SECOND-ORDER NECESSARY CONDITIONS FOR OPTIMAL IMPULSIVE CONTROL

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Abstract: Second-order conditions of optimality for an impulsive control problem are presented and derived from those for an appropriate abstract optimization problem with equality, inequality and set inclusion constraints. An important feature of these conditions is that they do not degenerate for abnormal control processes in spite of the absence of a priori normality assumptions on the data of the problem. Another salient feature concerns the fact that these conditions improve previous results by taking into account the second-order curvature effect of the constraints in the form of inclusion.

Keywords: Second-order necessary conditions, optimal impulsive control, nondegeneracy

1. INTRODUCTION

In this article, we present and discuss non degenerate second-order necessary conditions of optimality for an optimal impulsive control problem. By impulsive control system, it is meant one whose state variable is driven not only by conventional dynamics, which are absolutely continuous with respect to the Lebesgue measure, but also by measures which play a role of a control and give rise to a component of its evolution which is not absolutely continuous. This evolution may correspond to jumps in the state trajectory which are naturally considered in a number of applications such as aerospace navigation, (Lawden, 1963), resources management, finance, (Clark *et al.*, 1979; Dykhta and Samsonyuk, 2000), quantum electronics, (Dykhta and Samsonyuk, 2000; Miller and Rubinovitch, 2002), impact mechanics, (Brogliato, 1996), etc. There is already a vast literature (see (Miller and Rubinovitch, 2002; Dykhta and Samsonyuk, 2000; Vinter and Pereira, 1988; Pereira and Silva, 2000), to cite just a few references) addressing optimality conditions for this class of problems extending the body of results for conventional problems, (Vinter, 2000), to control systems with trajectories of bounded variation.

Here, we will consider the following optimal control problems with equality and inequality constraints on the endpoints of the state variable and constraints on the control measure.

$$
\begin{aligned}
(P)\ &\text{Minimize } e^0(x(t_0), x(t_1)) \\
&\text{subject to } dx(t) = f(t, x(t), u(t))dt \\
&\qquad\qquad + G(t)d\mu(t),\ t \in [t_0, t_1], \\
&\qquad e^1(x(t_0), x(t_1)) \leq 0,
\end{aligned}
$$

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$$e^2(x(t_0), x(t_1)) = 0,$$
$$d\mu \in \mathcal{M}.$$

Here, t_0, and t_1, with $t_0 < t_1$ are fixed. The functions $f : [t_0, t_1] \times R^n \times R^m \to R^n$, $G : [t_0, t_1] \to R^{n\times q}$, $e^i : R^n \times R^n \to R^{d(e_i)}$, $i = 0, 1, 2$, $d(e^0) = 1$ [3], are given and assumed to satisfy the following standard assumptions:

- The functions e^i, $i = 0, 1, 2$, are twice continuously differentiable.
- The function f is twice continuously differentiable w.r.t. (x, u) for all $t \in [t_0, t_1]$.
- The matrix function G is continuous.
- Functions f and its first and second order derivatives are bounded on any bounded subset and measurable w.r.t. t.

The function $u \in L_\infty([t_0, t_1]; R^m)$ [4] is the conventional measurable control, and the impulsive control $d\mu$ is a q-dimensional Borel measure taking values in the cone $\mathcal{M}$, given by

$$\Big\{ d\mu \in C^{*q} : \forall \phi \in C^q \text{ s.t. } \phi(t) \in M^0, \forall t, \int_B \phi(t) d\mu(t) \geq 0, \ \forall \text{ Borel } B \subset [t_0, t_1] \Big\}.$$

Here, M^0 is the dual of a given convex, closed, pointed cone $M \subset R^q$.

These conditions improve those available in most of the current optimal control literature on second-order conditions for both impulsive and conventional optimal control problems (see, for example, (Ledzewicz and Schaettler, 1998; Dykhta, 1997), and references therein) in two ways.

