CONSTRAINED INPUT-TO-STATE STABILITY OF NONLINEAR SYSTEMS

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Abstract: Input-to-state stability (ISS), integral-ISS (iISS) and their related notions of input/output stability and detectability have become a useful framework for nonlinear feedback analysis and design. In this paper, a new notion, that of constrained input-to-state stability (cISS), is presented and several characterizations of cISS are obtained. cISS, unlike ISS or iISS, is not confined to forward complete systems and generalizes the small-signal $\mathcal{L}_\infty$ stability. Moreover, for a class of nonlinear systems, constrained input-to-state stabilizability implies the solvability of an inverse optimal problem. The paper shows that cISS is a natural concept to improve the ISS framework.

Keywords: Stability analysis, nonlinear systems, nonlinear control systems, Lyapunov methods, optimal control, inverse problems.

1. INTRODUCTION

As remarked in (Sontag, 1998), input-to-state stability (ISS) can be seen as a nonlinear generalization of $\mathcal{L}_\infty$ stability and reflects the qualitative property of small overshot when the disturbance is uniformly bounded. In contrast with ISS, integral ISS (iISS) (Angeli *et al.*, 2000) is a weaker concept and reflects the qualitative property of small overshot when the disturbance has finite energy. ISS, iISS and their related notions of input/output stability and detectability have become a very useful framework—ISS framework (Sontag, 2000) for nonlinear feedback analysis and design. However, the current ISS framework has some deficiency. As an example, consider the following one-dimensional system

$$\dot{x} = -x + xu. \tag{1}$$

From (Sontag, 1998), system (1) is not ISS but iISS, since the constant input u with $u > 1$ produces unbounded trajectories. But it's easy to see that any constant input u with $|u| < 1$, which obviously has infinite energy, produces bounded trajectories for any initial state. Thus the stability property of system (1) can't be explained precisely by iISS. Another more interesting example is

$$\dot{x} = -x^3 + zx^3,\ \dot{z} = u \tag{2}$$

which is used to motivate *peaking phenomenon* in (Sepulchre *et al.*, 1997). Although the first subsystem of (2) is not forward complete, it actually exhibits the ISS-like property when $|z| < 1$.

Thus such property is named as constrained input-to-state stability (cISS), which reflects the qualitative property of small overshot when the magnitude of disturbances is constrained below a threshold. cISS, unlike ISS or iISS, is not confined to forward complete systems. This property can also be seen as a generalization of small-signal $\mathcal{L}_\infty$ stability (Khalil, 2002). Clearly, cISS is a natural concept to improve the ISS framework. Moreover, cISS is a property with broad applicability. Especially, it is shown that the PD-controlled manipulator used in (Angeli *et al.*, 2000) to motivate iISS, is also cISS, thus it can handle some bounded disturbance with constrained magnitude, which can't be achieved under the assumption of iISS. In addition, for a class of nonlinear systems, constrained input-to-state stabilizability, like ISS, implies the solvability of an inverse optimal problem. Finally, it's remarked that there are some notions similar with cISS, such as ISS with restriction ((ε, δ) ISS) (Teel and Praly, 1995). cISS can be seen as a special case of (ε, δ) ISS with $\varepsilon = +\infty$. However, there doesn't exist Lyapunov characterizations of (ε, δ) ISS, thus the study about (ε, δ) ISS only concentrates on the small gain theorem by using the input-output formulation. In contrast, we obtain several necessary and sufficient Lyapunov characterizations of cISS, and study the cISS stabilization and inverse optimality, which are beyond the existing results about (ε, δ) ISS and show that cISS is indeed a meaningful property.

The rest of the paper is organized as follows: Section 2 introduces the notion of cISS and studies the Lyapunov like characterization; Section 3 provides the notions related with cISS and a sufficient condition for cISS property; the inverse optimal problem is discussed in Section 4; finally, Section 5 summarizes the conclusion of this paper.

2. NOTION AND CHARACTERIZATIONS

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If in addition α is unbounded, then it is said to be of class $\mathcal{K}_\infty$. A continuous function $\gamma : [0, c) \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_C$ if it is strictly increasing and satisfies $\gamma(0) = 0$ and $\gamma(s)$ increases to $+\infty$ as $s \to c$, where c is a positive constant. Obviously, the inverse function of any $\mathcal{K} \backslash \mathcal{K}_\infty$ function belongs to class $\mathcal{K}_C$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{3}$$

where $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz, $f(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$ and $\boldsymbol{u} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is of measurable locally essentially bounded functions. The set of all such functions, endowed with the (essential) supremum norm $\|\boldsymbol{u}\| = \sup\{|\boldsymbol{u}(t)|, t \geq 0\} < +\infty$, is denoted by $\mathcal{L}_\infty^m$($| \cdot |$ denotes the standard Euclidean norm). Let $\boldsymbol{x}(t, \boldsymbol{\xi}, \boldsymbol{u})$ denote the solution at time t of (3) with $\boldsymbol{x}(0) = \boldsymbol{\xi}$ and $\boldsymbol{u}$. This is defined on some maximal interval $(T^-(\boldsymbol{\xi}, \boldsymbol{u}), T^+(\boldsymbol{\xi}, \boldsymbol{u}))$ with $-\infty \leq T^-(\boldsymbol{\xi}, \boldsymbol{u}) < 0 < T^+(\boldsymbol{\xi}, \boldsymbol{u}) \leq +\infty$. System (3) is said to be *forward complete* if $T^+(\boldsymbol{\xi}, \boldsymbol{u}) = +\infty$ for all $\boldsymbol{\xi}$ and $\boldsymbol{u}$.

