin{aligned}\bar{\mathcal{L}} &= \int_{t_1}^{t_2} \bar{l} \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left(\frac{m_1 + m_2}{2} \left(q^d_{[1]}\right)^2 + F q^d \right) \, \mathrm{d}t \, + \\ &\int_{t_1}^{t_2} \left(\frac{1}{2} \left(-c \left(q^o\right)^2 + \frac{m_1 m_2}{m_1 + m_2} \left(q^o_{[1]}\right)^2 \right) \right. \\ &\left. -F_d q^o - F q^o \, \frac{m_2}{m_1 + m_2} \right) \, \mathrm{d}t,\end{aligned}$$

on $J\bar{\mathcal{E}} = J\left(\mathcal{B} \times \bar{\mathcal{M}}_C\right)$. Thus the necessary decoupling of the oscillation and desired system is realized.

3.3 Controller Design

Based on the modelling of this simple system, we are able to develop a controller by means of energy considerations. The stored energy function of the Mass-Spring-Mass system is given by

$$H_{sys} = \frac{m_1}{2} \left(q^1_{[1]}\right)^2 + \frac{m_2}{2} \left(q^2_{[1]}\right)^2 + \frac{c_{sys}}{2} \left(q^2 - q^1\right)^2 .$$

The energy stored in the oscillation motion – the motion between the desired and the original system – can be stated as

$$H_o = H_{sys} - \underbrace{\frac{m_1 + m_2}{2} \left(q^d_{[1]}\right)^2}_{\bar{H}_d} \circ j^1 \phi_d,$$

where the prolongation $j^1 \phi_d$ of the previously defined map ϕ_d is used. The controller is aimed to decrease the energy H_o along the solution of the system. This demand makes sense, if the function is smooth on $J\mathcal{E}$ and bounded from below. In fact the resulting function

$$H_o = \frac{c_{sys}}{2} \left(q^2 - q^1\right)^2 + \frac{m_1 m_2 \left(q^2_{[1]} - q^1_{[1]}\right)^2}{2\left(m_1 + m_2\right)}, \tag{4}$$

is smooth and positive semi-definite. This result enables us to define a control law, that extracts energy from the unmeant oscillation motion, which is equivalent to a decrease of H_o.

Oscillation System: The derived oscillation energy (4) has to be minimized by a control law. Therefore we consider its time derivative along the trajectories of the system

$$\frac{\mathrm{d}}{\mathrm{d}t} H_o = -d_{sys} \left(q^2_{[1]} - q^1_{[1]}\right)^2 - \frac{m_2 \left(q^2_{[1]} - q^1_{[1]}\right)}{m_1 + m_2} F.$$

The introduction of the control law $F = F_o = d_o \left(q^2_{[1]} - q^1_{[1]}\right)$ admits the injection of additional damping into the oscillation motion. This control law is applied to the original system by the controller R_o and thus both systems – the oscillation system and the desired system – are affected. Consequently we have to analyze the stability of the original system.

Original System: The asymptotic stabilization of the original system (2) will be achieved by

$$F = -c \; q^1 - d \; q^1_{[1]} + d_o \left(q^2_{[1]} - q^1_{[1]}\right),$$

which represents the simple PD controller extended by R_o, which is depicted in Fig. 1. As a Lyapunov function we will use the adapted stored energy function $H = H_{sys} + \frac{c}{2} \, {q^1}^2$. The total time derivative of the positive definite function H results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H = -d_{sys} \Bigg(&\left(1 + \frac{d_o + d}{d_{sys}}\right) \, {q^1_{[1]}}^2 \\ &- \left(2 + \frac{d_o}{d_{sys}}\right) q^2_{[1]} q^1_{[1]} + {q^2_{[1]}}^2 \Bigg),\end{aligned}$$

which is negative semi-definite, if $d \geq \frac{{d_o}^2}{4 \; d_{sys}}$ is met. By taking LaSalle's invariants theorem into account, it can be shown, that the closed loop system is asymptotically stable.

4. THE MASS-BEAM-MASS SYSTEM

The Mass-Beam-Mass system (see Fig. 3) consists of a mass m_1, a vertically fixed Bernoulli-Euler beam (Meirovitch, 1967) – with a cross sectional area A, geometrical moment of inertia J_y and Young's module E – and a mass m_2 on the tip of the beam.