One important feature of the optimality conditions presented here is the fact that they do not degenerate even for abnormal control processes, (Arutyunov, 2000; Arutyunov, 2002). Second-order conditions previously obtained for optimal control do degenerate (see (Dykhta, 1997) for impulsive control, and (Ledzewicz and Schaettler, 1998) for conventional control problems). Furthermore, in contrast with publications concerning nondegenerate second-order necessary conditions of optimality (see, for example (Ben-Tal and Zowe, 1982; Kawasaki, 1988; Cominetti, 1990)), we do not require any a priori normality assumptions. The basic idea consists in using additional information from second-order conditions in order to select a subset of the set of multipliers satisfying the local necessary conditions of optimality in such a way that the conditions remain informative even for abnormal control processes.

[3] $d(f)$ denotes the dimension of the range space of the function f.

[4] Unless otherwise stated, all function spaces will be referred to the interval $[t_0, t_1]$ and its notation simplified to L^m_∞. Other examples are C^n, C^{*n}, BV^n, ...

Another pertinent feature which strengths the conditions derived by the authors in (Arutyunov *et al.*, 2003), consists in the fact that, now, our second-order conditions take into account the second-order effect of the curvature of the set in the measure control inclusion constraints. Such an effect has been noticed in a few publications in the optimization literature, (Ben-Tal and Zowe, 1982; Kawasaki, 1988; Kawasaki, 1991; Cominetti and Penot, 1997)) and transposed here to the optimal control context.

The approach to the proof of the result presented here consists in writing the above optimal control problem (P) as an abstract optimization problem (A) with equality, inequality constraints and constraints in the form of inclusion in a set, defined in appropriate spaces. Then, the necessary conditions of optimality derived in (Arutyunov and Pereira, to appear in 2005) are applied to (A) and decoded in terms of the data of (P).

This article is organized as follows: In the next section, we present a set of preliminary definitions having in mind not only to precise the problem under consideration, but also specify the concepts required for the statement of the main result. In section 3, we provide additional concepts, state the main result, and present a brief outline the proof which essentially is based on an abstract extremum principle. These abstract second-order conditions are stated in section 4. For this, we start by formulating the abstract optimization problem and corresponding second-order conditions of optimality presented and derived in (Arutyunov and Pereira, to appear in 2005), and compare these conditions with related results obtained previously.

2. PRELIMINARY DEFINITIONS

We start this section by presenting a number of definitions which will be essential to state our main result.

We say that a pair (u, μ) is an *admissible control* if $u \in L^m_\infty$ and $\mu \in BV^q$ such that $d\mu \in \mathcal{M}$.

An *admissible control process* is a triple (x_0, u, μ), where (u, μ) is an admissible control and the corresponding trajectory satisfies the given endpoint constraints.

For any given triple (x_0, u, μ), where x_0 is an initial value of the state variable, i.e., and (u, μ) is an admissible control, the *trajectory* associated with (1) is the unique right continuous function $x(\cdot)$ of bounded variation on $(t_0, t_1]$ satisfying $x(t_0) = x_0$ and, for $t > 0$,

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), u(s)) ds + \int_{[t_0, t]} G(s) d\mu(s) \quad (1)$$

We say that the admissible process (x_0^*, u^*, μ^*) is a *local minimizer* of the problem (P) if there exists $\varepsilon > 0$ and, for any finite-dimensional subspace $\mathbf{R} \subset L_\infty^m$, $\varepsilon_\mathbf{R} > 0$ such that the process (x_0^*, u^*, μ^*) yields a minimum to the problem (P) with the additional constraints $\|u - u^*\|_{L_\infty^m} < \varepsilon_\mathbf{R}$, $u \in \mathbf{R}$, $\|x_0 - x_0^*\| < \varepsilon$, and $\|d\mu - d\mu^*\|_{C^{*q}} < \varepsilon$.

Moreover, since (x_0^*, u^*, μ^*) is investigated for local minimum only, then there is no loss of generality in assuming that all endpoint inequality constraints are active at the optimal trajectory, i.e., $e^1(x^*(t_0), x^*(t_1)) = 0$.

In order to state the second-order conditions, we introduce next a local maximum principle, and define a critical cone, and an appropriate quadratic form.

Let $\psi \in R^n$, $\lambda = (\lambda^0, \lambda^1, \lambda^2) \in R^1 \times R^{d(e^1)} \times R^{d(e^2)}$.

In order to shorten the notation, we denote by x_i the point $x(t_i)$ [5], for $i = 0, 1$. When some arguments of a given function are missing, this means that the considered function is evaluated along the given reference process. The dot over the function label means the total derivative with respect to time. An argument variable appearing in sub index means that a partial derivative is being considered.

