MULTI CRITERIA SAMPLED-DATA $\mathcal{H}_\infty$ CONTROL APPLIED TO THE CHOICE OF SAMPLING RATE AND ANTI-ALIAS FILTER

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Abstract: A systematic evaluation procedure for the choice of sampling rate and anti-alias filter in sampled-data control is presented. It is based on a multi criteria $\mathcal{H}_\infty$ optimization procedure, where sampled-data measures are introduced to capture the intersample behavior. The optimization scheme is based on linear matrix inequalities (LMIs) formulated in the delta operator, and the result is a low order controller of PID type. Both low frequency performance, mid frequency stability margins and high frequency control activity are taken into account. The evaluation procedure suggests to use higher sampling rates than typical text books recommend.

Keywords: anti-alias filter, delta operator, $\mathcal{H}_\infty$ optimization, sampled-data, sampling period.

1. INTRODUCTION

Since the introduction of sampled-data control the question of a proper sampling rate has been discussed. Very few systematic approaches have been introduced. Typical rules of thumb are related to the desired closed loop bandwidth, see e.g. (Franklin *et al.*, 1998). Recently an evaluation procedure based on multi-criteria $\mathcal{H}_\infty$ controller design has been used for synthesis and evaluation of PI and PID controllers, see (Kristiansson and Lennartson, 2000). This approach includes minimization of load performance with constraints on control activity, stability margins and sensitivity to high frequency (HF) sensor noise.

This general and systematic evaluation procedure is now generalized to sampled-data systems and applied to the choice of sampling rate and anti-alias filter. A new lifted sampled-data model is introduced to properly model a continuous-time integral load disturbance. This sampled-data model captures the intersample behavior, where the alias phenomenon shows up especially at high frequencies.

The multi criteria constrained $\mathcal{H}_\infty$ optimization problem is formulated as an iteration between two sets of LMIs. Generally the optimization is a bilinear matrix inequality problem. For other related works see e.g. (Grigoriadis and Skelton, 1996; Wortelboer *et al.*, 1999). In this paper a special routine for the choice of an initial controller is a crucial step in the minimization strategy.

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The delta operator is also introduced to get a proper convergence for short sampling periods. Observe that the more well known shift operator exhibits bad numerical behavior for short sampling periods, see (Middleton and Goodwin, 1990). The results in this paper is a continuation of the results in (Xiao-Long *et al.*, 2002), but here with a focus on algorithmic and numerical aspects. Related results on $\mathcal{H}_\infty$ control using the delta operator can be found in e.g. (Middleton and Goodwin, 1990; Shor and Perkins, 1991; Collins and Song, 1999; Erwin and Bernstein, 2002).

2. DIFFERENT MODELS IN THE DELTA OPERATOR FORM

Consider the following state space model in the delta operator form, see (Lennartson *et al.*, 2004*b*):

$$\begin{bmatrix} \delta x \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \qquad (1)$$

This model includes both the input disturbance w, the control signal u, the performance signal z as well as the measured output signal y. The corresponding dimensions of these signals are n_w, n_u, n_z, and n_y respectively. The discrete-time updates occur at times t_k, $k = 0, 1, 2, \ldots$. The time interval between two updates is the sampling period $h = t_{k+1} - t_k$. A subscript δ is included in the matrices and signals when it is needed to distinguish from other models, cf. Table 1.

2.1 Generalized delta operator models

By making use of the generalized delta operator

$$\delta x(t_1, t_0) = \frac{x(t_1) - x(t_0)}{h}, \qquad t_1 > t_0 \qquad (2)$$

related models can also be formulated in the delta operator form. In sampled-data control a continuous-time plant is controlled by a discrete-time controller. This is a special case of a mixed continuous/discrete-time system where state jumps occur at the discrete-time instants. Then the update of the state vector at times t_k is expressed in the generalized delta operator form as $\delta x(t_k^+, t_k^-)$. By adding the notation $\delta x(t) = \dot{x}(t)$ in the continuous model when $t \neq t_k$, a compact description of mixed continuous/discrete-time systems is obtained, see Table 1, and further details in (Lennartson *et al.*, 2004*b*). This implies that (1) also can be interpreted as a mixed continuous/discrete-time system.