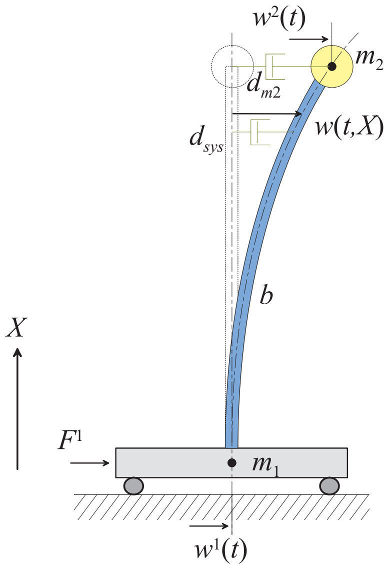

Fig. 3. Mass-Beam-Mass system

4.1 Modelling

The assumptions of linear elasticity and the Bernoulli-Euler beam theory allow us to formulate the kinetic energy density $T_b = \frac{A\rho}{2}\left(w^1_{[10]} + w_{[10]}\right)^2$ and the potential energy density $V_b = \frac{EJ_y}{2}\left(w_{[02]}\right)^2$ in order to extract the partial differential equation of motion for this distributed parameter system. A possible functional for Hamilton's principle is given by

$$\mathcal{L} = \int_{t_1}^{t_2}\int_0^L l \;\mathrm{d}X \wedge \mathrm{d}t = \tag{5}$$

$$\int_{t_1}^{t_2}\int_0^L (T_b - V_b + \frac{m_1}{2L} w^1_{[10]} + \frac{F^1}{L} w^1 + \frac{F^2}{L}\; w|_L$$

$$+\frac{m_2}{2L}\left(w^1_{[10]} + w_{[10]}|_L\right)^2 + F_d w) \;\mathrm{d}X \wedge \mathrm{d}t.$$

The forces F_d and F^2 are used for the introduction of internal damping. Finally the system is described by the partial differential equation

$$\rho A\;\left(w^1_{[20]} + w_{[20]}\right) + EJ_y\; w_{[04]} = -d_{sys} w_{[10]},$$

where $F_d = -d_{sys} w_{10}$ is applied. Additionally the boundary conditions are determined by

$$F^1 + d_{m_2}\left(w^2_{[10]} - w^1_{[10]}\right) + \int_0^L d_{sys} w_{[10]} \mathrm{d}X$$

$$= m_1 w^1_{[20]} + Q\,(t,0) = m_1 w^1_{[20]} + EJ_y\; w_{[03]}\big|_0$$

at $X = 0$, and

$$-d_{m_2}\left(w^2_{[10]} - w^1_{[10]}\right) = m_2 w^2_{[20]} - Q(t,L)$$

$$= m_2 w^2_{[20]} - EJ_y\; w_{[03]}\big|_L$$

at $X = L$, where $F^2 = -d_{m_2}\left(w^2_{[10]} - w^1_{[10]}\right)$ is used. The only interaction of the system with the surrounding is given by the force F^1 horizontally acting on the mass m_1.

4.2 System Decoupling

As shown in the introductory Mass-Spring-Mass example, the design of an add-on controller requires a decoupled representation of the investigated system.

The Lagrangian l of the Mass-Beam-Mass system is formulated on $J^2\mathcal{E}$ of the bundle $\left(\mathbb{R}^2 \times \mathcal{M}_C,\; pr_1,\; \mathbb{R}^2\right)$ as shown in (5). The local coordinates of the configuration manifold $\mathbb{R}^2 \times \mathcal{M}$ are given by $X^1 = t$, $X^2 = X$, $u^1 = w^1$, $u^2 = w$. The local coordinates $X^1 = t$, $X^2 = X$ are used for the reference manifold $\mathcal{B} = \mathbb{R}^2$.