The Pontryagin function, H, and the endpoint Lagrangian, l^λ, are, respectively, given by:

$$H(t, x, \psi, u) = \langle \psi, f(t, x, u) \rangle,$$
$$l^\lambda(x_0, x_1) = \lambda^0 e^0(x_0, x_1) + \sum_{i=1}^{2} \langle \lambda^i, e^i(x_0, x_1) \rangle.$$

The *local maximum principle* is satisfied by the process (x_0^*, u^*, μ^*) if there exists $\lambda \neq 0$, such that

$$\lambda^0 \geq 0, \quad \lambda^1 \geq 0, \quad \langle \lambda^1, e^1(x_0^*, x_1^*) \rangle = 0 \quad (2)$$

and a vector function $\psi \in BV^n$, solution to the adjoint system

$$\begin{cases} -\dot{\psi}(t) = H_x(t, x^*(t), \psi(t), u^*(t)) \\ (-\psi(t_1), \psi(t_0)) = l^\lambda_{(x_0, x_1)}(x_0^*, x_1^*) \end{cases} \quad (3)$$

which satisfy the following conditions:

$$H_u(t, x^*(t), \psi(t), u^*(t)) = 0 \quad \mathcal{L}\text{-a.e.},$$
$$\langle \psi(t), G(t)v \rangle \leq 0 \quad \forall (t, v) \in [t_0, t_1] \times M,$$
$$\langle \psi(t), G(t)\omega^*(t) \rangle = 0 \quad d\mu^*\text{-a.e.},$$

where $\omega^*(t) = \frac{d\mu^*(t)}{d|\mu^*(t)|}$ is the Radon Nicodym derivative of the measure $d\mu^*$ with respect to its total variation measure. Remark that any adjoint trajectory $\psi(t)$ and the function $H(\cdot)$ depend on λ due to the transversality condition in (3).

[5] This notation extends to the optimal reference trajectory, x^*, and to the variation δx.

Let $\Lambda = \Lambda(x_0^*, u^*, \mu^*)$, denote the set of all Lagrange multipliers λ satisfying the local maximum principle and $\|\lambda\| = 1$. Then (see (Arutyunov, 2000; Arutyunov *et al.*, 2003)), we have the following result:

Theorem 1. (First order necessary conditions of optimality). If (x_0^*, u^*, μ^*) is a local minimum for problem (P), then $\Lambda \neq \emptyset$.

A variation $(\delta x_0, \delta u, \delta \mu) \in R^n \times L_\infty^m \times C^{*q}$ is called *critical* if the corresponding state trajectory variation, $\delta x \in BV^n$, satisfies the following conditions:

$$\begin{cases} e^i_{(x_0, x_1)}(x_0^*, x_1^*)(\delta x_0, \delta x_1)^T \leq 0, \quad i = 0, 1, \\ e^2_{(x_0, x_1)}(x_0^*, x_1^*)(\delta x_0, \delta x_1)^T = 0 \end{cases} \quad (4)$$

$$d(\delta x)(t) = [f_x(t)\delta x(t) + f_u(t)\delta u(t)]dt + G(t)d(\delta\mu), \quad (5)$$

with $d(\delta\mu) \in T_\mathcal{M}(d\mu^*)$.

Denote by $\mathcal{K}_{cr}$, the cone of all critical variations.

For each $\lambda \in \Lambda(x_0^*, u^*, \mu^*)$, define the *quadratic form* $\Omega_\lambda : R^n \times L_\infty^m \times \mathcal{M} \to R^1$, by

$$\Omega_\lambda(\xi, \delta u, \delta\mu) = -\int_{t_0}^{t_1} \frac{\partial^2 H(t)}{\partial(x, u)^2}[(\delta x(t), \delta u(t))]^2 dt + \frac{\partial^2 l^\lambda}{\partial(x_0, x_1)^2}(x_0^*, x_1^*)[(\delta x_0, \delta x_1)]^2. \quad (6)$$

Here, δx is the solution to the above variational equation corresponding to δu and $\delta\mu \in T_\mathcal{K}(\mu^*)$ with $\delta x_0 = \xi$, and $Q[a]^2$ denotes the quadratic form $a^T Q a$.

