A STABLE RECURSIVE FILTER FOR STATE ESTIMATION OF LINEAR MODELS IN THE PRESENCE OF BOUNDED DISTURBANCES

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Abstract: This contribution proposes a robust recursive algorithm for state estimation of linear multi-output systems with unknown but bounded disturbances corrupting both the state and measurement vectors. A novel approach based on state bounding techniques is presented. The proposed algorithm can be decomposed into two steps : *time updating* and *observation updating* that uses a switching estimation Kalman-like gain matrix. Particular emphasis will be given to the design of a weighting factor that ensures consistency of the estimated state vectors with the input-output data and the noise constraints and that enforces convergence properties.

Keywords: State estimation, Multivariable linear systems, Recursive method, Bounded noise, Ellipsoids, Stability.

1. INTRODUCTION

State estimation of stochastic dynamical systems has been extensively studied during the last decades and the problem is usually solved by assuming white and Gaussian noises on model and measurements (Kalman filter). However, when the statistical properties of the noises are unknown or not satisfied, an alternative approach consists in considering that only bounds on the possible magnitude of the disturbances are available, the so-called set-membership estimation was first introduced by Schweppe (Schweppe, 1968; Schlaepfer and Schweppe, 1972) using ellipsoidal bounding techniques. The aim is to determine a set of state estimate vectors compatible with the bounds on the process disturbance and measurement noise. Since these pioneer works, a vast literature is dedicated to this subject in the context of parameter identification (Fogel and Huang, 1982; Canudas-De-Wit and Carrillo, 1990) and (Becis-Aubry *et al.*, 2003; Becis-Aubry *et al.*, 2004) or state estimation (Norton, 1994; Norton, 1995; Kapoor *et al.*, 1996; Maksarov and Norton, 1996*a*; Durieu *et al.*, 2001). The most of works of literature in the set-membership estimation framework aim to design the “smallest” set that contains all the possible values of the quantity to estimate and that is consistent with the model equations, input/output data, and the noises bounds. To our knowledge, very few works have been developed with a concern for the stability of some “state estimator” vector. Our goal in this paper is twofold : first, we aim to find a “guaranteed” estimator – *i.e.*, an estimator consistent with the model equations, input/output data, and the noises bounds – in other words, *acceptable* estimator ; secondly the estimated vector to be designed must *converge* to the true state vector and this is the main contribution of this paper.

As the Kalman filter, the algorithm proposed here can be decomposed into two steps : *time updating* and *observation updating*. These two steps will be presented successively, next some properties of the designed algorithm will be established and finally, the effectiveness of the algorithm will be demonstrated through a numerical example.

2. PROBLEM FORMULATION

Notations : $\mathcal{E}(c,P) := \{x \in \mathbb{R}^s \mid (x-c)^T P^{-1}(x-c) \leq 1\}$ is an ellipsoid in $\mathbb{R}^s$ ($s \in \mathbb{N}^*$), where $\boldsymbol{c} \in \mathbb{R}^s$ is its center and $P \in \mathbb{R}^{s\times s}$ is a symmetric positive definite (SPD) matrix that defines its shape, size and orientation in the $\mathbb{R}^s$ space; $\|\boldsymbol{x}\| = (\boldsymbol{x}^T\boldsymbol{x})^{\frac{1}{2}}$ is the Euclidean norm of the vector $\boldsymbol{x}$ and $\|\boldsymbol{x}\|_W = (\boldsymbol{x}^T W\boldsymbol{x})^{\frac{1}{2}}$ is its weighted Euclidean norm (W is a SPD matrix of appropriate dimension).

Let us consider the following discrete-time linear [1] system written in the state space :

$$\boldsymbol{x_k^*} = \Phi_{k-1}\boldsymbol{x_{k-1}^*} + H_{k-1}\boldsymbol{u_{k-1}} + \boldsymbol{w_{k-1}} \quad (1a)$$
$$\boldsymbol{y_k} = F_k\boldsymbol{x_k^*} + \boldsymbol{v_k} \quad (1b)$$