The mixed signal z is then a continuous-time signal $z_c(t)$ except at the discrete-time instants $t = t_k$ where it takes the values of the corresponding discrete-time signal, i.e. $z(t_k) = z_\delta(t_k)$. Note that the dimensions of these signals n_{z_c} and n_{z_δ} normally are not the same. A pure continuous-time signal means e.g. that $n_{z_\delta} = 0$.

Table 1. Different notations related to the delta operator model (1). The generalized delta operator $\delta x(t_1, t_0)$ is defined in (2).

Model	Signal, matrix	State update
continuous	$z_c,\ A_c$	$\dot{x}(t)$
discrete shift	$z_q,\ A_q$	$x(t_{k+1})$
discrete delta	$z_\delta,\ A_\delta$	$\delta x(t_k) = \delta x(t_{k+1}^-, t_k^-)$
lifted delta	$\breve{z},\ \breve{A}$	$\delta x(t_{k+1}^-, t_k^+)$
mixed cont/disc	$z,\ A =$	$\delta x(t) =$
	$\begin{cases} z_c, A_c & t \neq t_k \\ z_\delta,\ A_\delta & t = t_k \end{cases}$	$\begin{cases} \dot{x}(t) & t \neq t_k \\ \delta x(t^+, t^-) & t = t_k \end{cases}$

Another example where the generalized delta operator is useful is a lifted discrete-time model, representing the continuous-time behavior between two sampling instants (from time t_k^+ just after the discrete time update to time t_{k+1}^- just before next update). The state update is then expressed as $\delta x(t_{k+1}^-, t_k^+)$, and the matrices and signals in this lifted model are distinguished according to Table 1 by a $\breve{}$ above corresponding variable names. These models will be applied in the next section where a new sampled-data model is introduced.

2.2 Transformation from q to δ

The input signals in the delta model are related to the inputs in the shift operator model as $w_\delta = w_q/\sqrt{h}$ and $u_\delta = u_q$, and the corresponding output signals are defined as $z_\delta = z_q/\sqrt{h}$ and $y_\delta = y_q$. A consequence of the normalization of the performance signal z_δ with respect to the sampling period is that the size of this signal can be expressed by the norm

$$\|z\|^2 = \sum_{k=0}^{\infty} z_q'(t_k) z_q(t_k) = \sum_{k=0}^{\infty} z_\delta'(t_k) z_\delta(t_k)\, h$$

The second sum converges to a corresponding (Riemann) integral when $h \to 0$, which motivates the normalization factor $1/\sqrt{h}$. Similar arguments hold for the input signal w_δ, see (Middleton and Goodwin, 1990).

The delta operator $\delta x(t_k) = \delta x(t_{k+1}^-, t_k^-)$ and the normalized signals z_δ and w_δ together define the transformation from a shift operator to a delta operator model. A compact formulation of this transformation was recently introduced in (Lennartson *et al.*, 2004*b*).

2.3 Induced norms of mixed systems

The norm of a mixed signal is defined as

$$\|z\|^2 = \int_0^\infty z_c'(t) z_c(t)\, dt + \sum_{k=0}^{\infty} z_\delta'(t_k) z_\delta(t_k) h$$

When a mixed system (1) is controlled by a discrete-time controller

$$\begin{bmatrix} \delta x_K \\ u_\delta \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K \\ y_\delta \end{bmatrix} \triangleq K \begin{bmatrix} x_K \\ y_\delta \end{bmatrix} \quad (3)$$

the resulting closed loop system is denoted G_{zw}. It means that the induced norm of this mixed continuous/discrete-time closed loop system with input signal w and corresponding output signal z can be expressed as

$$\|G_{zw}\| = \sup_{\|w\|\neq 0} \frac{\|z\|}{\|w\|} \quad (4)$$

3. SAMPLED-DATA CONTROL WITH INTEGRAL ACTION

A non-standard sampled-data model is introduced in this section to properly handle continuous-time load disturbances. Sampled-data control generally means that a mixed continuous/discrete-time system (typically a continuous-time plant including discrete-time sensor noise) is controlled by a discrete-time controller. The restriction compared to a general mixed system is that no continuous-time feedback is included, i.e. the dimensions of y_c and u_c are zero ($n_{y_c} = n_{u_c} = 0$).