3. THE MAIN RESULT

The statement of our main result requires some additional definitions which enable the specification of a subset of $\Lambda_a(x_0^*, u^*, \mu^*) \subseteq \Lambda(x_0^*, u^*, \mu^*)$ ensuring that the optimality conditions remain informative even for abnormal control processes.

Denote by $\mathcal{K}_\pi \subset R^n \times L_\infty^m \times L_\infty^q \times R^q$ the set of the triples $(\xi, \delta u, \delta w, h)$, where δx is the solution to the following modification of (5),

$$\begin{cases} \dot{\delta x}(t) = f_x(t)\delta x(t) + f_u(t)\delta u(t) - G(t)\pi\delta w, \\ \qquad\qquad t \notin \text{supp}\{\mu_s^*\} \\ d(\delta w) \in T_\mathcal{M}(dw^*) \\ \delta x_0 = \xi, \ \delta u \in L_\infty^m, \end{cases} \quad (7)$$

being w^* and δw, functions of bounded variation associated with measures $d\mu^*$ and $d(\delta\mu)$, respectively, $\text{supp}\{\mu_s^*\}$ the support of the singular component of the measure μ^*, and π the matrix of orthogonal projection from R^q into $M \cap (-M)$, and that satisfy

$$e_{(x_0,x_1)}(x_0^*, x_1^*)(\delta x_0, \delta x_1)^T + e_{x_1}(x_0^*, x_1^*)G(t_1)\pi h = 0$$

where $e = (e_1, e_2)^T$, and $h = \delta w(t_1)$.

Define Ω_a^λ on $R^n \times L_\infty^m \times L_\infty^q \times R^q$ from (6) by formally replacing $\delta w(t_1)$ by h [6] and define the linear operator $\mathcal{A} : \mathcal{K}_\pi \to R^{d(e_1)+d(e_2)}$, by

$$\mathcal{A}(\delta x_0, \delta u, \delta w, h) := e_{(x_0,x_1)}(x_0^*, x_1^*)(\delta x_0, \delta x_1)^T + e_{x_1}(x_0^*, x_1^*)G(t_1)\pi h$$

where $(\delta x, \delta u, \delta w)$ satisfies (7).
Let $d = \mathrm{codim}(Im\mathcal{A})$ [7] , and let $\Lambda_a(x_0^*, u^*, w^*)$ be the set of multipliers $\lambda \in \Lambda(x_0^*, u^*, w^*)$ such the index [8] of the quadratic form Ω_a^λ on $\mathcal{K}_\pi$ is not greater than d.

Let $O^2_{\mathcal{M}}(\mu^*, \delta\mu)$ denote the outer second-order tangent cone to the set $\mathcal{M}$ at the point μ^* along the direction $\delta\mu$. This is an instance of the definition of the outer second-order tangent cone given in the next section for the case in which $C = \mathcal{M}$, $x^0 = \mu^*$, $h = \delta\mu$.

Theorem 2. Let an admissible control process (x_0^*, u^*, μ^*) be a solution to the problem (P).

Then, the set $\Lambda_a(x_0^*, u^*, \mu^*)$ is nonempty and, for any $(\xi, \delta u, \delta\mu) \in \mathcal{K}_{cr}$, and any convex set $\mathcal{T}(\delta\mu) \subseteq O^2_{\mathcal{M}}(\mu^*, \delta\mu)$ we have

$$\max_{\substack{\lambda \in \Lambda_a \\ |\lambda| = 1}} \left\{ \Omega_\lambda(\xi, \delta u, \delta\mu) - \sigma\Big(q; \mathcal{T}(\delta\mu)\Big) \right\} \geq 0, \quad (8)$$

where $q : [t_0, t_1] \to R^q$ and $\sigma\Big(q; \mathcal{T}(\delta\mu)\Big)$ are defined by, respectively, $q(t) = -G^T(t)\psi(t)$, and

$$\sigma\Big(q; \mathcal{T}(\delta\mu)\Big) = \sup_{\nu \in \mathcal{T}(\delta\mu)} \left\{ \int_{[t_0,t_1]} q^T(t) d\nu(t) \right\}.$$

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[6] Notice that, now, δx has a different meaning. The second term in (6) takes the form $l^\lambda_{(x_0,x_1)^2}(x_0^*, x_1^*)[(\delta x_0, \delta x_1)]^2 + 2l^\lambda_{x_0 x_1}(x_0^*, x_1^*)[(x_0, G(t_1)\delta w_1)] + l^\lambda_{x_1^2}(x_0^*, x_1^*)[G(t_1)\delta w_1]^2$.