where $\boldsymbol{x_k^*} \in \mathbb{R}^n$ is the unknown state vector to be estimated, $\boldsymbol{u_{k-1}} \in \mathbb{R}^m$ is a known control vector, $\boldsymbol{y_k} \in \mathbb{R}^p$ is a measurable system output vector, $\Phi_{k-1} \in \mathbb{R}^{n\times n}$, $H_k \in \mathbb{R}^{n\times m}$ and $F_k \in \mathbb{R}^{p\times n}$ are the state, input and output matrices, where Φ_{k-1} and F_k are of full rank and $\boldsymbol{w_k} \in \mathbb{R}^n$ and $\boldsymbol{v_k} \in \mathbb{R}^p$ are unobservable bounded noise vectors with unknown statistical characteristics that may include the modeling inaccuracies, the discretization errors or the computer round-off errors. $\boldsymbol{v_k}$ can represent the measurement noise and $\boldsymbol{w_{k-1}}$ can be viewed as unknown but bounded inputs. The only properties verified by $\boldsymbol{v_k}$ and $\boldsymbol{w_{k-1}}$ are

$$\boldsymbol{w_k} \in \mathcal{E}(\boldsymbol{0}, W_k) \Leftrightarrow \boldsymbol{v_k}^T W_k^{-1}\boldsymbol{v_k} \leq 1,\ \forall k \in \mathbb{N}^* \quad (2a)$$
$$\boldsymbol{v_k} \in \mathcal{E}(\boldsymbol{0}, V_k) \Leftrightarrow \boldsymbol{v_k}^T V_k^{-1}\boldsymbol{v_k} \leq 1,\ \forall k \in \mathbb{N}^*. \quad (2b)$$

Without loss of generality, we assume that $\boldsymbol{u_k} = \boldsymbol{0}$, $\forall k \in \mathbb{N}^*$ in (1a). If this is not the case, it suffices to add $H_{k-1}\boldsymbol{u_{k-1}}$ to the expression of the state prediction $\boldsymbol{\hat{x}_{k/k-1}}$ given by (3). Let $\boldsymbol{\hat{x}_k} \in \mathbb{R}^n$ be the estimate of $\boldsymbol{x_k^*}$. Our aim in the sequel is to design an estimation algorithm for the system (1a)–(1b) such that

- ***i.*** the set that contains all possible values of the true state vector $\boldsymbol{x_k^*}$ is quantified at each instant k;
- ***ii.*** the (*a posteriori*) output error vector $y_k - F_k\boldsymbol{\hat{x}_k}$ is *acceptable*, *i.e.*, it remains in the interior of the ellipsoid (2b) enclosing all possible values of the disturbance vectors $\boldsymbol{v_k}$, *i.e.*
  $(\boldsymbol{y_k} - F_k\boldsymbol{\hat{x}_k})^T V_k^{-1}(\boldsymbol{y_k} - F_k\boldsymbol{\hat{x}_k}) \leq 1,\ \forall k \in \mathbb{N}^*$;
- ***iii.*** the estimator is stable and convergent.

In the bounded error estimation context, each estimated state vector for which the output error is *acceptable* is the best estimate one can have.

[1] in the sense that its output vector is affine with respect to its state vector which dynamics are linear.

It is assumed that the state vector $\boldsymbol{x_k^*}$ belongs to a *known* ellipsoid $\mathcal{E}(\boldsymbol{\hat{x}_0}, \sigma_0^2 P_0)$, where $\boldsymbol{\hat{x}_0}$ is the initial estimate of $\boldsymbol{x_0^*}$, P_0 is a SPD matrix and σ_0 is a positive scalar. At the sampling time k, the ellipsoid containing all presumed values of the true state vector $\boldsymbol{x_k^*}$ is $\mathcal{E}(\boldsymbol{\hat{x}_k}, \sigma_k^2 P_k)$, where $\sigma_k > 0$, $P_k > 0$, $P_k^T = P_k$ and the center of the ellipsoid is the state estimate vector $\boldsymbol{\hat{x}_k}$. Note that the eigenvalues of the shape matrix, $\sigma_k^2\lambda_i(P_k)_{i=1,\ldots,n}$ correspond to the squared semi-lengths of its axes, the directions of which are defined by the associated (orthogonal) eigenvectors.

In what follows, we have to determine a progression law for the ellipsoid $\mathcal{E}_k := \mathcal{E}(\boldsymbol{\hat{x}_k}, \sigma_k^2 P_k)$ such that the aims ***i.–iii.*** are fulfilled.