Definition 1. System (3) is *constrained input-to-state stable* (cISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_C$ such that, for all $\boldsymbol{\xi} \in \mathbb{R}^n$ and all $\boldsymbol{u} \in \mathcal{L}_\infty^m$ with $\|\boldsymbol{u}\| < c$, it holds that

$$|\boldsymbol{x}(t, \boldsymbol{\xi}, \boldsymbol{u})| \leq \beta(|\boldsymbol{\xi}|, t) + \gamma(\|\boldsymbol{u}\|). \tag{4}$$

Definition 2. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a cISS-Lyapunov function for system (3) if there exist $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$ and $\chi : \mathbb{R}_{\geq 0} \to [0, c) \in \mathcal{K} \backslash \mathcal{K}_\infty$, such that for all $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\boldsymbol{\mu} \in \mathbb{R}^m$, it holds that

$$\alpha_1(|\boldsymbol{\xi}|) \leq V(\boldsymbol{\xi}) \leq \alpha_2(|\boldsymbol{\xi}|) \tag{5}$$

$$|\boldsymbol{\mu}| \leq \chi(|\boldsymbol{\xi}|) \Rightarrow DV(\boldsymbol{\xi})\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\mu}) \leq -\alpha(|\boldsymbol{\xi}|) \tag{6}$$

Remark 2.1. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a cISS-Lyapunov function for system (3) if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $H : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_C$, such that for all $\boldsymbol{\xi}, \boldsymbol{\mu}$, (5) and the following inequality hold

$$DV(\boldsymbol{\xi})\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\mu}) \leq -\alpha_3(|\boldsymbol{\xi}|) + H(|\boldsymbol{\mu}|) \tag{7}$$

This provides a dissipation type characterization for the cISS property. Clearly (7) implies (6). Suppose now that (6) holds. Define $H(s) = \max\{0, \bar{H}(|s|)\}$, where $\bar{H}(|s|) = \max\{DV(\boldsymbol{\xi})\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\mu}) + \alpha(|\boldsymbol{\xi}|) : |\boldsymbol{\mu}| \leq s, \chi(|\boldsymbol{\xi}|) \leq s\}$. Then H is continuous, $H(0) = 0$, and $H(s) \to \infty$ as $s \to c$. Thus one can assume that $H \in \mathcal{K}_C$. Note then that (7) holds with α_3 in replace of α, because $H(s) \geq \sup_{|\boldsymbol{\mu}|=s}\{DV(\boldsymbol{\xi})\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\mu}) + \alpha(|\boldsymbol{\xi}|)\}$.

The following result establishes that the existence of a smooth cISS-Lyapunov function is necessary as well as sufficient for the system to be cISS.

Theorem 1. System (3) is cISS if and only if it admits a smooth cISS-Lyapunov function.

Proof: Since the proof of sufficiency is almost same as Lemma 2.14 in (Sontag and Wang, 1995), only the necessity part is proved here. Firstly some concepts are introduced, then two lemmas are given to prove the theorem. Consider the system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{d}(t)\varphi(\boldsymbol{x}(t))) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{d}(t)) \tag{8}$$

where φ is any fixed smooth function, $\boldsymbol{d}(t) \in \mathcal{M}_D$ and $\mathcal{M}_D$ is the set of all measurable functions from $\mathbb{R}$ to $D = [-1, 1]^m$. Let $\boldsymbol{x}_\varphi(t, \boldsymbol{\xi}, \boldsymbol{d})$ denote the solution of (8) with $\boldsymbol{\xi}$ and $\boldsymbol{d}$. Then system (8) is *uniformly globally asymptotic stable* (UGAS) (Lin *et al.*, 1996) if it is forward complete and the following two properties hold: 1. uniform stability. There exists a $\mathcal{K}_\infty$ function $\delta(\cdot)$ such that for any $\varepsilon \geq 0$, $|\boldsymbol{x}_\varphi(t, \boldsymbol{\xi}, \boldsymbol{d})| \leq \varepsilon$ for all $\boldsymbol{d} \in \mathcal{M}_D$, whenever $|\boldsymbol{\xi}| \leq \delta(\varepsilon)$ and $t \geq 0$; 2. uniform attraction. For any $l, \varepsilon > 0$, there is a $T > 0$, such that for every $\boldsymbol{d}(t) \in \mathcal{M}_D$, $|\boldsymbol{x}_\varphi(t, \boldsymbol{\xi}, \boldsymbol{d})| < \varepsilon$ whenever $|\boldsymbol{\xi}| < l$ and $t \geq T$. System (3) is *constrained robustly stable* if there exist $\kappa \in \mathcal{K} \backslash \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, such that for every feedback law $\boldsymbol{k}(t, \boldsymbol{\xi})$ satisfies $|\boldsymbol{k}(t, \boldsymbol{\xi})| \leq \kappa(|\boldsymbol{\xi}|)$, it holds that $|\boldsymbol{x}(t)| \leq \beta(|\boldsymbol{\xi}|, t)$ for $t \geq 0$ and for all the solutions of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{k}(t, \boldsymbol{x}))$.

System (3) is *constrained weakly robustly stable* if there exists a smooth function φ satisfying $\psi(|\boldsymbol{\xi}|) \leq \varphi(\boldsymbol{\xi}) \leq \rho(|\boldsymbol{\xi}|)$ for some $\psi, \rho \in \mathcal{K}\backslash\mathcal{K}_\infty$, so that the corresponding system (8) is UGAS.

Lemma 2.1. If system (3) is cISS, then it is constrained weakly robustly stable.