[7] It can be shown that d is the dimension of the kernel of the block matrix operator $[A|B|G(t_1)\pi]^T : R^{d(e)} \to R^{n+d(e)+q}$, where $d(e) = d(e_1) + d(e_2)$,

$$A = e_{x_0}(x_0^*, x_1^*)^T + \Phi(t_1)e_{x_1}(x_0^*, x_1^*)^T,$$

$$B = e_{x_1}(x_0^*, x_1^*)\int_{t_0}^{t_1} \Psi(t_1, t)\Gamma(t)\times\Gamma(t)^T\Psi^{-1}(t_1, t)^T dt e_{x_1}(x_0^*, x_1^*)^T.$$

Here, $\Psi(t, s) = \Phi(t)\Phi^{-1}(s)$, Φ is the solution to the system $\dot{\Phi}(t) = f_x(x^*(t), u^*(t), t)\Phi(t)$, $[t_0, t_1]$-a.e., with $\Phi(t_0) = I$, $\Gamma(t) = [f_u(t)|G(t)\pi]$, and A^T denotes the transpose of A.

[8] The index of a quadratic form on a given subspace V is the dimension of the subspace of V of maximum dimension where the quadratic form is negative definite, (Arutyunov, 2000).

Remark. In the second-order necessary conditions obtained in (Arutyunov *et al.*, 2003), the term $-\sigma\Big(q; \mathcal{T}(\delta\mu)\Big)$ is absent in the inequality corresponding to (8) and the variations $\delta\mu$ are considered only in the set $\mathcal{M} + \{\mu^*\}$ which is a subset of the set $T_{\mathcal{M}}(\mu^*)$ considered here. Therefore, the conditions in (Arutyunov *et al.*, 2003) can not take into account the second-order curvature effect of the set inclusion constraints. This is the most significant improvement of the above result.

The proof of this result consists in casting the considered optimal control problem (P) in a abstract minimization problem (Q) and apply the second-order necessary conditions of optimality proved in (Arutyunov and Pereira, to appear in 2005). Then, the resulting conditions are decoded in terms of the data of (P).

First, let us remark that with a change of state variable (see (Clarke, 1983)), it is not difficult to define an optimal control problem equivalent to (P) whose cost functional depends only on the state variable at the initial time.

Let $\bar{x} = (x_0, u(\cdot), \mu)$, $X = R^n \times L_\infty^m \times C^{*q}$, and $C = R^n \times L_\infty^m \times \mathcal{M}$. Then, by defining $f : X \to R^1$, $F_1 : X \to R^{k_1}$, and $F_2 : X \to R^{k_2}$ (here, $k_1 = d(e^1)$, and $k_2 = d(e^2)$), respectively, by $f(\bar{x}) = e^0(x_0, x(t_1))$, $F_1(\bar{x}) := e^1(x_0, x(t_1))$, and $F_2(\bar{x}) := e^2(x_0, x(t_1))$, problem (P) becomes

$$\begin{array}{lrl} (Q) & \text{Minimize} & f(\bar{x}) \\ & \text{subject to} & F_1(\bar{x}) \leq 0 \\ & & F_2(\bar{x}) = 0 \\ & & \bar{x} \in C \subset X. \end{array}$$

This is an abstract optimization problem for which second-order necessary conditions of optimality are proved in (Arutyunov and Pereira, to appear in 2005). For convenience, we state this result in the next section. By following arguments similar to the ones in (Arutyunov *et al.*, 2003) it is straightforward to decode these conditions in terms of the data of (P) in order to obtain Theorem 2.

