ON HYBRID STOCHASTIC SINGULAR CONTROL

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Abstract: We consider stochastic hybrid systems controlled by a bounded variation process leading to a singular control. A typical example is considered (i.e., characterization and properties of the value function) and a more general class is discussed in a paper to appear. *Copyright© 2005 IFAC.*

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1. INTRODUCTION

Let us consider the following production/inventory system with only one product

$$\mathrm{d}x(t) = -b\mathrm{d}t + y(t)\mathrm{d}t + \sqrt{2}\mathrm{d}w(t), \tag{1}$$

where $x(t)$ is the inventory level, $b > 0$ is the average demand rate, $y(t)$ is the production rate and $w = w(t)$ is a given Wiener process used as the random disturbance as the time $t \geq 0$ evolves. The production system is subject to failures, and when there is not failure the rate can be controlled, i.e., if $y(0) = y$ and the rate is changed to y' there is a cost proportional to the change, namely, $c|y' - y|$, for a constant $c > 0$. Failures can occurs at times exponentially distributed with parameter λ, and then repair takes a random time which is also exponentially distributed with parameter μ. Times between failures and repairs are mutually independent, and also independent of the driving process w. We denote by $i(t)$ the variable indicating the failure/repair state, i.e., $i(t) = 1$ if the production is working (so, if a failure has occurs, it has been repaired) and $i = 0$ if the production fails (and therefore the system enter in a repair mode). Then, when $i(t) = 1$

$$\begin{cases} y(t) = y + v(t) & \text{as long as } \ i(t) = 1, \\ y(t) = 0 & \text{if } i(t) = 0, \end{cases} \tag{2}$$

i.e., in general

$$y(t) = i(t)\big(y + v(t)\big),$$

with $v(t)$ a process of bounded variation (by coordinates), the control of the production rate, and y in $\mathbb{R}^{d_2}$.

In the case of an infinite horizon discounted problem, one could consider minimizing the total cost

$$\begin{cases} J_{xyi}(v) = \mathbb{E}_{xyi}\Big\{ \int_0^\infty e^{-\alpha t} f(x(t))\mathrm{d}t + \\ \qquad\qquad + \int_0^\infty e^{-\alpha t} c i(t) \mathrm{d}|v(t)| \Big\}, \end{cases} \tag{3}$$

where $\mathrm{d}|v(t)|$ denotes the variation of $v(\cdot)$ on $[0, t]$, and more precise assumptions will be given later. The additive control in the equation for y and J above allows to see the previous problem as a so-called *singular control* in the sense of (Chow *et al.*, 1985) and references therein.

A formal dynamic programming argument can be used to obtain the following Hamilton-Jacobi-Bellman (HJB) equation for the optimal value function $u(x, y, i)$ denoted by $u_0(x, y) = u(x, y, 0)$ and $u_1(x, y) = u(x, y, 1)$ below

$$\begin{cases} L_1^y u_1 + \lambda(u_1 - u_0) \leq f, \qquad |\partial_y u_1| \leq c, \\ \big(L_1^y u_1 + \lambda(u_1 - u_0) - f\big)\big(|\partial_y u_1| - c\big) = 0, \end{cases} \tag{4}$$

with $L_0 u_0 + \mu(u_0 - u_1(\cdot,0)) = f$, and for every (x,y) in $\mathbb{R} \times [0,Y]$, some given $Y > 0$, and where L_0 and L_1^y are the following differential operators in the variable x,

$$\begin{cases} L_0\varphi(x) = -\partial_x^2\varphi(x) + b\partial_x\varphi(x) + \alpha\varphi(x), \\ L_1^y\varphi(x) = -\partial_x^2\varphi(x) + \\ \qquad +(b-y)\partial_x\varphi(x) + \alpha\varphi(x), \end{cases} \tag{5}$$

and assuming that after a repair, the machine restart with a rate $y = 0$, and therefore, u_0 does not depend on y in $[0,Y]$, for some constant Y previously selected.

Comparing (4) with the HJB equations obtained in (Chow *et al.*, 1985) (among others), one can see that we are in the *optimal corrections* case, with a state (x,y) for which the second component is degenerate. Note that one could also consider the case where the repair is controlled, i.e., a second control variable would act on $i(t)$.

This control problem is a simple particular case of hybrid system with singular control. Hybrid control refer to systems where there are both continuous and discrete dynamics, as well as continuous and discrete controls. Since more than ten years, many works have been devoted to hybrid control systems. One can refer to (Antsaklis *et al.*, 1993), (Bensoussan and Menaldi, 1997; Bensoussan and Menaldi, 2000), (Branicky *et al.*, 1998), (Menaldi, 2001) among others. On the other hand, the singular control of stochastic systems relates to situation where the effect of the control can lead to discontinuous variations of the state, the purely impulsive control being a particular case, for instance, we refer to (Fleming and Soner, 1992) and (Menaldi and Robin, 1983; Menaldi and Robin, 1984) for details on singular control.