*Proof:*Assume system (3) is cISS, and notice that it is easy to prove that system (3) is cISS if and only if there exist $\beta \in \mathcal{KL}$ and $\gamma : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_C$ such that, for $\boldsymbol{u} \in L_\infty^m$ with $\|\boldsymbol{u}\| < c$, $\boldsymbol{\xi} \in \mathbb{R}^n$ and $t \geq 0$, it holds that $|\boldsymbol{x}(t, \boldsymbol{\xi}, \boldsymbol{u})| \leq \max\{\beta(|\boldsymbol{\xi}|, t), \gamma(\|\boldsymbol{u}\|)\}$. Let $\bar{\beta}(s) = \beta(s, 0)$, then $\bar{\beta} \in \mathcal{K}$. Without loss of generality, one can always assume that $\bar{\beta}(s) > s$ for all $s > 0$, and thus $\bar{\beta} \in \mathcal{K}_\infty$ and $\bar{\beta}^{-1}(s) < s$ for all $s > 0$. Since $\gamma \in \mathcal{K}_C$, $\gamma^{-1} \in \mathcal{K}\backslash\mathcal{K}_\infty$. Now let ρ be a $\mathcal{K}\backslash\mathcal{K}_\infty$ function satisfying $\rho(s) < \gamma^{-1}(\frac{1}{4}\bar{\beta}^{-1}(s))$ for all $s > 0$. Note that for any $\rho \in \mathcal{K}\backslash\mathcal{K}_\infty$, there exist a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a $\mathcal{K}\backslash\mathcal{K}_\infty$ function ψ such that $\psi(|\boldsymbol{\xi}|) \leq \varphi(\boldsymbol{\xi}) \leq \rho(|\boldsymbol{\xi}|)$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$. Now for the fixed function φ and following the same lines as Lemma 2.12 in (Sontag and Wang, 1995), it is direct to show that system (8) is UGAS. Therefore, system (3) is constrained weakly robustly stable. □

Lemma 2.2. If system (3) is constrained weakly robustly stable, then there exists a smooth cISS-Lyapunov function for the system.

*Proof:*Following the proof of Lemma 2.1, it then comes from the Converse Lyapunov Theorem (Lin *et al.*, 1996) that there exists a smooth Lyapunov function V for system (8) such that (5) holds with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $DV(\boldsymbol{\xi})\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{d}\varphi(\boldsymbol{\xi})) \leq -\alpha(|\boldsymbol{\xi}|)$ holds for all $\boldsymbol{\xi} \in \mathbb{R}^n$,$|\boldsymbol{d}| \leq 1$ and $\alpha \in \mathcal{K}_\infty$. Then

$$DV(\boldsymbol{\xi})\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{v}) \leq -\alpha(|\boldsymbol{\xi}|), \tag{9}$$

when $|\boldsymbol{v}| \leq \varphi(\boldsymbol{\xi})$. Since $\psi(|\boldsymbol{\xi}|) \leq \varphi(\boldsymbol{\xi})$, (9) holds when $|\boldsymbol{v}| \leq \psi(|\boldsymbol{\xi}|)$. Let $\chi(s) = \psi(s)$, thus V is a cISS-Lyapunov function for system (3). □

Thus the proof of Theorem 1 is completed. ■

Remark 2.2. There actually exists another simpler but not independent way to prove the necessity by considering the fact: cISS of system (3) is equivalent to ISS of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{\gamma}}(\boldsymbol{v}))$, where $\bar{\boldsymbol{\gamma}} : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and satisfies $\gamma_1(|\boldsymbol{v}|) \leq |\bar{\boldsymbol{\gamma}}(\boldsymbol{v})| \leq \gamma_2(|\boldsymbol{v}|)$, $\gamma_1, \gamma_2 \in \mathcal{K}\backslash\mathcal{K}_\infty$ and $\gamma_2 = \gamma^{-1} \circ \varrho$, $\varrho \in \mathcal{K}_\infty$. However, such proof may conceal some useful property of cISS, such as system (3) is cISS if and only if it is constrained robustly stable (due to limitation of space, the detail, which will be studied further in another paper, is omitted).

Corollary 1. Consider system (3). If there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that for some α_1, α_2, $\delta \in \mathcal{K}_\infty$, $\alpha : \mathbb{R}_{\geq 0} \to [0, c) \in \mathcal{K}\backslash\mathcal{K}_\infty$, and for all $\boldsymbol{\xi}, \boldsymbol{\mu}$, (5) and the following hold

$$DV(\boldsymbol{\xi})\boldsymbol{f}(\boldsymbol{\xi}, \boldsymbol{\mu}) \leq -\alpha(|\boldsymbol{\xi}|) + \delta(|\boldsymbol{\mu}|) \tag{10}$$

Then (3) is cISS and forward complete.

*Proof:*By Theorem 1 and Corollary 2.12 in (Angeli and Sontag, 1999), (10) implies cISS and forward completeness. ■

Example 1. Consider system (1). Let $V(x) = \log(1 + x^2)$. Obviously, V is proper and $\dot{V} \leq -2x^2/(1+x^2)+2|u|$. Since $2x^2/(1+x^2) \in \mathcal{K}\backslash\mathcal{K}_\infty$, (1) is cISS and forward complete. As for the first subsystem of (2), let $V(x) = x^2$, then $|z| \leq \chi(|x|) \Rightarrow \dot{V} < 0$ for any $\chi : \mathbb{R}_{\geq 0} \to [0, 1) \in \mathcal{K}\backslash\mathcal{K}_\infty$, thus it is cISS but not forward complete.