Again we introduce a coordinate transformation $f : \mathcal{M}_C \rightarrow \bar{\mathcal{M}}_C$ defined by the two maps

$$\phi_d : w^d = \frac{m_1\, w^1 + m_2\, w^2 + \int_0^L \rho A\;\left(w^1 + w\right)\,\mathrm{d}X}{m_{sys}},$$

where $m_{sys} = m_1 + m_2 + \int_0^L \rho A\,\mathrm{d}X$ is used, and $\phi_o :\; w^o = w$. Also in this case, the new coordinate w^d equals the center of gravity of the original system.

Now the second prolongation of f^{-1} allows us to transform the Lagrangian l from equation (5). Consequently the following functional

$$\bar{\mathcal{L}} = \int_{t_1}^{t_2}\int_0^L \left(j^2 f^{-1}\right)^* (l)\;\;\mathrm{d}X \wedge \mathrm{d}t =$$

$$\int_{t_1}^{t_2}\left(\frac{m_{sys}}{2}\; {w^d_{[10]}}^2 + F^1 w^d\right)\mathrm{d}t +$$

$$\frac{1}{2}\int_{t_1}^{t_2}\int_0^L\left(\frac{m_2 m_1}{2L m_{sys}}\;\left(w^o_{[10]}\Big|_L\right)^2 + \frac{m_1 \rho A}{2m_{sys}}\;\left(w^o_{[10]}\right)^2\right.$$

$$+\frac{m_2 \rho A}{2m_{sys}}\;\left(w^o_{[10]}\Big|_L - w^o_{[10]}\right)^2 - E\,J_y\;\left(w^o_{[02]}\right)^2$$

$$+\frac{\rho^2 A^2}{2m_{sys}}\left(L\left(w^o_{[10]}\right)^2 - \frac{1}{L}\left(\int_0^L w^o_{[10]}\mathrm{d}X\right)^2\right)$$

$$-\frac{F^1}{L\; m_{sys}} m_2\; w^o|_L + \frac{F^2}{L}\; w^o|_L$$

$$\left. + F_d w^o - \frac{F^1 \rho A\, w^o}{m_{sys}}\right)\;\;\mathrm{d}X \wedge \mathrm{d}t,$$

is generated. Thus the splitting of the motion in a rigid body motion and oscillations is shown.

4.3 Controller Design

In the introductory Mass-Spring-Mass example a control law, consisting of a classical PD component extended by an oscillation damping controller, was introduced. This procedure is now extended to the infinite dimensional case.

The stored energy function of the system is given by

$$H_{sys} = \frac{m_1}{2}\left(w^1_{[10]}\right)^2 + \frac{m_2}{2}\left(w^2_{[10]}\right)^2 + \frac{1}{2}\int_0^L \left(\rho\, A\left(w^1_{[10]} + w_{[10]}\right)^2 + E\,J_y\left(w_{[02]}\right)^2\right) \mathrm{d}X,$$

where $w^2 = w^1 + w|_L$ is used. As in the introductory example, the desired system is a rigid body system. Thus the energy stored in the desired system is given by $\bar{H}_d = \frac{m_{sys}}{2}\, {w^d_{[10]}}^2$. Furthermore the oscillation energy can be formulated as $H_o = \bar{H}_o \circ j^2\phi_o = H_{sys} - \bar{H}_d \circ j^1\phi_d$ and results in

$$\begin{aligned} H_o = &\frac{1}{2}\int_0^L E\,J_y\left(w_{[02]}\right)^2 \mathrm{d}X + \frac{m_2 m_1}{2m_{sys}}\left(w_{[10]}\big|_L\right)^2 \\ &+\frac{m_1\rho A}{2m_{sys}}\int_0^L \left(w_{[10]}\right)^2 \mathrm{d}X \\ &+\frac{m_2\rho A}{2m_{sys}}\int_0^L \left(w_{[10]}\big|_L - w_{[10]}\right)^2 \mathrm{d}X \\ &+\frac{\rho^2 A^2}{2m_{sys}}\left(L\int_0^L \left(w_{[10]}\right)^2 \mathrm{d}X - \left(\int_0^L w_{[10]}\mathrm{d}X\right)^2\right). \end{aligned}$$

By applying the Cauchy-Schwarz and triangle inequality it follows, that

$$L\int_0^L \left(w_{[10]}\right)^2 \mathrm{d}X \geq \left(\int_0^L w_{[10]}\,\mathrm{d}X\right)^2$$

is met and consequently it is shown, that H_o is smooth and bounded from below.