2. A CLASS OF HYBRID-SINGULAR PROBLEMS

Taking into account the preliminary example, one could treat a class of hybrid monotone follower (to simplify the presentation) namely:

$$\begin{cases} \mathrm{d}x(t) = g_1(x(t),y(t))\mathrm{d}t + \\ \qquad +\sigma_1(x(t),y(t))\mathrm{d}w_1(t), \\ x(0) = x, \end{cases} \tag{6}$$

where x belongs to $\mathbb{R}^{d_1}$, and a Markov chain $i(t)$ with values in $\{0,1\}$. If $i(0) = 1$ then the sequence of switching of i is denoted by τ_1, τ_1', τ_2, τ_2', $\ldots$, with $\tau_0' = 0$ and τ_1 the first transition from 1 to 0 and so on, but if $i(0) = 0$ then the sequence of switching of θ is denoted by τ_1', τ_2, τ_2', $\ldots$, with $\tau_1 = 0$ and τ_1' the first transition from 0 to 1 and so on. Thus,

$$\begin{cases} \mathrm{d}y(t) = \mathrm{d}\nu(t) + g_2(x(t),y(t))\mathrm{d}s + \\ \qquad +\sigma_2(x(t),y(t))\mathrm{d}w_2(t), \quad t \in (\tau_j', \tau_{j+1}) \\ y(\tau_0') = y, \quad y(t) = 0, \quad t \in [\tau_j, \tau_j'],\ j \geq 1, \end{cases} \tag{7}$$

and minimize

$$J_{xyi}(\nu) = \mathbb{E}_{xyi}\Big\{ \int_0^\infty e^{-\alpha t} f(x(t),y(t))\mathrm{d}t \Big\}. \tag{8}$$

Note that $y(\tau_0') = y$ is only defined when $i(0) = 1$.

Let us consider the uncontrolled process

$$z(t) = \big(x(t), y(t), i(t)\big) \in \mathbb{R}^{d_1} \times \mathbb{R}_+ \times \{0,1\}$$

and the corresponding semigroup

$$\Phi(t)\varphi(x,y,i) = \mathbb{E}\big\{\varphi\big(x(t),y(t),i(t)\big)\big\}.$$

Also assume that f is uniformly continuous with polynomial growth of degree p with respect to x and y, i.e, for each $\varepsilon > 0$ there exists δ such that $|x - x'| + |y - y'| < \delta$ implies $f(x,y,i) - f(x',y',i)| \leq \varepsilon(1 + |x| + |y|)^p$ for every x, x', y, y', and i. Denote by B_p the space of Borel measurable functions with polynomial growth of degree $p \geq 0$. If

$$u(x,y,i) = \inf_{\nu} \big\{ J_{xyi}(\nu) \big\}$$

then the following result is obtained: the optimal cost u is the maximum solution of the set of relations

$$\begin{aligned} & w \leq e^{-\alpha t}\Phi(t) + \int_0^t e^{-\alpha s}\Phi(s)\mathrm{d}s, \quad \forall t \geq 0, \\ & w(x,y,i) \leq w(x, y+\xi, 1), \quad \forall \xi \geq 0, \\ & L_0 w(x,0,0) + \mu\big(w(x,0,0) - w(x,0,1)\big) = \\ & \qquad\qquad = f(x,0), \quad \forall x, \end{aligned}$$

where L_0 is the generator corresponding to (5) with $y = 0$ as in (6).

In order to establish this result, we can approximate the initial problem by an impulsive control problem like in (Menaldi and Robin, 1983), i.e.,

$$\begin{aligned} & u_\varepsilon(x,y,i) = \inf\big\{ J_{xyi}(\nu) : \nu \text{ impulse} \big\} \\ & J_{xyi}^\varepsilon(\nu) = \mathbb{E}_{xyi}\Big\{ \int_0^\infty e^{-\alpha t} f(x(t),y(t))\mathrm{d}t + \\ & \qquad\qquad + \varepsilon \sum_k e^{-\alpha\tau_k} \Big\}. \end{aligned}$$

Certainly, besides the above semigroup formulation one may use the viscosity approach, and depending on the assumptions, one may have to revise the comparison arguments for viscosity solutions to be applied to this situation.