3. RELATED NOTIONS AND FURTHER CHARACTERIZATION

In this section, consider system (3) with output

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \ \boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) \tag{11}$$

where $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^p$ is continuous and $\boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{0}$. System (11) is assumed to be forward complete. Let $\boldsymbol{y}(t, \boldsymbol{\xi}, \boldsymbol{u})$ denote the output of system (11) for any $\boldsymbol{\xi}, \boldsymbol{u}$, that is, $\boldsymbol{y}(t, \boldsymbol{\xi}, \boldsymbol{u}) = \boldsymbol{h}(\boldsymbol{x}(t, \boldsymbol{\xi}, \boldsymbol{u}))$.

In contrast with input-output-to-state stability (IOSS) (Sontag, 2000), system (11) is *constrained input-output-to-state stable* (cIOSS) if there exist $\beta \in \mathcal{KL}$ and $\gamma_1 : [0, c_1) \to \mathbb{R}_{\geq 0}, \gamma_2 : [0, c_2) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_C$ such that

$$|\boldsymbol{x}(t, \boldsymbol{\xi}, \boldsymbol{u})| \leq \beta(|\boldsymbol{\xi}|, t) + \gamma_1(\|\boldsymbol{u}\|) + \gamma_2(\|\boldsymbol{y}\|) \tag{12}$$

for all $t \geq 0$, $\boldsymbol{\xi} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathcal{L}_\infty^m$ with $\|\boldsymbol{u}\| < c_1$ and $\boldsymbol{y} \in \mathcal{L}_\infty^p$ with $\|\boldsymbol{y}\| < c_2$. A sufficient condition for cIOSS, is the existence of a cIOSS-Lyapunov function, that is of a differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that (5) holds for $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ and

$$DVf(\boldsymbol{\xi}, \boldsymbol{\mu}) \leq -\alpha(|\boldsymbol{\xi}|)+\delta(|\boldsymbol{\mu}|)+\lambda(|\boldsymbol{y}|), \forall \boldsymbol{\xi}, \boldsymbol{\mu} \tag{13}$$

holds for $\delta, \lambda \in \mathcal{K}_\infty$ and $\alpha : \mathbb{R}_{\geq 0} \to [0, c) \in \mathcal{K}\backslash\mathcal{K}_\infty$. Analogous with input-to-output stability (IOS) (Sontag, 2000), system (11) is said to be *constrained input-to-output stable* (cIOS) if there exist $\beta \in \mathcal{KL}$ and $\gamma : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_C$ such that

$$|\boldsymbol{y}(t, \boldsymbol{\xi}, \boldsymbol{u})| \leq \beta(|\boldsymbol{\xi}|, t) + \gamma(\|\boldsymbol{u}\|), \tag{14}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$ and all $\boldsymbol{u} \in \mathcal{L}_\infty^m$ with $\|\boldsymbol{u}\| < c$.

Besides, cISS is closely related to *small-signal $\mathcal{L}_\infty$ stability* (Khalil, 2002), which is defined as follows: system (11) is said to be *small-signal $\mathcal{L}_\infty$ stable* if there exist $\gamma : [0, c) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_C$ and a nonnegative constant β such that

$$\|\boldsymbol{y}\| \leq \gamma(\|\boldsymbol{u}\|) + \beta \tag{15}$$

for all $\boldsymbol{u} \in \mathcal{L}_\infty^m$ with $\|\boldsymbol{u}\| < c$.

Theorem 2. Consider system (11) and suppose that: 1. system (11) is cISS; 2. $\boldsymbol{h}$ satisfies

$$|\boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{u})| \leq \alpha_1(|\boldsymbol{x}|) + \alpha_2(|\boldsymbol{u}|) + \eta \tag{16}$$

for all $t, \boldsymbol{x}, \boldsymbol{u}$ and for some class $\mathcal{K}$ functions α_1, α_2, and a nonnegative constant η. Then system (11) is small-signal $\mathcal{L}_\infty$ stable.

*Proof:*Since system (11) is cISS, there exist $\beta \in \mathcal{KL}$ and $\gamma : [0,c) \to \mathbb{R}_{\geq 0} \in \mathcal{K}_C$ such that (4) holds. Using (16), we obtain

$$|\boldsymbol{y}(t)| \leq \alpha_1(\beta(|\boldsymbol{\xi}|,t) + \gamma(\|\boldsymbol{u}\|)) + \alpha_2(|\boldsymbol{u}|) + \eta$$
$$\leq \alpha_1(2\beta(|\boldsymbol{\xi}|,t)) + \alpha_1(2\gamma(\|\boldsymbol{u}\|)) + \alpha_2(|\boldsymbol{u}|) + \eta$$

Thus, $\|\boldsymbol{y}\| \leq \gamma_0(\|\boldsymbol{u}\|) + \beta_0$, where $\gamma_0 = \alpha_1 \circ 2\gamma + \alpha_2$, $\beta_0 = \alpha_1(2\beta(|\boldsymbol{\xi}|), 0) + \eta$. Obviously, $\gamma_0 \in \mathcal{K}_C$, thus system (11) is small-signal $\mathcal{L}_\infty$ stable. ■

In particular, if assume $\boldsymbol{h} = \boldsymbol{x}$ in Theorem 2, then it is direct to draw the conclusion. Therefore, cISS can be seen as a generalization of small-signal $\mathcal{L}_\infty$ stability. On the other hand, since it is sometimes difficult to directly find a cISS Lyapunov function for general systems, another sufficient 'Lasalle' type condition for cISS is provided below.