3.1 Lifting the continuous-time model

Without any continuous-time feedback control, the continuous-time part of a mixed system, modelling the behavior between the sampling instants, reduces to

$$\begin{bmatrix} \dot{x}(t) \\ z_c(t) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x(t) \\ w_c(t) \end{bmatrix} \quad (5)$$

This model can be replaced by a lifted norm preserving discrete-time model in the delta operator form. As in the shift operator case, the lifted model is easily obtained by integrating the Hamiltonian matrix for the continuous-time system.

The Hamiltonian matrix and its (backward) transition matrix $\Pi(t, t_{k+1}^-)$ from t_{k+1}^- to t, $t \in (t_k, t_{k+1})$ associated with the continuous-time model (5) are defined as follows, where $Q_c = \gamma^2 I - D_c' D_c$, see e.g. (Green and Limebeer, 1995)

$$\mathcal{H}_S = \begin{bmatrix} A_c + B_c Q_c^{-1} D_c' C_c & B_c Q_c^{-1} B_c' \\ -C_c'(I + D_c Q_c^{-1} D_c') C_c & -(A_c + B_c Q_c^{-1} D_c' C_c)' \end{bmatrix}$$

and $\dot{\Pi}(t, t_{k+1}^-) = \mathcal{H}_S(t)\Pi(t, t_{k+1}^-), \Pi(t_{k+1}^-, t_{k+1}^-) = I$.

To obtain a delta formulation we introduce

$$\exp \begin{bmatrix} -h\mathcal{H}_S & -h\mathcal{H}_S \\ 0 & 0 \end{bmatrix} \triangleq \begin{bmatrix} \Pi(h) & h\Gamma(h) \\ 0 & I \end{bmatrix}$$

where $\Pi(h) = I_{2n} + h\Gamma(h)$. The minus sign before $\mathcal{H}_S$ is due to the backward transition. With $\Gamma(h)$ partitioned into sub-matrices of size $n \times n$ according to $\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12}; & \Gamma_{21} & \Gamma_{22} \end{bmatrix}$, the following lemma generates a corresponding lifted norm preserving discrete-time model in the delta operator, see (Lennartson *et al.*, 2004*b*).

Lemma 1. The induced norm of the continuous-time system (5) is bounded as $\sup_{\|w_c\|\neq 0} \|z_c\|/\|w_c\| < \gamma$ if and only if the following lifted discrete-time model in the delta operator

$$\begin{bmatrix} \delta x(t_k) \\ \breve{z}(t_k) \end{bmatrix} = \begin{bmatrix} \breve{A}(h) & \breve{B}_w(h) \\ \breve{C}_z(h) & 0 \end{bmatrix} \begin{bmatrix} x(t_k) \\ \breve{w}(t_k) \end{bmatrix} \quad (6)$$

where

$$\breve{A}(h) = -\Gamma_{11}(h)\big(I + h\Gamma_{11}(h)\big)^{-1} \quad (7)$$

$$\breve{B}_w(h)\breve{B}_w'(h) = -\gamma^{-2}\big(I + h\Gamma_{11}(h)\big)^{-1}\Gamma_{12}(h) \quad (8)$$

$$\breve{C}_z'(h)\breve{C}_z(h) = \Gamma_{21}(h)\big(I + h\Gamma_{11}(h)\big)^{-1} \quad (9)$$

satisfies the bound $\sup_{\|\breve{w}\|\neq 0} \|\breve{z}\|/\|\breve{w}\| < \gamma$. □

Observe that the lifted model matrices $\breve{A}$, $\breve{B}_w$, and $\breve{C}_z$ depend on γ.

3.2 Lifted sampled-data model including integral action

Integral action in sampled-data control is obtained by introducing a discrete time integrator in the loop. However, in the optimization criterion we want to include an integrated *continuous-time* load disturbance (added to the control signal) to take care of the inter-sample behavior. Discrete-time integral action with an ordinary sampled-data model means however that the load disturbance either becomes piece-wise constant (i.e. discrete-time) or a continuous-time integrator approximation has to be introduced outside the loop (a weighting filter with a pole close to zero). The first approach implies an approximation while the latter one results in numerical problems.