3. TIME UPDATE

At this stage, we compute, at each step k, the ellipsoid $\mathcal{E}_{k/k-1}$ containing the "reach set" from $\mathcal{E}_{k-1}$ of the current state vector $\boldsymbol{x_k^*}$, that evolves obeying to the plant dynamics described by (1a) and affected by the unknown noise vector $\boldsymbol{w_{k-1}}$. This is done by performing the vector sum of the ellipsoid $\mathcal{E}(\boldsymbol{0}, W_{k-1})$ and a linear transformation (of matrix F_k) of $\mathcal{E}_{k-1}$:

$$\mathcal{E}_{k/k-1} \supseteq \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \Phi_{k-1}\boldsymbol{x_1} + \boldsymbol{x_2},\ \boldsymbol{x_1} \in \mathcal{E}_{k-1},\ \boldsymbol{x_2} \in \mathcal{E}(\boldsymbol{0}, W_{k-1})\}.$$

Lemma 1. Let $\boldsymbol{x_{k-1}^*} \in \mathcal{E}(\boldsymbol{\hat{x}_{k-1}}, \sigma_{k-1}^2 P_{k-1})$ obeying to (1a) where $\boldsymbol{w_{k-1}} \in \mathcal{E}(\boldsymbol{0}, W_{k-1})$ and let

$$\boldsymbol{\hat{x}_{k/k-1}} = \Phi_{k-1}\boldsymbol{\hat{x}_{k-1}}, \quad (3)$$
$$P_{k/k-1} = \frac{\Phi_{k-1}P_{k-1}\Phi_{k-1}^T}{1-\rho} + \frac{W_{k-1}}{\rho\sigma_{k-1}^2}, 0 < \rho < 1, \quad (4)$$
$$\sigma_{k/k-1}^2 = \sigma_{k-1}^2\,; \quad (5)$$

then

$\forall \rho \in]0,1[, \boldsymbol{x_k^*} \in \mathcal{E}(\boldsymbol{\hat{x}}_{k/k-1}, \sigma_{k/k-1}^2 P_{k/k-1}) = \mathcal{E}_{k/k-1}$. ■

Proof. At time $k-1$,

$$\mathcal{E}_{k-1} = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{\hat{x}_{k-1}}\|^2_{P_{k-1}^{-1}} \leq \sigma_{k-1}^2\right\}$$
$$= \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{\hat{x}_{k-1}} + \sigma_{k-1}P_{k-1}^{\frac{1}{2}}\boldsymbol{y},\ \boldsymbol{y} \in \mathcal{B}_n\right\};$$

we have

$$\boldsymbol{x_{k-1}^*} \in \mathcal{E}_{k-1} \Leftrightarrow \boldsymbol{x_{k-1}^*} = \boldsymbol{\hat{x}_{k-1}} + \sigma_{k-1}P_{k-1}^{\frac{1}{2}}\boldsymbol{y},\ \boldsymbol{y} \in \mathcal{B}_n$$
$$\Leftrightarrow \Phi_{k-1}\boldsymbol{x_{k-1}^*} = \Phi_{k-1}\boldsymbol{\hat{x}_{k-1}} + \sigma_{k-1}\Phi_{k-1}P_{k-1}^{\frac{1}{2}}\boldsymbol{z},\ \boldsymbol{z} \in \mathcal{B}_n$$
$$\Leftrightarrow \Phi_{k-1}\boldsymbol{x_{k-1}^*} = \Phi_{k-1}\boldsymbol{\hat{x}_{k-1}} + \sigma_{k-1}\Phi_{k-1}P_{k-1}^{\frac{1}{2}}\boldsymbol{z},\ \boldsymbol{z} \in \mathcal{B}_n$$
$$\Leftrightarrow \Phi_{k-1}\boldsymbol{x_{k-1}^*} \in \mathcal{E}(\Phi_{k-1}\boldsymbol{\hat{x}_{k-1}}, \sigma_{k-1}^2\Phi_{k-1}P_{k-1}\Phi_{k-1}^T) \quad (6)$$

where $\mathcal{B}_n$ is a unit ball in $\mathbb{R}^n$ centered at $\boldsymbol{0}$.