4. AN EXTREMUM PRINCIPLE

Let X be agiven vector space, and $x^* \in X$ be a solution to problem (Q) in the previous section. From now on, we will drop the bar in $\bar{x}$. We assume that the set $C \subseteq X$ is closed, the mappings $F_i : X \to R^{k_i}$, $i = 1, 2$, and the function $f : X \to R^1$ are twice continuously differentiable in a neighborhood of x^* with respect to the finite topology τ. This means that the restrictions of f, F_1 and F_2 to an arbitrary finite dimensional subspace $S \in \mathcal{S}$ such that $x^* \in S$ are twice continuously differentiable in a neighborhood (that depends on

S) of x^*. For a more detailed discussion on the assumptions, see (Arutyunov, 2000; Arutyunov and Pereira, to appear in 2005).

The critical cone, $\mathcal{K}(x^*)$, of problem (Q) at the point x^* is defined by

$$\Big\{h \in T_C(x^*) : \langle f_x(x^*), h\rangle \leq 0,\ F_{2x}(x^*)h = 0,\\ \langle F_{1,jx}(x^*), h\rangle \leq 0\ \forall j \text{ s. t. } F_{1,j}(x^*) = 0\Big\}.$$

Here $T_C(x^*) = \bigcup_{S\in\mathcal{S}} T_{C\cap S}(x^*)$, where $S \in \mathcal{S}$ is an arbitrary finite dimensional linear subspace of X such that $x^* \in S$, denotes the tangent cone to C at x^*, being $T_{C\cap S}(x^*)$ the contingent (Bouligand) cone, given by the set

$$\Big\{x \in h\ :\ \exists\{\varepsilon_n\} \downarrow 0,\ dist_S(x^*+\varepsilon h, C) = o(\varepsilon_n)\Big\}.$$

Here, $dist_S(c, C) = \inf_{\xi\in C\cap S}\{\|\xi - c\|\}$.

We will need the outer second-order tangent cone to C at x^* in the direction h, $O^2_C(x^*, h)$ which, in the context of this problem, is defined as $O^2_C(x^*, h) = \bigcup_{S\in\mathcal{S}} O^2_{C\cap S}(x^*, h)$, for some $h \in S$ with S as above, and

$$O^2_{C\cap S}(x^*, h) = \Big\{w : \exists\{\varepsilon_n\} \downarrow 0 \text{ s.t.}\\ dist_S(x^* + \varepsilon_n h + \frac{1}{2}\varepsilon_n^2 w, C) = o(\varepsilon^2)\Big\}.$$

Note that if $h \in T_C(x^*)$ then $O^2_C(x^*, h) \neq \emptyset$, (Bonnans *et al.*, 1999). See (Kawasaki, 1988; Cominetti, 1990; Kawasaki, 1991; Cominetti and Penot, 1997; Arutyunov and Pereira, to appear in 2005) for further details concerning the definition, properties, and computation.

Let $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ with $\lambda_0 \in R^1$, $\lambda_1 \in R^{k_1}$, and $\lambda_2 \in R^{k_2}$, and denote by $\mathcal{L}$ the generalized Lagrangian defined by

$$\mathcal{L}(x, \lambda) = \lambda_0 f(x) + \langle\lambda_1, F_1(x)\rangle + \langle\lambda_2, F_2(x)\rangle.$$

The set of Lagrange of multipliers associated with the point x^* according to the Lagrange multiplier rule, i.e., such that

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial x}(x^*, \lambda) \in -N_C(x^*) \\ \lambda_0 \geq 0, \lambda_1 \geq 0,\ \langle\lambda_1, F_1(x^*)\rangle = 0,\ |\lambda| = 1. \end{cases} \tag{9}$$

is denoted by $\Lambda = \Lambda(x^*)$.

Here, $N_C(x^*)$ is the normal cone in the sense of Mordukhovich, (Mordukhovich, 1993), which can be defined as $N_C(x^*) = \bigcup_{S\in\mathcal{S}} N^S_C(x^*)$ where S is as above,

$$N^S_C(x^*) = \limsup_{\substack{x\in S\\ x\to x^*}} \bigcup_{r>0}\{r[x - W_{S\cap C}(x)]\}$$

where $W_{S\cap C}(x) = \inf_{\xi\in C\cap S}\{\|\xi - x\|\}$.

By *invariant linear subspace* (*ILS*) relatively to C at x, it is meant a linear subspace of X, denoted by $\mathcal{I}_C(x)$, such that $x + \mathcal{I}_C(x) \subseteq C$, $\forall x \in C$.