The solution is to include a continuous-time integrator in the loop and then modify the hold circuit model for the discrete-time control signal, see Figure 1. A lifted sampled-data model including this modified hold function will therefore be derived.

Assume that the control signal is included as the last states in the state vector, i.e. $u = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} x$. At the sampling instants the change of control signal

$$u_\delta(t_k) = \frac{u(t_k) - u(t_{k-1})}{h}$$

is the input to the plant model. All together the discrete-time update at the sampling instants includes

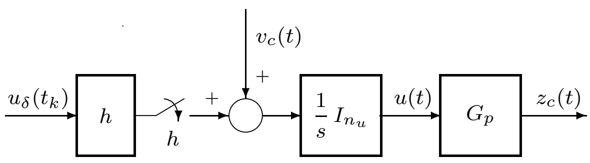

Fig. 1. Plant model G_p with continuous-time integrated load disturbance v_c and discrete-time control signal $u_\delta(t_k) = (u(t_k) - u(t_{k-1}))/h$.

$$\begin{bmatrix} \delta x(t_k^+, t_k^-) \\ z_\delta(t_k) \\ y_\delta(t_k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} \\ 0 & 0 & D_{zu_\delta} \\ C_{y_\delta} & D_{yw_\delta} & D_{yu_\delta} \end{bmatrix} \begin{bmatrix} x(t_k^-) \\ w_\delta(t_k) \\ u_\delta(t_k) \end{bmatrix} \quad (10)$$

Between these updates the system is assumed to be described by the continuous-time model (5), which in its lifted form is given by the discrete-time model (6)-(9). A lifted norm-preserving discrete-time model in the delta operator for this mixed continuous/discrete-time model is now presented. First, introduce the extended discrete-time input and output signals $\bar{z}_\delta = \begin{bmatrix} \breve{z}' & z_\delta' \end{bmatrix}'$ and $\bar{w}_\delta = \begin{bmatrix} \breve{w}' & w_\delta' \end{bmatrix}'$.

Theorem 2. Consider the mixed continuous/discrete-time system (5), (10) where the control signal u is included as the last states in the state vector, i.e. $u = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix} x$. The induced norm of this mixed system controlled by a general discrete-time controller (3) is bounded as $\sup_{\|w\| \neq 0} \|z\|/\|w\| < \gamma$ iff the following lifted discrete-time model in the delta operator

$$\begin{bmatrix} \delta x \\ \bar{z}_\delta \\ y_\delta \end{bmatrix} = \begin{bmatrix} \breve{A} & \begin{bmatrix} \breve{B}_w & 0 \end{bmatrix} & \breve{B}_u \\ \begin{bmatrix} \breve{C}_z \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} \breve{D}_{zu} \\ D_{zu_\delta} \end{bmatrix} \\ C_{y_\delta} & \begin{bmatrix} 0 & D_{yw_\delta} \end{bmatrix} & D_{yu_\delta} \end{bmatrix} \begin{bmatrix} x \\ \bar{w}_\delta \\ u_\delta \end{bmatrix} \quad (11)$$

where

$$\breve{B}_u = (I + h\breve{A}) \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} \quad \breve{D}_{zu} = h\breve{C}_z \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} \quad (12)$$

controlled by the same controller, satisfies the discrete-time bound $\sup_{\|\bar{w}_\delta\| \neq 0} \|\bar{z}_\delta\|/\|\bar{w}_\delta\| < \gamma$.