Considering (6) and the fact that $\boldsymbol{w_{k-1}} \in \mathcal{E}(\boldsymbol{0}, W_{k-1})$, then applying the result that gives the expression of the ellipsoid that contains the sum of two ellipsoids (Schweppe, 1973), it can be deduced that

$\forall \rho \in]0,1[, (\Phi_{k-1}\boldsymbol{x^*_{k-1}} + \boldsymbol{w_{k-1}}) \in$
$\mathcal{E}\left(\Phi_{k-1}\boldsymbol{\hat{x}_{k-1}}, \frac{\sigma^2_{k-1}}{1-\rho}\Phi_{k-1}P_{k-1}\Phi^T_{k-1} + \frac{1}{\rho}W_{k-1}\right)$ (7)
(7) $\Leftrightarrow \boldsymbol{x^*_k} \in \mathcal{E}(\boldsymbol{\hat{x}_{k/k-1}}, \sigma^2_{k/k-1}P_{k/k-1}) = \mathcal{E}_{k/k-1}$.□

The optimal value of ρ is the one that minimizes either the volume of the ellipsoid $\mathcal{E}_{k/k-1}$ (*i.e.*, the determinant of $\sigma^2_k P_{k/k-1}$) or the squared sum of its axes lengths (*i.e.*, the trace of $\sigma^2_k P_{k/k-1}$). These values and the methods of their obtention are given in details in the excellent paper (Maksarov and Norton, 1996*a*).

4. OBSERVATION UPDATE

The observation equation (1b) and the inequality (2b) define an other bounding set for the vector $\boldsymbol{x^*_k}$. Indeed, it is clear that $\boldsymbol{x^*_k} \in \mathcal{S}_k$, where $\mathcal{S}_k := \left\{\boldsymbol{x} \in \mathbb{R}^n | (\boldsymbol{y_k} - F_k\boldsymbol{x})^T V_k^{-1}(\boldsymbol{y_k} - F_k\boldsymbol{x}) \le 1\right\}$. The aim of the algorithm to be presented is to compute recursively the state estimate vector $\boldsymbol{\hat{x}_k}$, the SPD matrix P_k and the positive scalar σ^2_k which define the ellipsoid containing the presumed values of $\boldsymbol{x^*_k}$ in light of the current measurements. This is carried out by performing, at each iteration, the intersection between the ellipsoid $\mathcal{E}_{k/k-1}$ (obtained in the previous section) and the set $\mathcal{S}_k$. This intersection does not result, in general, in an ellipsoid and has to be circumscribed by an ellipsoid :

Lemma 2. If

$$\boldsymbol{\hat{x}_k} = \boldsymbol{\hat{x}_{k/k-1}} + K_k\boldsymbol{\delta_k}, \tag{8}$$
$$P_k = (I_n - K_kF_k)\,P_{k/k-1}, \tag{9}$$
$$\sigma^2_k = \sigma^2_{k/k-1} + \omega[1 - \boldsymbol{\delta}^T_k(\omega F_kP_{k/k-1}F^T_k + V_k)^{-1}\boldsymbol{\delta_k}] \tag{10}$$

where $K_k \in \mathbb{R}^{n\times p}$ is the gain matrix of the estimator and $\boldsymbol{\delta_k} \in \mathbb{R}^p$ is the innovation vector :

$$K_k = \omega P_{k/k-1}F^T_k\left(\omega F_kP_{k/k-1}F^T_k + V_k\right)^{-1}, \tag{11}$$
$$\boldsymbol{\delta_k} = \boldsymbol{y_k} - F_k\boldsymbol{\hat{x}_{k/k-1}}\,; \tag{12}$$

then $\forall\omega \in \mathbb{R}_+$,

$$\left(\mathcal{E}(\boldsymbol{\hat{x}_{k/k-1}}, \sigma^2_{k/k-1}P_{k/k-1}) \cap \mathcal{S}_k\right) \subseteq \mathcal{E}(\boldsymbol{\hat{x}_k}, \sigma^2_kP_k).\ \blacksquare$$

This lemma is the summarized version of some propositions enounced in (Becis-Aubry *et al.*, 2003) applied to the state estimation problem.

