SUCCESSIVE POLE SHIFTING USING SAMPLED-DATA LQ REGULATORS

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Abstract:
Design of sampled-data control systems is considered, where the closed-loop poles of the discretized system are shifted in a successive manner, in order to achieve a desired overall pole allocation. Modal extraction and the linear quadratic optimal regulator for sampled-data systems are employed in each step of the pole shifting operation, using both continuous-time and discrete-time representation of the controlled plant and the performance index to be minimized. Unfortunately, the optimality of the overall closed-loop system cannot be guaranteed, due to repeated conversion between continuous-time and discrete-time systems in the iterative method. However, the performance index in each step can be utilized as a guide for choosing reasonable location of the closed-loop poles from practical point of view.

Keywords: LQ control method, Sampled-data systems, Pole assignment, Control system design, Modes, Performance indices

1. INTRODUCTION

The linear quadratic (LQ) regulator is widely used in the design of control systems. The basic theory has been well established, and the stability of the closed-loop system is guaranteed for arbitrary weighting matrices of the performance index, provided the assumptions regarding positive definiteness, controllability and observability are satisfied. Thus, the optimal solution can be readily calculated if a specific performance index to be minimized is given as a design parameter. This is not always the case, however, and it often happens in a practical situation that the weighting matrices of the performance index need to be adjusted as the tuning parameters. The performance index alone may not be sufficient for describing the design objective, and the weighting matrices may be adjusted according to the response of the closed-loop systems, for instance. The location of the closed-loop poles may be regarded as a guide in the selection of the weighting matrices. However, the relationship between the weighting matrices and the closed-loop poles is not straightforward except for special cases (Harvey and Stein, 1978). Moreover, the specification of the closed-loop alone is not always sufficient for achieving desired response. This often leads to trial and error in the choice of weighting matrices.

There has been some work which employs modal decomposition and successive shifting of a real pole or a pair of complex conjugate poles (Solheim, 1972; Amin, 1984; Medanic *et al.*, 1988; Saif, 1989; Tharp, 1992; Fujinaka and Shibata, 1996) for continuous-time systems and discrete-time systems. These methods restrict the choice of the weighting matrices to some extent, but it may be effective in eliminating too much freedom in the design parameters.

This paper is concerned with the design of sampled-data system by the successive pole shifting method using sampled-data LQ regulators. The sampled-data LQ regulator considers the inter-sample behavior of the controlled plant, whereas the discrete-time LQ regulators applied to sampled-data systems only treat the behavior at the sampling instants. Thus, the use of sampled-data LQ regulators may have some advantage when the inter-sample behavior of the controlled plant is crucial in the design of the control system.

Basically, the sampled-data LQ regulator problem can be solved by converting the original problem into an equivalent discrete-time LQ regulator problem. Thus, the controlled plant and the performance index are given in terms of continuous-time representation, for which the optimal solution is given in terms of discrete-time feedback law. This leads to a difficulty when successive pole shifting is considered, since the intermediate system results in a sampled-data control system which cannot be described as a pure continuous-time or discrete-time system treated in the standard LQ regulator theory. In order to consider the inter-sample behavior of the controlled system in the intermediate stage, the discrete-time representation of the sampled-data control system is temporarily converted to an equivalent continuous-time system. Then the weighting matrices for the continuous-time performance index are converted to those of an equivalent discrete-time representation, using the coefficient matrices of the temporary continuous-time system. The intended pole shifting is achieved with the optimal feedback gain obtained for the converted discrete-time LQ regulator problem. This procedure may be repeated as many times as desired, and the feedback gain in each stage is accumulated to obtain the final result.

2. SAMPLED-DATA LQ REGULATOR

Basic results regarding the LQ regulator theory are briefly reviewed in this section. Consider a continuous-time linear time-invariant system described by the state equation

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) \tag{1}$$

where $x_c \in \mathrm{R}^n$ is the state, $u_c \in \mathrm{R}^m$ is the input, and $A_c \in \mathrm{R}^{n\times n}$ and $B_c \in \mathrm{R}^{n\times m}$ are the coefficient matrices. It is assumed that the pair (A_c, B_c) is stabilizable and that the state variable $x(t)$ is directly available for observation. Thus, state feedback control is considered throughout this paper, and the output of the system is irrelevant for the design purpose.