Proof: The state jump at time t_k in (10) can also be expressed as

$$x(t_k^+) = x(t_k^-) + h \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} u_\delta(t_k)$$

which together with the norm preserving lifted model (6) from Lemma 1, where $\delta x(t_k) = \delta x(t_{k+1}^-, t_k^+)$, implies that the update of the state vector from time t_k^- to t_{k+1}^- can be expressed as, cf. Table 1

$$\begin{aligned} \delta x(t_k) &= \delta x(t_{k+1}^-, t_k^-) = \delta x(t_{k+1}^-, t_k^+) + \delta x(t_k^+, t_k^-) \\ &= \breve{A} x(t_k^+) + \breve{B}_w \breve{w}(t_k) + \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} u_\delta(t_k) \\ &= \breve{A} x(t_k^-) + (I + h\breve{A}) \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} u_\delta(t_k) \\ &\quad + \breve{B}_w \breve{w}(t_k) \end{aligned}$$

In the same way the lifted performance output $\breve{z}$ is rewritten as

$$\breve{z} = \breve{C}_z x(t_k^+) = \breve{C}_z x(t_k^-) + h\breve{C}_z \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} u_\delta(t_k)$$

The notations in (12), including the additional discrete inputs and outputs in (10), then gives the lifted model (11). From an induced norm point of view this model includes both the continuous-time behavior (5) and the discrete-time update (10). □

3.3 Closed loop system

To complete this section we note that the closed loop system obtained when (11) is controlled by (3) has the general structure

$$\begin{bmatrix} \delta \bar{x}(t_k) \\ \bar{z}_\delta(t_k) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} \bar{x}(t_k) \\ \bar{w}_\delta(t_k) \end{bmatrix} \quad (13)$$

with the extended state vector $\bar{x} = \begin{bmatrix} x' & x_K' \end{bmatrix}'$, Denote this closed loop system $G_{\bar{z}_\delta \bar{w}_\delta}$, and observe that the dependency on the controller K (3) generally can be expressed as

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} K \begin{bmatrix} C_2 & D_2 \end{bmatrix} \quad (14)$$

This shows that the closed loop matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are affine in the controller K, and Theorem 2 can be reformulated as the following corollary.

Corollary 3. The induced norm of the mixed system (5), (10) controlled by a discrete-time controller (3) is bounded as $\|G_{zw}\| < \gamma$ iff the $\mathcal{H}_\infty$ norm

$$\|G_{\bar{z}_\delta \bar{w}_\delta}\|_\infty < \gamma \quad (15)$$

4. MULTI CRITERIA SAMPLED-DATA $\mathcal{H}_\infty$ CONTROLLERS

In the synthesis of low order multi criteria sampled-data controllers we first show how an iteration between two linear matrix inequalities (LMIs) can be used to optimize an $\mathcal{H}_\infty$ criterion. This scheme is then extended to a multi criteria $\mathcal{H}_\infty$ problem.

4.1 PK iteration between two LMIs

Assume that a candidate controller K in (14) is given. Consider the mixed system (5), (10) controlled by K. The induced norm of the resulting closed loop mixed system G_{zw} is bounded as $\|G_{zw}\| < \gamma$ if and only if there exists a $P = P' > 0$ such that

$$L_P = \begin{bmatrix} \mathcal{A}'P + P\mathcal{A} & P\mathcal{B} & \mathcal{C}' \\ \mathcal{B}'P & -\gamma I & \mathcal{D}' \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{bmatrix} + h \begin{bmatrix} \mathcal{A}' \\ \mathcal{B}' \\ 0 \end{bmatrix} P \begin{bmatrix} \mathcal{A} & \mathcal{B} & 0 \end{bmatrix} < 0 \quad (16)$$

where $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ in (14) depends on the lifted γ dependent model (11).

Based on the resulting P we are searching for an updated version of K. However, introducing (14) for a given P but unknown K, we observe that (16) is not an LMI anymore, since it includes quadratic expressions in K.

Replacing P in the second term of (16) with $PP^{-1}P$ and applying a Schur complement yields however the following LMI in K, cf. (14)

$$L_K = \begin{bmatrix} \mathcal{A}'P + P\mathcal{A} & P\mathcal{B} & \mathcal{C}' & \sqrt{h}\mathcal{A}'P \\ \mathcal{B}'P & -\gamma I & \mathcal{D}' & \sqrt{h}\mathcal{B}'P \\ \mathcal{C} & \mathcal{D} & -\gamma I & 0 \\ \sqrt{h}P\mathcal{A} & \sqrt{h}P\mathcal{B} & 0 & -P \end{bmatrix} < 0 \tag{17}$$

These two LMIs open up for an iterative algorithm solving both P, K and minimizing γ.