Now, we are interested in the derivation of the "optimal" value of ω with respect to a criterion to be chosen. Contrary to some algorithms of the literature (Maksarov and Norton, 1996*a*; Durieu *et al.*, 2001) that minimize the size of the ellipsoid $\mathcal{E}(\boldsymbol{\hat{x}_k}, \sigma^2_kP_k)$, the optimal value of ω chosen here is the one that guarantees the stability of the algorithm given by the equations (3),(4),(5) and (8),(9), (10), (11), (12) in the sense of Lyapunov and that minimizes the quantity $\max_{v_k\in\mathcal{E}(0,V_k)} \mathcal{V}_k - \mathcal{V}_{k-1}$, where $\mathcal{V}_k$ is a Lyapunov function of the estimation error $\boldsymbol{x^*_k} - \boldsymbol{\hat{x}_k}$. This amounts to minimize σ^2_k defined in (10) with respect to ω on $\mathbb{R}_+$ and leads to the following result :

Lemma 3. The value of ω that solves $\min\limits_{\omega\in\mathbb{R}_+} \sigma^2_k$ is

$$\omega^*_k = \begin{cases} 0 & \text{if } \|\boldsymbol{\delta_k}\|_{V_k^{-1}} \le 1, \\ \varpi_k & \text{otherwise}\,; \end{cases} \tag{13}$$

where ϖ_k is the *unique* real positive solution of the equation $\beta \sum\limits_{i=1}^{p} \dfrac{\alpha_i^2}{(\gamma_i\omega + 1)^2} = 1$ in the unknown ω, with $\alpha_i = \alpha_{k_i} = \dfrac{\boldsymbol{u_{k_i}}^T\bar{V}_k\boldsymbol{\delta_k}}{\|\boldsymbol{\delta_k}\|_{V_k^{-1}}}$, $\beta = \beta_k = \|\boldsymbol{\delta_k}\|^2_{V_k^{-1}}$ and $\gamma_i = \gamma_{k_i} \in \mathbb{R}_+$, $i \in \{1, 2, \ldots, p\}$ satisfy

$$\det\left(F_kP_{k/k-1}F^T_k - \gamma_{k_i}V_k\right) = 0, \tag{14}$$

where $\boldsymbol{u_{k_i}} \in \mathbb{R}^p$ are such that

$$\bar{V}_kF_kP_{k/k-1}F^T_k\bar{V}^T_k\boldsymbol{u_{k_i}} = \gamma_i\boldsymbol{u_{k_i}} \tag{15}$$

and $\bar{V}_k$ satisfies $\bar{V}^T_k\bar{V}_k = V_k^{-1}$ (*e.g.* $\bar{V}_k = V_k^{-\frac{1}{2}}$). ■

Proof. To prove this lemma we need to state the following proposition (without proof for lack of place) :

Proposition 1. The equation $\beta \sum\limits_{i=1}^{p} \dfrac{\alpha_i^2}{(\gamma_i\omega + 1)^2} = 1$, where $\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$, with $\sum_{i=1}^p \alpha_i^2 = 1$, $\beta \in \mathbb{R}_+$ and $\gamma_1, \gamma_2, \ldots, \gamma_p \in \mathbb{R}^*_+$, has one and only one real strictly positive solution *if and only if* $\beta > 1$. Furthermore, letting $\boldsymbol{\alpha} = (\alpha_1\ \alpha_2 \cdots \alpha_p)$ and $\boldsymbol{\gamma} = (\gamma_1\ \gamma_2 \cdots \gamma_p)$, the function $\mathscr{F}_{\boldsymbol{\alpha},\beta,\boldsymbol{\gamma}} : \mathbb{R}_+ \to \mathbb{R}, \omega \mapsto \mathscr{F}_{\boldsymbol{\alpha},\beta,\boldsymbol{\gamma}}(\omega) = \omega - \omega\beta \sum\limits_{i=1}^{p} \dfrac{\alpha_i^2}{\gamma_i\omega + 1}$ has a global minimum on $\mathbb{R}_+$ for $\omega = \omega^*$ such that $\omega^* = \begin{cases} \varpi \text{ if } \beta > 1, \\ 0 \ \text{ otherwise}\,; \end{cases}$ where $\varpi \in \mathbb{R}^*_+$ is the solution of the equation above. ■

From (10), we have

$$\sigma^2_k = \sigma^2_{k-1} - \omega\boldsymbol{\bar{\delta}_k}^T(\omega X_k + I_p)^{-1}\boldsymbol{\bar{\delta}_k} + \omega \tag{16}$$