Theorem 3. System (11) is cISS and forward complete provided that: 1. it admits a quasi ISS-Lyapunov function (Angeli, 1999) V_1 with $\sigma, \delta_1, \underline{\alpha}_1, \bar{\alpha}_1 \in \mathcal{K}_\infty$ satisfy $\underline{\alpha}_1(|\boldsymbol{x}|) \leq V_1(\boldsymbol{x}) \leq \bar{\alpha}_1(|\boldsymbol{x}|)$ and $DV_1\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \leq -\sigma(|\boldsymbol{y}|) + \delta_1(|\boldsymbol{u}|)$; 2. it admits a cIOSS-Lyapunov function V_2 with $\lambda, \delta_2, \underline{\alpha}_2, \bar{\alpha}_2 \in \mathcal{K}_\infty$ and $\alpha : \mathbb{R}_{\geq 0} \to [0,c) \in \mathcal{K}\backslash\mathcal{K}_\infty$ satisfy $\underline{\alpha}_2(|\boldsymbol{x}|) \leq V_2(\boldsymbol{x}) \leq \bar{\alpha}_2(|\boldsymbol{x}|)$ and $DV_2\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \leq -\alpha(|\boldsymbol{x}|) + \delta_2(|\boldsymbol{u}|) + \lambda(|\boldsymbol{y}|)$; 3. $\limsup_{s\to+\infty} \lambda(s)/\sigma(s) < +\infty$.

Proof: Firstly, define $q := \bar{q} \circ \theta_{\boldsymbol{y}}^{-1}$, where $\bar{q}(\cdot) > 0$ is a nondecreasing continuous function, and $\theta_{\boldsymbol{y}} := \bar{\alpha}_2 \circ \alpha^{-1} \circ \lambda$. Notice that $\theta_{\boldsymbol{y}}^{-1} \in \mathcal{K}\backslash\mathcal{K}_\infty$ because α is, thus q is a positive nondecreasing bounded function. Now it's claimed that, for $\rho(s) := \int_0^s q(r)dr$, $W = \rho(V_2)$ is a cIOSS Lyapunov function. Since $DW(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \leq -q[V_2(\boldsymbol{x})]\alpha(|\boldsymbol{x}|) + B\delta_2(|\boldsymbol{u}|) + \bar{q}(|\boldsymbol{y}|)\lambda(|\boldsymbol{y}|)$, whenever $\lambda(|\boldsymbol{y}|) \leq \alpha(|\boldsymbol{x}|)$ or $\lambda(|\boldsymbol{y}|) \geq \alpha(|\boldsymbol{x}|)$, where B is an upper bound of q. Define $\bar{q}(s) := \inf_{s\leq r} \sigma(r)/(1+\lambda(r))$. Obviously, $\bar{q}$ is a positive nondecreasing function. Then it is easy to prove that V defined as $V = V_1 + W$ satisfies assumptions of Corollary 1. Thus system (11) is cISS and forward complete. ■

Example 2. In the following, it's shown that the PD-controlled manipulator used in (Angeli *et al.*, 2000) to motivate iISS is also cISS by Theorem 3. A simple model of the manipulator is given as

$$(mr^2 + ML^2/3)\ddot{\theta} + 2mr\dot{r}\dot{\theta} = \tau$$
$$m\ddot{r} - mr\dot{\theta}^2 = F \qquad (17)$$

where F, τ indicate external torques. The PD-control law used in (Angeli *et al.*, 2000) is

$$\tau = -k_{d_1}\dot{\theta} - k_{p_1}(\theta - \theta_d), F = -k_{d_2}\dot{r} - k_{p_2}(r - r_d) \qquad (18)$$

where $k_{p_1}, k_{p_2}, k_{d_1}, k_{d_2} > 0$ and θ_d, r_d can be seen as the input. For notational simplicity, we also write $\boldsymbol{q} = [\theta, r]^T$ and $\boldsymbol{q}_d = [\theta_d, r_d]^T$. Let K and P be the kinetic and potential energy of the system: $K = (ML^2/3 + mr^2)\dot{\theta}^2/2 + m\dot{r}^2/2$, $P = k_{p_1}\theta^2/2 + k_{p_2}r^2/2$. Firstly, choose the mechanical energy $V_1 = K + P$ of (17) as a candidate Lyapunov function, then the time derivative of V_1 is

$$\dot{V}_1 = -k_{d_1}\dot{\theta}^2 - k_{d_2}\dot{r}^2 + k_{p_1}\dot{\theta}\theta_d + k_{p_2}\dot{r}r_d$$
$$\leq -N_1|\dot{\boldsymbol{q}}|^2 + N_2|\boldsymbol{q}_d|^2, \qquad (19)$$

where N_1, N_2 are sufficiently small and large constants respectively. Thus V_1 can be seen as a quasi ISS-Lyapunov function with $\boldsymbol{y} = \dot{\boldsymbol{q}}$. Define V_2 as

$$V_2 = K + P + \varepsilon\frac{mr\dot{r} + \theta(ML^2/3 + mr^2)\dot{\theta}}{1 + r^2 + \theta^2}$$

where ε is a sufficiently small constant (to keep V_2 positive definite). The time derivative of V_2 is

$$\dot{V}_2 = -k_{d_1}\dot{\theta}^2 - k_{d_2}\dot{r}^2 + k_{p_1}\dot{\theta}\theta_d + k_{p_2}\dot{r}r_d$$
$$+\varepsilon\frac{m\dot{r}^2 + (ML^2/3 + 2mr^2)\dot{\theta}^2 + rF + \theta\tau}{1 + r^2 + \theta^2}$$
$$-\varepsilon\frac{2r\dot{r} + 2\theta\dot{\theta}}{(1 + r^2 + \theta^2)^2}[mr\dot{r} + \theta(ML^2/3 + mr^2)\dot{\theta}]$$
$$\leq M_1|\boldsymbol{q}_d|^2 + M_2|\dot{\boldsymbol{q}}|^2 + \varepsilon_1\frac{rF + \theta\tau}{1 + r^2 + \theta^2}$$