Algorithm 1

(1) Compute an initial guess of a low order stabilizing controller K and the related minimal γ value in (15), see further comments below.
(2) Compute the lifted γ dependent model (11) and the resulting closed loop system $G_{\bar{z}_\delta \bar{w}_\delta}$.
(3) For the given K and γ, solve the LMI (16) for $P > 0$.
(4) For the given P in Step 3, solve the LMI (17) for K, including minimization of γ.
(5) Iterate 2-4 until γ has converged.

In Step 1 an initial minimal value of γ is required. This is preferably obtained by γ iteration of the LMI (16), where the lifting procedure needs to be repeated in each iteration. A more efficient alternative is to replace the LMI by solving the corresponding Riccati equation, cf. (Gahinet and Apkarian, 1994), (Lennartson *et al.*, 2004*a*).

This alternating LMI scheme, which we call PK iteration (c.f. DK iteration in μ synthesis), was recently presented for discrete time delta operator models, see (Lennartson *et al.*, 2004*a*). In the examples we have studied the alternating iteration has converged rapidly. Most improvement in γ is in fact achieved during the first 10 iterations. No significant improvement is normally achieved after 20-30 iterations.

The key to avoid local minima is to obtain an initial low order controller K not too far from the optimal solution. One successful approach is to apply controller reduction based on a full order controller, see e.g. (Anderson and Liu, 1989) and (Lennartson *et al.*, 2004*a*).

4.2 Multi criteria minimization

To obtain a fair comparison between different controllers, as will be demonstrated in the next section, it is desirable to add a number of constraints in the control design. More explicitly consider a set of mixed continuous/discrete-time closed loop systems $G_{z_i w_i}$ for $i = 0, \ldots, \ell$ and the following bounds on their induced norms $\|G_{z_i w_i}\| < \gamma_i$. The related multi criteria optimization problem is then formulated as

$$\min_K \gamma_0 \qquad \gamma_i \le c_i \quad i = 1, \ldots, \ell \tag{18}$$

This constrained nonlinear optimization problem can be solved by extending the PK-iteration in Algorithm 1 as follows:

Algorithm 2

(1) Compute an initial guess of a low order stabilizing controller K and a minimal γ_0, see comments after Algorithm 1, and let $\gamma_i = c_i$ for $i = 1, \ldots, \ell$.
(2) For each $G_{z_i w_i}$ for $i = 0, \ldots, \ell$ compute the lifted γ_i dependent model (11) and the resulting closed loop system matrices $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i$, cf. (13), (14).
(3) For the given K and γ_i, solve the LMIs $P_i > 0$, $L_{P_i} < 0$ (16) for $i = 0, \ldots, \ell$.
(4) For the given P_i in Step 3 solve the coupled LMIs $L_{K_i} < 0$ (17) with K as unknown, including minimization of γ_0.
(5) Iterate 2-4 until γ_0 has converged.

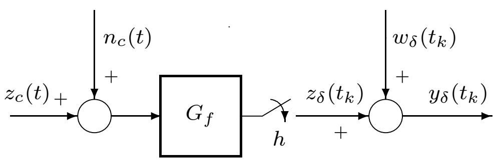

Fig. 2. Anti-alias filter G_f including continuous as well as discrete sensor noise n_c and w_δ.

5. CHOICE OF SAMPLING RATE AND ANTI-ALIAS FILTER

The multi criteria design method presented above is now applied as a systematic evaluation procedure for the choice of proper sampling rates and anti-alias filters. Consider the combined sampled-data system in Fig. 1 and Fig. 2, including an integrated continuous-time load disturbance v_c and anti-alias filter G_f. Introduce the criteria, cf. (4)

$$\begin{aligned} J_v &= \|G_{z_c v_c}\| \qquad & M_S &= \|G_{z_\delta w_\delta}\| \\ J_u &= \|G_{u n_c}\| \qquad & J_{HF} &= \|G_{u_\delta n_c}\| \end{aligned} \tag{19}$$

These criteria are sampled-data generalizations of the ones used for evaluation of continuous-time PID type controllers in (Kristiansson and Lennartson, 2000). They measure load disturbance performance (J_v), stability margin (M_S)(inverse of the minimal distance to the point -1 in the Nyquist curve), control activity (J_u), and high frequency sensitivity to sensor noise and uncertainties (J_{HF}). In the following example J_v will be minimized subject to constraints on the other criteria. Since these criteria together capture the most important demands on a feedback system, a fair comparison between different control strategies is achieved. Especially observe that the alias phenomenon is captured, since the criteria include the intersample behavior of the closed loop system.