where $X_k = \bar{V}_kF_kP_{k-1}F^T_k\bar{V}^T_k$ and $\boldsymbol{\bar{\delta}_k} = \bar{V}_k\boldsymbol{\delta_k}$. As X_k is symmetric, there exists $U_k = \left(\boldsymbol{u_{k_1}} \ldots \boldsymbol{u_{k_p}}\right)$ such that $U_kU^T_k = I_p$, $U^T_kX_kU_k$ is diagonal and $X_k\boldsymbol{u_{k_i}} = \gamma_{k_i}\boldsymbol{u_{k_i}}$ for all $i \in \{1, \ldots, p\}$, where γ_{k_i} and $\boldsymbol{u_{k_i}}$ are the i^{th} eigenvalue (given in (14) and (15)) and the associated eigenvector of X_k. It is obvious that $\boldsymbol{u_{k_i}}^T(\omega^*_kX_k + I_p)^{-1}\boldsymbol{u_{k_i}} = \dfrac{1}{(\omega^*_k\gamma_{k_i} + 1)}$, thus (16) becomes

$$\begin{aligned}\sigma^2_k &= \sigma^2_{k-1} + \omega - \omega\boldsymbol{\bar{\delta}_k}^TU_kU^T_k(\omega X_k + I_p)^{-1}U_kU^T_k\boldsymbol{\bar{\delta}_k} \\ &= \sigma^2_{k-1} + \omega\left(1 - \sum_{i=1}^{p}\frac{\left(\boldsymbol{u_{k_i}}^T\boldsymbol{\bar{\delta}_k}\right)^2}{\omega\gamma_{k_i} + 1}\right) \\ &= \sigma^2_{k-1} + \omega\left(1 - \beta_k\sum_{i=1}^{p}\frac{\alpha^2_{k_i}}{\omega\gamma_{k_i} + 1}\right).\end{aligned} \tag{17}$$

Firstly, as P_{k-1} is symmetric positive definite, as $\bar{V}_k$ is non singular and as F_k is a full row rank matrix, $X_k = \bar{V}_k F_k P_{k-1} F_k^T \bar{V}_k^T$ is SPD so $\gamma_{k_i} > 0$, $\forall i \in \{1, \ldots, p\}$; secondly, as $\sum_{i=1}^{p} \left(\boldsymbol{u_{k_i}}^T \bar{\boldsymbol{\delta}}_{\boldsymbol{k}} \right)^2 = \bar{\boldsymbol{\delta}}_{\boldsymbol{k}}^T U_k U_k^T \bar{\boldsymbol{\delta}}_{\boldsymbol{k}} = \left\| \bar{\boldsymbol{\delta}}_{\boldsymbol{k}} \right\|^2$, we have $\sum_{i=1}^{p} \alpha_{k_i}^2 = \frac{1}{\left\| \bar{\boldsymbol{\delta}}_{\boldsymbol{k}} \right\|^2} \sum_{i=1}^{p} \left(\boldsymbol{u_{k_i}}^T \bar{\boldsymbol{\delta}}_{\boldsymbol{k}} \right)^2 = 1$; and thirdly $\beta_k = \left\| \bar{\boldsymbol{\delta}}_{\boldsymbol{k}} \right\|^2 \geq 0$. Consequently, we can use the function $\mathscr{F}_{\boldsymbol{\alpha}_k, \beta_k, \boldsymbol{\gamma}_k}$ from the *Proposition* 1 to rewrite (17) as $\sigma_k^2 = \sigma_{k-1}^2 + \mathscr{F}_{\boldsymbol{\alpha}_k, \beta_k, \boldsymbol{\gamma}_k}(\omega)$; and we show that

$$\omega_k^* = \arg\left(\min_{\omega \geq 0} \sigma_k^2 \right) = \arg\left(\min_{\omega \geq 0} \mathscr{F}_{\boldsymbol{\alpha}_k, \beta_k, \boldsymbol{\gamma}_k}(\omega) \right).$$

We can therefore use the *Proposition* 1 to deduce the value of ω_k^* and the *Lemma* 3 is thus proved. ❑

Corollary 1. The value of ω that minimizes σ_k^2 is given by (13) where ϖ_k is the *unique* real positive eigenvalue of the matrix $\Xi_k \in \mathbb{R}^{2p \times 2p}$:

$$\Xi_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{\xi_{k_0}}{\xi_{k_{2p}}} & -\frac{\xi_{k_1}}{\xi_{k_{2p}}} & -\frac{\xi_{k_2}}{\xi_{k_{2p}}} & \cdots & -\frac{\xi_{k_{2p-1}}}{\xi_{k_{2p}}} \end{pmatrix}$$

where ξ_{k_i}, $i \in \{0, \ldots, 2p\}$, are the components of

$$\boldsymbol{\xi_k} = \sum_{i=1}^{p} \alpha_{k_i}^2 \boldsymbol{q_{k_1}} * (\boldsymbol{q_{k_2}} \cdots * (\boldsymbol{q_{k_{i-1}}} * (\boldsymbol{q_{k_i}} - \boldsymbol{d_k}) * (\boldsymbol{q_{k_{i+1}}} \cdots * (\boldsymbol{q_{k_{p-1}}} * \boldsymbol{q_{k_p}})))),$$