for a sufficiently small ε_1 and sufficiently large M_1, M_2. Substituting F and τ as (18) into the previous inequality, we can show that

$$\dot{V}_2 \leq \tilde{M}_1|\boldsymbol{q}_d|^2 + \tilde{M}_2|\dot{\boldsymbol{q}}|^2 - \varepsilon_2\frac{|\boldsymbol{q}|^2}{1 + |\boldsymbol{q}|^2} \qquad (20)$$

for some sufficiently large constants $\tilde{M}_1, \tilde{M}_2$ and sufficiently small constant ε_2. Obviously, V_2 is a cIOSS-Lyapunov function with $\boldsymbol{y} = \dot{\boldsymbol{q}}$. Moreover, since the terms involving $\dot{\boldsymbol{q}}$ in (19) and (20) are both quadratic, assumption of Theorem 3 is trivially satisfied. Thus, system (17) is cISS and forward complete.

Fig.1 shows the simulation result for $\theta_d = 1.2\tanh(\dot{\theta})$, $r_d = 0$ and initial state $(0, 0.1, 0.1, 0.1)^T$. Notice that the difference with (Angeli *et al.*, 2000) is the scaling involved in θ_d, which is reduced from 3 to 1.2, the stability property of resultant systems are different: when $\theta_d = 3\tanh(\dot{\theta})$, it results in unbounded trajectories (see Fig.3 in (Angeli *et al.*, 2000)); when $\theta_d = 1.2\tanh(\dot{\theta})$, which has infinite energy and hence can't be dealt with iISS, it yields bounded trajectories.

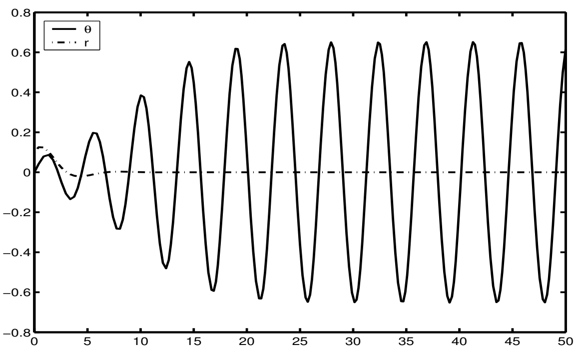

Fig. 1. Bounded state trajectories

4. INVERSE OPTIMALITY

In this section, consider the following system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})\boldsymbol{d} + \boldsymbol{g}_2(\boldsymbol{x})\boldsymbol{u} \tag{21}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{d} \in \mathbb{R}^q$ is a disturbance, $\boldsymbol{u} \in \mathbb{R}^m$ is a control input, $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{g}_1 : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, $\boldsymbol{g}_2 : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz functions and $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. It's constrained input-to-state stabilizable if there exist a continuous map $\boldsymbol{k} : \mathbb{R}^n \to \mathbb{R}^m$ with $\boldsymbol{k}(\boldsymbol{0}) = \boldsymbol{0}$ and a constant c such that system (21) becomes cISS when $\|\boldsymbol{d}\| \le c$. If, in addition, $\lambda\boldsymbol{k}$ for $\lambda \in (1/2, +\infty)$, still constrained input-to-state stabilize (21), $\boldsymbol{k}$ achieves gain margin $(1/2, +\infty)$. Now we introduce the concept of cISS-control Lyapunov function (cISS-clf), whose existence leads to a Sontag type construction of constrained input-to-state stabilizing control laws, then it is shown that constrained input-to-state stabilizability implies the solvability of a modified *inverse optimal gain assignment* problem (Krstic and Li, 1998).

Definition 3. A continuously differentiable and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is said to be a cISS-clf for system (21) if there exists $\delta : \mathbb{R}_{\ge 0} \to [0, c) \in \mathcal{K}\backslash\mathcal{K}_\infty$ such that for all $\boldsymbol{x} \ne \boldsymbol{0}$ and all $\boldsymbol{d}$ with $|\boldsymbol{d}| \le \delta(|\boldsymbol{x}|)$, it holds that

$$\inf_{\boldsymbol{u}}\{L_{\boldsymbol{f}}V + L_{\boldsymbol{g}_1}V\boldsymbol{d} + L_{\boldsymbol{g}_2}V\boldsymbol{u}\} < 0 \tag{22}$$

A cISS-clf V satisfies the *small control property* if there exists a continuous control law $\alpha_c(\boldsymbol{x})$ such that $L_{\boldsymbol{f}}V + L_{\boldsymbol{g}_1}V\delta(|\boldsymbol{x}|) + L_{\boldsymbol{g}_2}V\alpha_c(\boldsymbol{x}) < 0, \forall \boldsymbol{x} \ne \boldsymbol{0}$.

Theorem 4. For system (21), if there exists a cISS-clf V with small control property, the following Sontag type control law $\boldsymbol{u} = \boldsymbol{k}_s(\boldsymbol{x})$ defined as

$$\boldsymbol{k}_s = \begin{cases} -\frac{w(\boldsymbol{x}) + \sqrt{w^2 + |L_{\boldsymbol{g}_2}V|^4}}{|L_{\boldsymbol{g}_2}V|^2}(L_{\boldsymbol{g}_2}V)^T, & L_{\boldsymbol{g}_2}V \ne \boldsymbol{0} \\ \boldsymbol{0}, & L_{\boldsymbol{g}_2}V = \boldsymbol{0} \end{cases} \tag{23}$$

where $w(\boldsymbol{x}) = L_{\boldsymbol{f}}V + |L_{\boldsymbol{g}_1}V|\delta(|\boldsymbol{x}|)$, constrained input-to-state stabilize (21) with gain margin $(1/2, +\infty)$. On the other hand, if (21) is constrained input-to-state stabilizable, there exists a cISS-clf with small control property.