Oscillation System: If the time derivative of the positive semi-definite function H_o along the solution of the distributed parameter system is negative semi-definite and some further conditions are met, then it can be assumed, that the oscillations vanish.

With the boundary conditions $w_{[01]}\big|_{X=0} = 0$ and $w_{[02]}\big|_{X=L} = 0$ it follows, that the derivation of H_o can be stated in form of

$$\begin{aligned} \frac{\mathrm{d}H_o}{\mathrm{d}t} = &-d_{m_2}\left(w^2_{[10]} - w^1_{[10]}\right)^2 - \int_0^L d_{sys}\left(w_{[10]}\right)^2 dX \\ &+\frac{F^1}{m_{sys}}\left(m_2\left(w^1_{[10]} - w^2_{[10]}\right) - \int_0^L \rho A\, w_{[10]}\,\mathrm{d}X\right). \end{aligned}$$

One immediately realizes, that the function H_o is invariant under a solution of the corresponding autonomous, undamped system.

A choice of the input force

$$F^1 = F_o = -\frac{d_o}{m_{sys}}\left(m_2\left(w^1_{[10]} - w^2_{[10]}\right) - \int_0^L \rho A\, w_{[10]}\,\mathrm{d}X\right) \tag{6}$$

guarantees, that additional damping is injected into the oscillation motion. The application of the developed control law causes the vibrating structure to converge to the desired rigid body motion. The measurement of $\int_0^L \rho A\, w_{[10]}\,\mathrm{d}X$ can be realized by applying a piezoelectric film along the Bernoulli-Euler beam.

Original System: The derived control law (6) is now applied to the original system and does obviously not guarantee the stability of a desired position of the whole structure. Thus this control law must be extended in order to achieve asymptotic stability. In this case the modified storage function $H = \frac{c}{2}\left(w^1\right)^2 + H_{sys}$ will be used as Lyapunov function and we have to analyse its total time derivative

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} = &\, cw^1 w^1_{[10]} + w^1_{[10]}F^1 - d_{m_2}\left(w^2_{[10]} - w^1_{[10]}\right)^2 \\ &- \int_0^L d_{sys}\left(w_{[10]}\right)^2\ \mathrm{d}X. \end{aligned}$$

With the input force $F^1 = -c\ w^1 - d\ w^1_{[10]} + F_o$ the derivative results in

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} = &-\left(d + d_e\frac{m_2}{m_{sys}} + d_{m_2}\right) {w^1_{[10]}}^2 - d_{m_2}\, {w^2_{[10]}}^2 \\ &+\left(\frac{m_2\, d_e}{m_{sys}} + 2d_{m_2}\right) w^1_{[10]} w^2_{[10]} + \\ &\frac{d_e}{m_{sys}} w^1_{[10]}\int_0^L \rho A\, w_{[10]}\,\mathrm{d}X - \int_0^L d_{sys}\, {w_{[10]}}^2\ \mathrm{d}X. \end{aligned}$$

In order to obtain a negative semi-definite function $\frac{\mathrm{d}}{\mathrm{d}t}H$, we have to fulfill

$$\begin{aligned} d \geq &\, d_{m_2}\left(\frac{m_2\, d_e}{2m_{sys}d_{m_2}} + 1\right)^2 - d_e\frac{m_2}{m_{sys}} - d_{m_2} \\ &+\frac{1}{d_{sys}}\left(\frac{d_e\rho A}{2m_{sys}}\right)^2. \end{aligned}$$

The positive definiteness of the function H and the negative semi-definiteness of its time derivative $\dot{H}$ along the solution of the system is a necessary condition for stability. The derivation of a sufficient condition will be part of future investigations.