$\boldsymbol{q_{k_i}} = (\gamma_{k_i}\ 1) * (\gamma_{k_i}\ 1) = \left(\gamma_{k_i}^2\ 2\gamma_{k_i}\ 1\right)$, $\boldsymbol{d_k} = (0\ 0\ \beta_k)$, $*$ is the convolution operation [2] and α_{k_i}, β_k and γ_{k_i} are defined in the *Lemma* 3. ■

Proof. The proof of this corollary is straightforward by considering the *Lemma* 3 and the fact that the roots of a polynomial are the eigenvalues of its companion matrix. ❑

5. ALGORITHM'S PROPERTIES

In this section, we show that the proposed algorithm with $\omega = \omega_k^*$ computed by the aid of *Lemma* 3 and/or *Corollary* 1 *fulfills the expectations* $\boldsymbol{i - iii}$ expressed in the section 2.

Theorem 1. Consider the state estimation algorithm (3)–(5) (where $\rho = \rho_{k-1}$) and (8)–(12) for the model (1a)–(1b). The following propositions are true for all $k \in \mathbb{N}^*$ and for all $0 < \rho < 1$

i. if $\boldsymbol{x_0^*} \in \mathcal{E}(\hat{\boldsymbol{x}}_{\boldsymbol{0}}, \sigma_0^2 P_0)$ then $\boldsymbol{x_k^*} \in \mathcal{E}(\hat{\boldsymbol{x}}_{\boldsymbol{k}}, \sigma_k^2 P_k)$ for all $\omega \in \mathbb{R}_+$;

ii. the worst case weighted norm of the estimation error vector, $\max_{v_k \in \mathcal{E}(\boldsymbol{0}, V_k)} \left\| \boldsymbol{x_k^*} - \hat{\boldsymbol{x}}_{\boldsymbol{k}} \right\|_{P_k^{-1}}$, is minimized with respect to ω at ω_k^* given by the *Lemma* 3 and/or the *Corollary* 1 ;

Furthermore, if $\omega = \omega_k^*$, the algorithm has the following properties

iii. the sequence $(\sigma_k)_{k \in \mathbb{N}^*}$ – which represents an upper bound on the weighted norm of the estimation error vector $\left\| \boldsymbol{x_k^*} - \hat{\boldsymbol{x}}_{\boldsymbol{k}} \right\|_{P_k^{-1}}$ – is decreasing and convergent on $\mathbb{R}_+$;

iv. the *a posteriori* output error vector is always *acceptable*, *i.e.*, $\|\boldsymbol{y_k} - F_k \hat{\boldsymbol{x}}_{\boldsymbol{k}}\|_{V_k^{-1}} \leq 1$.

Moreover, if there exists positive reals a_1, a_2, b_1 and b_2 such that the following inequalities are satisfied for a finite $m \in \mathbb{N}^*$ and for all $k \in \mathbb{N}$:

$$a_1 I_n \geq \sum_{i=k}^{k+m-1} \widetilde{\Phi}_{k+m,i+1} W_i \widetilde{\Phi}_{k+m,i+1}^T \geq a_2 I_n$$

$$b_1 I_n \leq \sum_{i=k}^{k+s_k(m)} \widetilde{\Phi}_{i,k+s_k(m)}^T F_i^T V_i F_i \widetilde{\Phi}_{i,k+s_k(m)} \leq b_2 I_n$$

where

$$\widetilde{\Phi}_{k+j,k} = \widetilde{\Phi}_{k+j,k+j-1} \widetilde{\Phi}_{k+j-1,k+j-2} \cdots \widetilde{\Phi}_{k+1,k},$$
$$\widetilde{\Phi}_{k+1,k} = \Phi_k, \quad \widetilde{\Phi}_{k,k+j} = \widetilde{\Phi}_{k+j,k}^{-1}, \text{ and } \widetilde{\Phi}_{k,k} = I_n$$