The standard continuous-time LQ regulator problem is to find a control law which minimizes the performance index

$$J_c = \int_0^\infty \{x_c(t)^T Q_c x_c(t) + u_c(t)^T R_c u_c(t)\} dt \tag{2}$$

where $Q_c \in \mathrm{R}^{n\times n}$ is a positive semidefinite weighting matrix for the state $x_c(t)$, and $R_c \in \mathrm{R}^{m\times m}$ is a positive definite weighting matrix for the input $u_c(t)$. The pair $(Q^{1/2}, A)$ is assumed to be detectable, which guarantees the stability of the corresponding closed-loop system. The optimal solution for the continuous-time system is given by the state feedback law

$$u_c(t) = F_c x_c(t) \tag{3}$$

Here, the matrix $F_c \in \mathrm{R}^{m\times n}$ is the optimal feedback gain, given by

$$F_c = -{R_c}^{-1} {B_c}^T P_c \tag{4}$$

and the matrix $P_c \in \mathrm{R}^{n\times n}$ is the stabilizing solution of the algebraic Riccati equation

$$P_c A_c + {A_c}^T P_c - P_c B_c {R_c}^{-1} {B_c}^T P_c + Q_c = 0 \tag{5}$$

The numerical solution of the algebraic Riccati equation can be readily obtained by using an existing software package.

Now, the sampled-data LQ regulator problem is considered for the same system and the same performance index stated above. With the introduction of the standard zero-order hold used in digital control systems, the manipulating input $u_c(t)$ is restricted to be constant during each sampling interval. Thus,

$$u_c(t) = u_k,\ kh \le t < (k+1)h \tag{6}$$

where $h > 0$ denotes the sampling period. The sampled-data LQ regulator problem is to minimize the continuous-time performance index J_c with the additional constraint (6). It can be solved by converting the given problem to an equivalent discrete-time LQ regulator problem. The standard discrete-time LQ regulator problem is summarized in the following.

The standard discrete-time LQ regulator problem is to consider a discrete-time system described by

$$x_{k+1} = Ax_k + Bu_k \tag{7}$$

where $x_k \in \mathrm{R}^n$ is the state, $u_k \in \mathrm{R}^m$ is the input, $A \in \mathrm{R}^{n\times n}$ and $B \in \mathrm{R}^{n\times m}$ are the coefficient matrices, and find a control law which minimizes the discrete-time performance index

$$J = \sum_{k=0}^{\infty} \left({x_k}^T Q x_k + 2{x_k}^T S u_k + {u_k}^T R u_k \right) \tag{8}$$

where $Q \in \mathrm{R}^{n\times n}$, $S \in \mathrm{R}^{n\times m}$ and $R \in \mathrm{R}^{m\times m}$ are weighting matrices. Note that a cross-term ${x_k}^T S u_k$ is introduced in the performance index, since it is mandatory for the sampled-data LQ regulator problem.

The optimal solution for the discrete-time system is given by the state feedback law

$$u_k = F x_k \tag{9}$$

Here, the matrix $F \in \mathrm{R}^{m\times n}$ is the optimal feedback gain, given by

$$F = -(R + B^T P B)^{-1}(S^T + B^T P A) \tag{10}$$

and the matrix $P \in \mathrm{R}^{n\times n}$ is the stabilizing solution of the discrete version of algebraic Riccati equation

$$\begin{aligned} P = {} & A^T P A - (S^T + B^T P A)^T \\ & \times (R + B^T P B)^{-1}(S^T + B^T P A) + Q \end{aligned} \tag{11}$$

Again, a numerical solution of the discrete algebraic Riccati equation can be calculated with an existing software package.

In the sampled-data LQ regulator problem, the parameters and variables of the original problem are converted to an equivalent discrete-time LQ regulator problem in the following manner. The discrete-time state equation (7) is obtained by a standard discretization with zero-order hold. Thus,

$$x_k = x_c(kh) \tag{12}$$

$$u_k = u_c(kh) \tag{13}$$

$$A = \exp(A_c h) \tag{14}$$

$$B = \int_0^h \exp(A_c \tau) B_c d\tau \tag{15}$$

The weighting matrices are converted with the following relationship.

$$\tilde{A}_c := \begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix} \tag{16}$$

$$\tilde{Q}_c := \begin{bmatrix} Q_c & 0 \\ 0 & R_c \end{bmatrix} \tag{17}$$

$$\tilde{Q} = \int_0^h \exp(\tilde{A}_c \tau)^T \tilde{Q}_c \exp(\tilde{A}_c \tau) d\tau \tag{18}$$

$$\tilde{Q} := \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \tag{19}$$

The numerical computation of the above conversion can be performed by an existing software package. The optimal solution to the sampled-data LQ regulator problem is given by

$$u_c(t) = F x_c(kh),\ kh \le t < (k+1)h \tag{20}$$

where the optimal feedback gain F is given by (10). Thus the optimal feedback gain F can be obtained for a specific choice of the weighting matrices Q_c and R_c. The present paper deals with a case where the weighting matrices are regarded as the tuning parameters during the control system design procedure.