3. RESULTS

Let us continue with the study of the example in Section 1, with $f(x) = x^2$. The optimal cost function (u_1, u_0) is the maximum solution of the HJB conditions:

$$
\begin{aligned}
&L_1^y u_1(x,y) + \lambda(u_1(x,y) - u_0(x)) \le x^2,\\
&\partial_y u_1(x,y) \le M u_1(x,y),\\
&L_0 u_0(x) + \mu(u_0(x) - u_1(x,0)) = x^2,
\end{aligned}
$$

for every (x,y) in $\mathbb{R} \times [0,Y]$, $Y > 0$, and where L_0 and L_1^y are differential operators in the variable x as in (4), and

$$
M\varphi(y) = \inf\big\{c|y' - y| + \varphi(y') \;: \\ : y' \in [0,Y],\ y' \ne y\big\}.
$$

Clearly, $u_1(x,y) = u(x,y,1)$ and $u_0(x) = u(x,y,0)$, which results independent of the variable y. We obtain the following regularity: u_1 is strictly convex in x, and u_0 and u_1 have locally bounded second derivatives in x, locally bounded first derivatives in y, with $u_i(x,y)(1+x^2)^{-1}$, $\partial_x u_i(x,y)(1+|x|)^{-1}$, $\partial_y u_i(x,y)(1+|x|)^{-1}$, i=1,2, and $\partial_x^2 u_1(x,y)$ are bounded in $\mathbb{R}\times[0,Y]$. Moreover, $|\partial_x u_1(x,y)| \to \infty$ as $|x| \to \infty$.

About the optimal policy, we have

Theorem 1. There exist $R > 0$ sufficiently large such that for every y in $[0,Y]$ we have $\partial_y u_1(x,y) = c$ if $x \ge R$ and $\partial_y u_1(x,y) = -c$ if $x \le -R$. Thus for $x > R$ (or $x < -R$) it is optimal to jump immediately from y to 0 (or M). Moreover, the *continuation set* $\{(x,y) : |\partial_y u_1(x,y)| < c\}$ is nonempty.

Proof. For a given $R > 0$ to be chosen later, define (w_1, w_0) as follows

$$
\begin{aligned}
&w_1(x,y) = u_1(x,0) + cy, \quad \forall x \ge R,\ \forall y \in [0,Y],\\
&w_1(x,y) = u_1(x,Y) + c(Y-y),\\
&\qquad\qquad\qquad\qquad \forall x \le -R,\ \forall y \in [0,Y],\\
&w_0(x) = u_0(x), \quad \forall x \in (-\infty,-R] \cup [R,+\infty),\\
&L_1^y w_1(x,y) + \lambda(w_1(x,y) - u_0(x)) \le x^2,\\
&\qquad\qquad\qquad\qquad \forall x \in [-R,R],\\
&|\partial_y w_1(x,y)| \le c, \quad \forall x \in [-R,R]\\
&L_0 w_0(x) + \mu(w_0(x) - u_1(x,0)) = x^2,\\
&\qquad\qquad\qquad\qquad \forall x \in (-R,R).
\end{aligned}
$$

Thus $w_1(x,y)$ is Lipschitz in y across $y = 0$ and $y = Y$, and $w_0(x)$ is Lipschitz across $x = -R$ and $x = R$, and for any y in $[0,Y]$ we have

$$
\begin{aligned}
&L_1^y w_1(x,y) = L_1^0 u_1(x,0) - y\partial_x u_1(x,0) + \lambda c y,\\
&\qquad\qquad\qquad\qquad \forall x \in [R,+\infty),\\
&L_1^y w_1(x,y) = L_1^Y u_1(x,Y) +\\
&\qquad + (Y-y)\partial_x u_1(x,Y) + \lambda c(Y-y),\\
&\qquad\qquad\qquad\qquad \forall x \in (-\infty,R],
\end{aligned}
$$

Since $L_1^0 u_1(x,0) + \lambda(u_1(x,0) - u_0(x)) \le x^2$ and $L_1^Y u_1(x,Y) + \lambda(u_1(x,Y) - u_0(x)) \le x^2$ we will have $L_1^y w_1(x,y) + \lambda(w_1(x,y) - w_0(x)) \le x^2$ for $|x| > R$ and $0 \le y \le Y$ if $-y\partial_x u_1(x,0) + \lambda c y \le 0$ for $x > R$ and $(Y-y)\partial_x u_1(x,Y) + \lambda c(Y-y) \le 0$ for $x < -R$. Because

$$
\begin{aligned}
&\partial_x u_1(x,0) \to +\infty \ \text{ as } \ x \to +\infty,\\
&\partial_x u_1(x,Y) \to -\infty \ \text{ as } \ x \to -\infty,
\end{aligned}
$$

we can choose R so large that $\partial_x u_1(x,0) \ge \lambda c$ for any $x > R$ and $\partial_x u_1(x,Y) \le \lambda c$ for any $x < -R$. Hence

$$
\begin{aligned}
&L_1^y w_1(x,y) + \lambda(w_1(x,y) - w_0(x)) \le x^2,\\
&\qquad \forall x \in (-\infty,-R] \cup [R,+\infty),\ \forall y \in [0,Y],
\end{aligned}
$$

which proves that (w_1, w_0) is a solution of the HJB equation.