Proof: The proof is omitted here. ■

Before the discussion about inverse optimal problem, let us introduce the notation and also some properties of *Legendre-Fenchel* transform for class $\mathcal{K}_C$ functions. For $H \in \mathcal{K}_C$ whose derivative H' exists and is also of class $\mathcal{K}_C$, let ℓH denotes the *Legendre-Fenchel* transform $\ell H(h) = h(H')^{-1}(h) - H((H')^{-1}(h))$, where $(H')^{-1}$ is the inverse function of H'.

Lemma 4.1. If $H, H' : [0, c) \to \mathbb{R}_{\ge 0} \in \mathcal{K}_C$, then the Legendre-Fenchel transform satisfies the following properties: (a) $\ell H(h) = \int_0^h (H')^{-1}(s)ds$; (b) $\ell\ell H(h) = H(h)$, for $h < c$; (c) $\ell H \in \mathcal{K}_\infty$; (d) $\ell H(H'(h)) = h(H')(h) - H(h)$, for $h < c$.

Lemma 4.2. (Young's Inequality, Theorem 156 (Hardy *et al.*, 1989)): For any vectors $\boldsymbol{x}$ that satisfies $|\boldsymbol{x}| < c$ and $\boldsymbol{y}$, the following inequality holds $\boldsymbol{x}^T\boldsymbol{y} \le H(|\boldsymbol{x}|) + \ell H(|\boldsymbol{y}|)$, and the equality is achieved if and only if $\boldsymbol{y} = H'(|\boldsymbol{x}|)\boldsymbol{x}/|\boldsymbol{x}| \; \forall \; |\boldsymbol{x}| < c$, that is, for $\boldsymbol{x} = (H')^{-1}(|\boldsymbol{y}|)\boldsymbol{y}/|\boldsymbol{y}|$.

Lemma 4.3. Assume the auxiliary system of (21):

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}_1(\boldsymbol{x})\ell H(2|L_{\boldsymbol{g}_1}V|)\frac{(L_{\boldsymbol{g}_1}V)^T}{|L_{\boldsymbol{g}_1}V|^2} + \boldsymbol{g}_2(\boldsymbol{x})\boldsymbol{u} \tag{24}$$

where $V(\boldsymbol{x})$ is a Lyapunov function candidate for (24), $H, H' : [0, c) \to \mathbb{R}_{\ge 0} \in \mathcal{K}_C$ and ℓH denotes the Legendre-Fenchel transform of H. Suppose that there exists a matrix-valued function $\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{R}^T(\boldsymbol{x}) > 0$ for all $\boldsymbol{x}$ such that the control law

$$\boldsymbol{u} = \boldsymbol{k}(\boldsymbol{x}) = -\boldsymbol{R}(\boldsymbol{x})^{-1}(L_{\boldsymbol{g}_2}V)^T \tag{25}$$

globally stabilizes (24) with respect to $V(\boldsymbol{x})$, then the control law $\boldsymbol{u} = \boldsymbol{k}^*(\boldsymbol{x}) = \beta\boldsymbol{k}(\boldsymbol{x}) = -\beta\boldsymbol{R}(\boldsymbol{x})^{-1}(L_{\boldsymbol{g}_2}V)^T$ with any $\beta \ge 2$ solves the following inverse optimal gain assignment problem for (21) by minimizing the cost function

$$J(\boldsymbol{u}) = \sup_{|\boldsymbol{d}| \le \lambda c}\{\lim_{t\to\infty}[2\beta V(\boldsymbol{x}(t)) + \int_0^t (L(\boldsymbol{x}) + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u} - \beta\lambda H(\frac{|\boldsymbol{d}|}{\lambda}))\boldsymbol{d}\tau]\}, \tag{26}$$

for any $\lambda \in [1, 2]$, where $L(\boldsymbol{x})$ is positive definite, radial unbounded and defined as $L(\boldsymbol{x}) = \beta(2-\lambda)\ell H(2|L_{\boldsymbol{g}_1}V|) + \beta(\beta-2)L_{\boldsymbol{g}_2}VR^{-1}(L_{\boldsymbol{g}_2}V)^T - 2\beta[L_{\boldsymbol{f}}V + \ell H(2|L_{\boldsymbol{g}_1}V|) - L_{\boldsymbol{g}_2}VR^{-1}(L_{\boldsymbol{g}_2}V)^T]$.

Proof: It has been shown in Lemma 4.1 and 4.2 that properties of the Legendre-Fenchel transform mentioned in the Appendix of (Krstic and Li, 1998) for class $\mathcal{K}_\infty$ functions also applies to class $\mathcal{K}_C$ functions, thus the proof is essentially same with Theorem 3.1 (Krstic and Li, 1998). ■

Theorem 5. If system (21) is constrained input-to-state stabilizable, the inverse optimal gain assignment problem is solvable.

Proof: By Theorem 4, $\boldsymbol{u} = \boldsymbol{k}_s(\boldsymbol{x})$ defined in (23), which can be rewritten as (25), can constrained input-to-state stabilize system (21) with gain margin $(1/2, +\infty)$. Therefore, if we can find an appropriate $H \in \mathcal{K}_C$ and show that $\boldsymbol{u} = \boldsymbol{k}_s(\boldsymbol{x})/2$ can globally stabilize the auxiliary system (24), then the proof is complete.