For a given closed set C, an *ILS* is not, in general, unique since any linear subspace of an *ILS* also is an invariant linear subspace. For any $x \in C$, put $\mathcal{I}_C(x) = \bigcap_{r\neq 0} r[C - x]$, being the intersection taken over all reals $r \in R^1$, $r \neq 0$.

If the set C is convex, then, $\forall x \in C$, $\mathcal{I}_C(x)$ is the maximal *ILS* relatively C and it does not depend on x.

Note also that, if the set C is a convex cone, then $\mathcal{I}_C = C \cap (-C)$ is the maximal *ILS*.

In what follows, we assume, for convenience, that $F_1(x^*) = 0$. This can always be achieved by omitting the nonactive components of the inequality constraints.

Take any linear subspace $S \subseteq X$ and consider the set of all Lagrange multipliers $\lambda \in \Lambda$ for which there exists a linear subspace $\Pi \subseteq S$ (depending on λ) such that

- $\text{codim}_S\Pi \leq k$,
- $\Pi \subseteq \text{Ker}\ \dfrac{\partial F}{\partial x}(x^*)$,
- $\dfrac{\partial^2\mathcal{L}}{\partial x^2}(x^*, \lambda)[h, h] \geq 0, \quad \forall h \in \Pi.$

Here, $F = [F_1|F_2]^T$, codim_S denotes the codimension relative to the subspace S. We denote this set of Lagrange multipliers by $\Lambda_k(x^*, S)$. Each set $\Lambda_k(x^*, S)$ is obviously compact (but might be empty).

For a given set $D \subseteq X$, we denote by $\sigma(\cdot, D)$ its support function, i.e., $\sigma(\chi, D) = \sup_{x\in D}\langle\chi, x\rangle$, for some $\chi \in X^*$.

Theorem 3 (Theorem 2.1 in (Arutyunov and Pereira, to appear in 2005)). Let x^* be a point of local minimum with respect to the finite topology τ of problem (Q).

Then, for each *ILS* $\mathcal{I}_C(x^*)$, the set $\Lambda_k(x^*, \mathcal{I}_C(x^*))$ is nonempty, and, moreover, for each $h \in \mathcal{K}(x^*)$ and any convex set $\mathcal{T}(h) \subseteq O^2_C(x^*, h)$,

$$\max_{\lambda\in\Lambda_a}\left(\frac{\partial^2\mathcal{L}}{\partial x^2}(x^*, \lambda)[h, h]\right.\\ \left. -\sigma\left(-\frac{\partial\mathcal{L}}{\partial x}(x^*, \lambda), \mathcal{T}(h)\right)\right) \geq 0.$$

Here, $\Lambda_a = \text{conv}\Lambda_k(x^*, \mathcal{I}_C(x^*))$ and $\text{conv}A$ denotes the convex hull of the set A .

Remark. The problem

$$\text{Minimize } f(x) \text{ subject to } F(x) \in C$$

has been widely investigated (see (Bonnans and Shapiro, 2000) and references therein) under the assumptions of convexity of the set C and Robinson's constraint qualification for arbitrary Banach space Y. In this context, they obtained the following optimality condition:

$$\max_{\lambda \in \Lambda} \left(\frac{\partial^2 \mathcal{L}}{\partial x^2}(x^*, \lambda)[h, h] - \sigma\left(\tilde{\lambda}, \mathcal{T}(h)\right) \right) \geq 0$$

for all h in the critical cone for this problem.

In (Arutyunov and Pereira, to appear in 2005), the version of theorem 3 above for this problem (theorem 4.1 in (Arutyunov and Pereira, to appear in 2005)) shows that, for the case of finite dimensional Y, the second-order necessary conditions of optimality from, (Ben-Tal and Zowe, 1982; Cominetti, 1990; Bonnans and Shapiro, 2000) holds for generalized Lagrangian without the Robinson's constraint qualification (which amounts to a normality assumption) and without the convexity assumption on C.

Note also that our condition in theorem 3 is, in general, stronger than the corresponding one in (Ben-Tal and Zowe, 1982; Cominetti, 1990; Bonnans and Shapiro, 2000) because $\Lambda_a^Q \subseteq conv\Lambda^Q$.

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