3. SUCCESSIVE POLE SHIFTING

An iterative design method using the LQ regulator theory is treated in this section. In the following, the coefficient matrix A_c of the controlled plant is assumed to have distinct eigenvalues, for simplicity.

First, a specific mode in A_c is chosen for pole shifting, and either a real pole or a pair of complex conjugate poles corresponding to the specified mode is extracted by applying a similarity transform. Namely, a nonsingular matrix $T \in \mathrm{R}^{n\times n}$ with the first one or two columns corresponding to the eigenspace of the specified mode can be chosen such that

$$T^{-1} A_c T = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \tag{21}$$

$$T^{-1} B_c = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{22}$$

where either $A_{11} \in \mathrm{R}^{1\times 1}$ or $A_{11} \in \mathrm{R}^{2\times 2}$ represents the specified mode.

It should be noted that the weighting matrix Q_c of the quadratic performance index J_c is transformed as well. Let Q_{11} be a positive semidefinite matrix with the same size as A_{11}, and let the whole weighting matrix Q_c be chosen as

$$Q_c = (T^{-1})^T \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \tag{23}$$

Then the eigenvalue of A_{11} can be shifted without altering any other eigenvalues of A_c (Solheim, 1972). The choice of Q_{11} is assumed to satisfy the observability of the pair $({Q_{11}}^{1/2}, A_{11})$ in addition to being positive semidefinite. Thus, a particular selection of $Q_{11} = 0$ is excluded.

Now, a new matrix V defined by

$$V := B_c {R_c}^{-1} {B_c}^T \tag{24}$$

is introduced. Applying the above transformation matrix T to V yields $T^{-1} V (T^{-1})^T$, which is partitioned as

$$T^{-1} V (T^{-1})^T = \begin{bmatrix} V_{11} & V_{12} \\ {V_{12}}^T & V_{22} \end{bmatrix} \tag{25}$$

where V_{11} has the same size as A_{11} and Q_{11}.

Thus, a pole shifting for the specified mode is accomplished by considering the LQ regulator problem for the first or second order system represented by the matrices A_{11}, Q_{11}, and V_{11}, and calculating the corresponding optimal feedback gain F_{11}. Then, the optimal solution is converted to that for the original system.

Next, another mode in the original system is considered for pole shifting, and the similarity transformation for extracting the mode is performed. The similarity transformation matrix needs to be calculated again, since the coefficient matrix A_c has been altered by the application of the state feedback. After the second mode is extracted, the same procedure is applied to the smaller system for desired pole shifting. This is repeated until all the modes have been treated. Furthermore, successive pole shifting can be accumulated in a way such that the optimality of the overall closed-loop system is guaranteed. The design procedure is summarized in the following.

Step 1 Choose a transformation matrix T to extract the partitioned matrices A_{11} and V_{11} which represent a real pole or a pair of complex conjugate poles to be shifted.

Step 2 Find a weighting matrix Q_{11} with which a desired pole positioning is accomplished.

Step 3 Calculate the weighting matrix Q_c and the corresponding optimal feedback gain F_c for the whole system, then form a closed-loop system with F_c.

Step 4 Go back to Step 1, while there are remaining poles to be shifted.

Step 5 Accumulate the matrices Q_c and F_c in each step to obtain the overall weighting matrix and optimal feedback gain which achieve the desired pole positioning.

The above method for continuous-time systems is valid for discrete-time systems as well. However, the optimality of the overall system is not guaranteed as it is, and a modification of the weighting matrix R is required in each step (Amin, 1984).

For sampled-data LQ regulators, the above procedure cannot be directly applied, since the conversion between continuous-time and discrete-time representation of the state equation as well as the performance index is required for each mode to be extracted. The present paper shows a method for achieving successive pole shifting using a first or second order system, at the cost of losing the optimality of the overall system.