5. SIMULATION

In the following simulations we consider plants, where $m_1 \gg m_2$ is met. This condition implies a bad convergence of the oscillations as shown in Fig. 4 for a simple *PD* controller. In contrary to this result, the introduced oscillation rejection controller R_o causes the trajectories w^1 and w^2 depicted in Fig. 5. As mentioned before, the oscillation controller R_o with the output force F_o (6) does not provide the stabilization of a certain position of the plant. Thus Fig. 6 shows the results achieved with the combination of the *PD* controller and the oscillation controller R_o. The

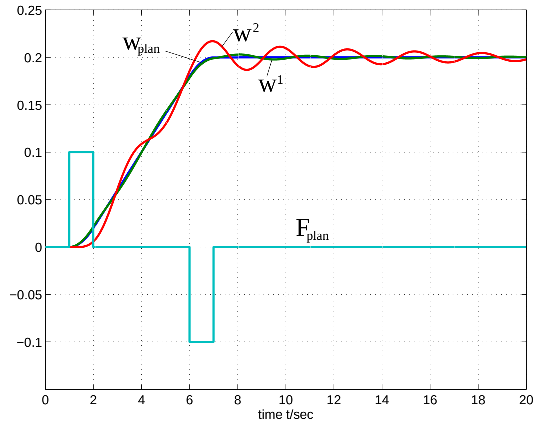

Fig. 4. PD controller without extension

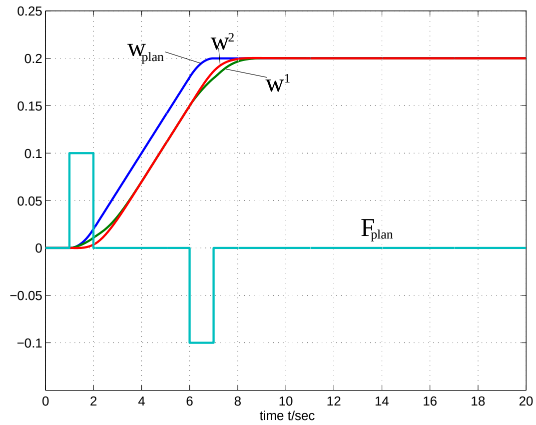

Fig. 5. The oscillation controller

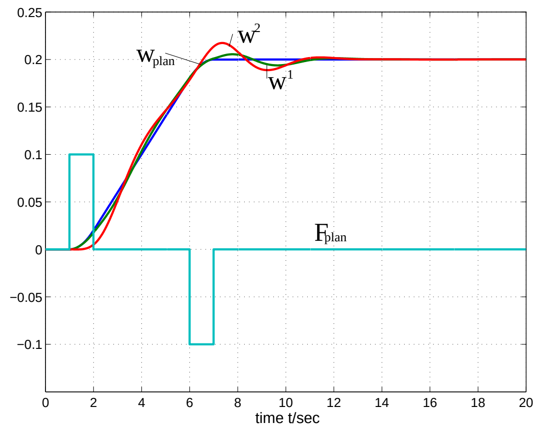

Fig. 6. PD controller with oscillation controller

final combination of PD and oscillation rejection controller R_o meets the requested stabilization of a desired position and fast fading oscillations.

6. CONCLUSIONS

This contribution presents a controller design method based on energy considerations. Thereby no approximations of partial differential equations by means of ordinary differential equations are necessary. The presented modelling leads to a system of partial and ordinary differential equations. The goal of the controller design is the asymptotic stabilization of the whole structure with respect to a desired reference position.
The control problem is solved by considering the stored energy of a desired system and the stored energy of the original elastic system. The analysis of the energy difference between the desired and actual motion yields a positive semi-definite function, if the position of the desired system is chosen as the center of gravity of the original system. The corresponding time derivative of the oscillation energy $\dot{H}_o$ supplies the collocated output to the input force F. Hence it is possible to define a control law, which extracts only the oscillation energy from the system. From physical observations one is able to assume, that this energy evolution is equivalent to the stabilization of the meant rigid body motion.
In a second step it is shown, that by the introduction of an extended control law, the original system can be stabilized towards a desired position. The used method represents only a necessary condition for stability in the infinite dimensional case. A rigorous proof of stability with respect to a certain norm of the closed loop Mass-Beam-Mass system will be a part of further investigations.
Taking the practical application of this control law into account, one immediately realizes the problem to determine the integral term by means of measurements. The application of a distributed piezoelectric sensor in order to overcome this problem is actually under investigation. This types of sensors provide intrinsically the required integration along the beam structure.
Finally some simulation results, where the impact of the controller on the motion of the system is presented, close this contribution.

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