Assume the range of δ is $[0, \lambda c)$, where $\lambda \in [1, 2]$. Since $|L_{\boldsymbol{g}_1}V(\boldsymbol{x})| = 0$ vanishes at the origin $\boldsymbol{x} = \boldsymbol{0}$, there exists $\pi \in \mathcal{K}_\infty$ such that

$$|L_{\boldsymbol{g}_1}V| \le \pi(|\boldsymbol{x}|) \tag{27}$$

and $\int_0^h \pi \circ \delta^{-1}(\lambda s)ds \to +\infty$ as $h \to c$. Since $\delta \in \mathcal{K}\backslash\mathcal{K}_\infty$, $\delta'(r) > 0$, for $r > 0$, it is easy to prove that there always exists $\pi \in \mathcal{K}_\infty$ such that $\lim_{r\to+\infty} \pi(r)\delta'(r) = +\infty$. Let α be any $\mathcal{K}_\infty$ function so that $\alpha(r) \le \pi(r)\delta'(r)$ for $r \ge 0$, then $\int_0^h \pi \circ \delta^{-1}(\lambda s)ds = \frac{1}{\lambda}\int_0^{\delta^{-1}(\lambda h)} \pi(t)\delta'(t)dt \ge \frac{1}{\lambda}\int_0^{\delta^{-1}(\lambda h)} \alpha(t)dt$ with $t = \delta^{-1}(\lambda s)$. Obviously, the last integral increases to $+\infty$ as h goes to c. Thus such a $\pi \in \mathcal{K}_\infty$ always exists. Since $\delta \circ \pi^{-1}(r) \in \mathcal{K}\backslash\mathcal{K}_\infty$, $\int_0^r \delta \circ \pi^{-1}(s)ds \in \mathcal{K}_\infty$, define

$$\xi'(2r) = \frac{1}{\lambda}\delta \circ \pi^{-1}(r),\ H'(h) = (\xi')^{-1}(h). \quad (28)$$

Then it is easy to check that both H and H' are $\mathcal{K}_C$ functions. From Lemma 4.1 and the definition of H, it follows that $\ell H(2h) =$

$$\xi(2h) = \frac{1}{\lambda}\int_0^h \delta \circ \pi^{-1}(s)ds \le \frac{1}{\lambda}h\delta \circ \pi^{-1}(h).$$

By Theorem 4, the time derivative of V along (24) with $\boldsymbol{u} = \boldsymbol{k}_s(\boldsymbol{x})/2$ for all $\boldsymbol{x} \ne \boldsymbol{0}$ is

$$\dot{V} = L_{\boldsymbol{f}}V + \frac{1}{2}L_{\boldsymbol{g}_2}V\boldsymbol{k}_s(\boldsymbol{x}) + \ell H(2|L_{\boldsymbol{g}_1}V|)$$

$$\le L_{\boldsymbol{f}}V + \frac{1}{2}L_{\boldsymbol{g}_2}V\boldsymbol{k}_s(\boldsymbol{x}) + \frac{1}{\lambda}|L_{\boldsymbol{g}_1}V|\delta\circ\pi^{-1}(|L_{\boldsymbol{g}_1}V|)$$

Notice that $\lambda \in [1, 2]$ and (27), then

$$\dot{V} \le L_{\boldsymbol{f}}V + |L_{\boldsymbol{g}_1}V|\delta(|\boldsymbol{x}|) + \frac{1}{2}L_{\boldsymbol{g}_2}V\boldsymbol{k}_s(\boldsymbol{x}) \quad (29)$$

which is negative definite by Theorem 4. Thus system (24) is asymptotically stabilized by $\boldsymbol{k}_s/2$. The same method (Krstic and Li, 1998) can be used to achieve radial unboundedness, if $L(\boldsymbol{x})$ is positive definite. Then by Lemma 4.3, $\boldsymbol{k}_s$ solves the inverse optimal gain assignment problem. ■

Example 3. Consider

$$\begin{aligned} \dot{x}_1 &= -x_1 + (x_1 - \cos x_1)d + u \\ \dot{x}_2 &= -x_2 + (x_2 + \cos x_1)d - u \end{aligned} \quad (30)$$

No matter what control law u is applied, $d \equiv 2$ gives $d(x_1+x_2)/dt = x_1+x_2$. This means that system (30) is not input-to-state stabilizable. On the other hand, let $\boldsymbol{x} = [x_1, x_2]^T$, then $V(\boldsymbol{x}) = \log(1+x_1^2) + \log(1+x_2^2)$ is a cISS-clf with small control property. Since for all $\boldsymbol{x}, d$, $L_{\boldsymbol{f}}V + L_{\boldsymbol{g}_1}Vd \le -2|\boldsymbol{x}|^2/(1+|\boldsymbol{x}|^2) + 6|d|$. Therefore, (30) is constrained input-to-state stabilizable. Let $\delta(r) = r^2/3(1+r^2)$. Following the lines of Theorem 4 and Theorem 5, (30) can be designed to handle some bounded disturbance with constrained magnitude and to solve an inverse optimal gain assignment problem, which both can't be achieved under the assumption of iISS.

5. CONCLUSION

The cISS notion is introduced as a natural concept to better deal with the stability of systems which are not ISS and can't be handled precisely by iISS, and to generalize the small signal $\mathcal{L}_\infty$ stability. Moreover, cISS, unlike ISS and iISS, is not confined to forward systems. Several characterizations and related notions of cISS are provided to show cISS is compatible with ISS framework. In addition, for a class of nonlinear systems, constrained input-to-state stabilizability implies the solvability of an inverse optimal problem. We believe that cISS is a natural concept to improve the ISS framework and has broad applicability.

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