4. SUCCESSIVE POLE SHIFTING FOR SAMPLED-DATA LQ REGULATOR

Suppose that a specific mode of the controlled plant has been extracted as in the previous section. Then the block diagonal structure of $T^{-1}A_cT$ in (21) is retained in the corresponding coefficient matrix $T^{-1}AT$ of the discretized system (7), since the relationship in (14) implies

$$\exp\left(T^{-1}A_c hT\right) = T^{-1}AT \tag{26}$$

and the matrices A_c and A share the same eigenvectors in general. Moreover, the matrix $\tilde{A}_c$ in (16) is given by

$$\tilde{A}_c = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & A_{22} & B_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{27}$$

and

$$\begin{aligned} \exp(\tilde{A}_c\tau) = & \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \\ & \times \exp\left(\begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & A_{22} & B_2 \\ 0 & 0 & 0 \end{bmatrix} \tau \right) \\ & \times \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \end{aligned} \tag{28}$$

Furthermore, the selection of the weighting matrix Q_c as given in (23) leads to the similar structure in the weighting matrices of the discretized performance index (8), since the relationship in (18) implies

$$\tilde{Q}_c = \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_c \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{29}$$

and

$$\begin{aligned} \tilde{Q} = & \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}^T \\ & \times \int_0^h \exp\left(\begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & A_{22} & B_2 \\ 0 & 0 & 0 \end{bmatrix} \tau \right)^T \\ & \times \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_c \end{bmatrix} \\ & \times \exp\left(\begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & A_{22} & B_2 \\ 0 & 0 & 0 \end{bmatrix} \tau \right) d\tau \\ & \times \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \end{aligned} \tag{30}$$

It should be noted that the block diagonal structure of the weighting matrix $\tilde{Q}_c$ is not retained in the discretized version of the weighting matrix $\tilde{Q}$, thus requiring the weighting matrix S in $\tilde{Q}$ for the cross term between x_k and u_k. However, the expansion of (30) reveals that the elements of $\tilde{Q}$ take the following form:

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{31}$$

$$S = \begin{bmatrix} S_1 \\ 0 \end{bmatrix} \tag{32}$$

Thus, the similarity transformation matrix T is valid for extracting the specific mode of the plant in both continuous-time representation and discrete-time representation. Finally, it can be concluded that the choice of the continuous-time weighting matrix Q_c in the form of (23) guarantees the manipulation of the specific mode in the discrete-time representation without altering any other ones.

Now that the shifting of a single real pole or a pair of complex conjugate poles is achieved, the iterative procedure is considered. There is a substantial difficulty in handling the successive pole shifting method of the sampled-data LQ regulators, compared to similar methods for purely continuous-time or discrete-time LQ regulators. This is due to the fact that the controlled plant in the intermediate stage of successive pole shifting is already a sampled-data control system, whereas it is a purely continuous-time plant in the initial stage. Although the behavior of the sampled-data control system at the sampling instants can be described as a purely discrete-time system, it is not sufficient when the weighting matrix of the continuous-time performance index is considered as a tuning parameter for pole shifting. In order to alleviate this difficulty, the discrete-time system

$$x_{k+1} = (A + BF)x_k + Bu_k \tag{33}$$

in the intermediate stage of successive pole shifting is converted to an equivalent continuous-time system

$$\dot{x}_c(t) = \bar{A}_c x_c(t) + \bar{B}_c u(t) \tag{34}$$

where

$$A + BF = \exp(\bar{A}_c h) \tag{35}$$

$$B = \int_0^h \exp(\bar{A}_c \tau)\bar{B}_c d\tau \tag{36}$$

and the corresponding continuous-time performance index

$$\bar{J}_c = \int_0^\infty \{x_c(t)^T \bar{Q}_c x_c(t) + u_c(t)^T \bar{R}_c u_c(t)\}dt \tag{37}$$

is considered.

Then the modal decomposition is applied to the coefficient matrix $\bar{A}_c$, using a new transformation matrix $\bar{T}$ for extracting a next mode as in (21). After choosing the weighting matrix $\bar{Q}_c$ and $\bar{R}_c$ with the structure of $\bar{Q}_c$ similar to that of Q_c in (23), the weighting matrix of the equivalent discrete-time system is calculated by applying the conversion as in (18).

The optimal feedback gain for the discrete-time LQ regulator is capable of shifting the specified mode without altering other ones, and the weighting matrices of the performance index is indeed derived from a sampled-data LQ regulator problem. This procedure can be repeated as many times as desired, and the successive pole shifting using the sampled-data LQ regulators is accomplished. It should be noted, however, that the final result may not correspond to the solution of a sampled-data LQ regulator problem which shifts all the specified modes in a single step. Thus, the overall optimality cannot be guaranteed.

5. CONCLUSIONS

A design of sampled-data system by successive pole shifting is considered, and a tuning method using the sampled-data LQ regulator is presented. A single real pole or a pair of complex conjugate poles can be shifted at a time, without altering other modes, using the sampled-data LQ regulators. The design method is more involved than a similar method based on the discrete-time LQ regulators, and it may be of interest to compare the capabilities of both methods, especially in terms of the region of the closed-loop poles accomplished.

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