Therefore $w_i \le u_i$, for $i = 0,1$ because (u_1,u_0) is the maximum solution. On the other hand, the condition $|\partial_y u_1| \le c$ yields (by definition) $u_1(x,y) \le u_1(x,0) + cy = w_1(x,y)$ for $x > R$ and $u_1(x,y) \le u_1(x,0) + c(Y-y) = w_1(x,y)$ for $x < -R$, which proves the desired result.

Now since y belongs to $[0,Y]$, the convexity in y is not sufficient to show that the *continuation set* $\{(x,y) : |\partial_y u_1(x,y)| < c\}$ is nonempty. Indeed we argue as follows.

Let x_m be a minimizer of $u_1(x,0)$, i.e., x_m in *argmin*$\{u_1(x,0)\}$ and consider the open interval $B_\eta = \{x : |x - x_m| < \eta\}$ for some $\eta > 0$ to be chosen later. Define (u_1^η, u_0^η) by modifying (u_1,u_0) inside the interval B_η as follows

$$
\begin{aligned}
&L_1^y u_1^\eta(x,y) + \lambda(u_1^\eta(x,y) - u_0^\eta(x)) = x^2\\
&\qquad\qquad\qquad\qquad \forall x \in B_\eta,\ \forall y,\\
&L_0 u_0^\eta(x,y) + \mu(u_0^\eta(x) - u_1^\eta(x,0)) = x^2\\
&\qquad\qquad\qquad\qquad \forall x \in B_\eta,\\
&(u_1^\eta, u_0^\eta) = (u_1,u_0) \ \text{ outside } \ B_\eta.
\end{aligned}
$$

For any $\varepsilon > 0$ we can choose $\eta > 0$ such that $|\partial_x u_1^\eta(x,0)| < \varepsilon$ for any x in B_η. If we set $w^\eta = \partial_y u_1^\eta(x,y)$ then

$$
\begin{aligned}
&L_1^y w^\eta + \lambda w^\eta = \partial_x u_1^\eta \ \text{ in } \ B_\eta,\\
&w^\eta = \partial_y u_1^\eta \ \text{ on } \ \partial B_\eta,
\end{aligned}
$$

which yields the representation

$$
\begin{aligned}
&w^\eta(x,y) = \mathbb{E}_{xy}\Big\{\int_0^\tau e^{-(\alpha+\lambda)s}\partial_x u_1^\eta(X_s,y)\mathrm{d}s +\\
&\qquad + e^{-(\alpha+\lambda)s} w^\eta(X_\tau,y)\Big\},\\
&\tau = \inf\{s \;:\; X_s \notin B^\eta\}.
\end{aligned}
$$

Since $w^\eta(X_\tau,y)| \le c$ we have

$$
\begin{aligned}
|w^\eta(x,y)| \le \frac{\varepsilon}{\alpha+\lambda}\big(1 - \mathbb{E}\{e^{-(\alpha+\lambda)\tau}\}\big) +\\
+ c\mathbb{E}\{e^{-(\alpha+\lambda)\tau}\}.
\end{aligned}
$$

Hence, by selection ε sufficiently small so that $\varepsilon < c(\alpha+\lambda)$ we have the strict inequality. This proves that (u_1^η, u_0^η) satisfies the HJB inequalities, and then $(u_1^\eta, u_0^\eta) \le (u_1, u_0)$, which is the maximum solution. On the other hand, the maximum principle applied on the couple (u_1^η, u_0^η) and (u_1, u_0) in the interval B^η yields $(u_1^\eta, u_0^\eta) \ge (u_1, u_0)$. Therefore $|\partial_y u_1(x,y)| = |\partial_y u_1^\eta(x,y)| < c$ for any x in B_η and any y in $[0,Y]$. $\square$

The functions

$$\begin{aligned}\overline{z}(x) &= \inf\{y \,:\, \partial_y u_1(x,y) = c\},\\ \underline{z}(x) &= \sup\{y \,:\, \partial_y u_1(x,y) = -c\},\end{aligned}$$

are the free boundary of the continuation region $D = \{(x,y) : |\partial) y u_1(x,y)| < c\}$. We know that for some constant $R > 0$ we have $\underline{z}(x) = 0$ if $x > R$ and $\overline{z}(x) = Y$ if $x < -R$, and D is bounded in $\mathbb{R} \times [0,Y]$.

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