Example 1. Consider the continuous-time plant model

$$G_p(s) = \frac{e^{-0.5s}}{(1+s)(1+0.7s)(1+0.7^2 s)(1+0.7^3 s)}$$

where the time delay is modelled as a second order Padé approximation. As anti-alias filter, first and fourth order Butterworth filters with different bandwidths

$$\omega_{bf} = \alpha^{1/n}\omega_N \qquad n = 1, 4 \tag{20}$$

normalized by the Nyquist frequency $\omega_N = \pi/h$. When the parameter $\alpha \ll 1$, then $\alpha = |G_f(j\omega_N)|$ is the filter gain (damping) at ω_N. A fourth order PID type sampled-data controller is designed such that J_v is minimized, subject to the constraints

$$M_S \leq 1.7 \qquad J_u \leq 10 \qquad J_{HF} \leq 50$$

Note that integral action is achieved due to the integrated load disturbance in the sampled-data model, cf. Fig. 1. The sampling period is chosen as $h = 0.1$, which according to Table 2 implies 9.5%-13.2% performance deterioration $J_v(h)/J_v(0)$ compared to the continuous-time performance $J_v(0)$. The different outcomes depend on the choice of the anti-alias filter.

Table 2. Load performance $J_v(h)$ relative to the corresponding continuous-time performance $J_v(0)$, and sampling frequency $\omega_s = 2\pi/h$ related to the closed loop bandwidth ω_b for different anti-alias filters of order n and bandwidth $\alpha^{1/n}\omega_N$ (20).

h	n	α	$J_v(h)/J_v(0)$	ω_b	ω_s/ω_b
0.1	1	0.1	10.0%	1.47	43
0.1	1	1	10.8%	1.48	42
0.1	4	0.1	13.2%	1.54	40
0.1	4	1	9.5%	1.63	39

Too large damping (small α and low bandwidth $\alpha^{1/n}\omega_N$) implies a significant phase lag in the control loop, which implies worse performance J_v, since the same stability margin $M_S \leq 1.7$ is required. Too small damping (larger α and higher bandwidth $\alpha^{1/n}\omega_N$) means on the other hand that the alias phenomenon shows up especially in the HF criterion J_{HF}. This implies that the control activity needs to be reduced, which also deteriorates the performance J_v.

An optimal α value for a fourth order anti-alias filter is close to one in this example. This can be compared to text book recommendations, where typically $\alpha = 0.1$ or less is suggested. On the other hand a first order filter gives slightly better performance for $\alpha = 0.1$ compared to $\alpha = 1$.

Table 2 also shows that with a performance deterioration of about 10%, the relation $\omega_s/\omega_b \approx 40$. In the classical text book on digital control (Franklin *et al.*, 1998) $20 \leq \omega_s/\omega_b \leq 40$ is recommended. In this example $\omega_s/\omega_b = 20$ implies a deterioration of about 35%, which indicates that the higher sampling rate selection is more reasonable, especially when the phase lag of the anti-alias filter and the intersample behavior are taken into account.

6. CONCLUSIONS

A systematic evaluation procedure for the choice of sampling rate and anti-alias filter has been presented. It is based on a multi criteria controller design using the delta operator. Sampled-data measures are introduced to capture the intersample behavior.

The evaluation shows that higher sampling rates than typical text book recommendations are reasonable, when anti-alias filter and the intersample behavior are taken into account. It is also observed that the design of an anti-alias filter is a compromise between too much damping resulting in significant phase lag, and too much HF noise caused by the alias phenomenon.

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