and $s_k(m)$ is such that [3]

$$\operatorname{Card} \mathcal{N}(\omega_k^*, \omega_{k+1}^*, \ldots, \omega_{k+s_k(m)-1}^*) = m$$
$$\forall k, s \in \mathbb{N}^*, \mathcal{N}(\omega_k^*, \ldots, \omega_{k+s-1}^*) := \{i \in \mathbb{N}^* | k \leq i < k+s, \omega_i^* > 0\}$$

then

v. the sequence $(\omega_k^*)_{k \in \mathbb{N}^*}$ is convergent in $\mathbb{R}_+$ and $\lim_{k \longrightarrow \infty} \omega_k^* = 0$;

vi. the sequences $\left(\hat{\boldsymbol{x}}_{\boldsymbol{k}} - \hat{\boldsymbol{x}}_{\boldsymbol{k/k-1}}\right)_{k \in \mathbb{N}^*}$ and $\left(P_k - P_{k/k-1}\right)_{k \in \mathbb{N}^*}$ are convergent in $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ respectively, with $\lim_{k \to \infty} \hat{\boldsymbol{x}}_{\boldsymbol{k}} - \hat{\boldsymbol{x}}_{\boldsymbol{k/k-1}} = \boldsymbol{0}$ and $\lim_{k \to \infty} P_k - P_{k/k-1} = 0_{n \times n}$;

vii. the innovation vector tends to the interior of the ellipsoid $\mathcal{E}(\boldsymbol{0}, V_k)$, *i.e.*, $\forall \varepsilon > 0, \exists k_\infty \in \mathbb{N}^*, \forall k > k_\infty, \boldsymbol{\delta}_k^T V_k^{-1} \boldsymbol{\delta}_k < 1 + \varepsilon$;

viii. Consider the dynamic system of state vector

$$\breve{\boldsymbol{x}}_{\boldsymbol{k}} = P_k P_{k/k-1}^{-1} \Phi_{k-1} \breve{\boldsymbol{x}}_{\boldsymbol{k-1}}; \qquad (18)$$

1. there exists positive reals c_1, c_2, c_3 and $\varrho < 1$ (*e.g.*, $\varrho = \min_{k \leq i \leq k+s_k(m)} \rho_i$) and a positive integer m such that the sequence $(\breve{\mathcal{V}}_k)_{k \in \mathbb{N}^*}$, where $\breve{\mathcal{V}}_k = \breve{\boldsymbol{x}}_{\boldsymbol{k}}^T P_k^{-1} \breve{\boldsymbol{x}}_{\boldsymbol{k}}$, satisfies the inequalities :

$$0 < c_1 \left\| \breve{\boldsymbol{x}}_{\boldsymbol{k}} \right\|^2 \leq \breve{\mathcal{V}}_k \leq c_2 \left\| \breve{\boldsymbol{x}}_{\boldsymbol{k}} \right\|^2,$$

for all $k \in \mathbb{N}^*$ such that $\breve{\boldsymbol{x}}_{\boldsymbol{k}} \neq \boldsymbol{0}$ and

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[2] If $\boldsymbol{x} = (x_r \cdots x_0)$ and $\boldsymbol{y} = (y_s \cdots y_0)$, then (assuming that $r \leq s$) $\boldsymbol{z} = \boldsymbol{x} * \boldsymbol{y} = (z_{r+s}\, z_{r+s-1} \cdots z_0)$ where $z_j = \sum_{i=0}^{j} x_i y_{j-i}$ if $j \leq \min(r,s) = r$ and $z_j = \sum_{i=0}^{r} x_i y_{j-i}$ otherwise

[3] In the sequence $\left\{\omega_k^*, \omega_{k+1}^*, \ldots, \omega_i^*, \ldots, \omega_{k+s_k(m)-1}^*\right\}$ of length $s_k(m)$, there must be exactly m times $\omega_i^* \neq 0$ (*i.e.* $\|\boldsymbol{\delta_i}\|_{V_k^{-1}} > 1$) and the rest $s_k(m) - m$ of ω_i^* are zero. Card S is the cardinal of the set $S = \{s_1, \ldots, s_m\}$ and is equal to the *finite* number m of its elements.

$$\breve{\mathcal{V}}_{k+s_k(m)+1} - (1-\varrho)^{s_k(m)+1}\breve{\mathcal{V}}_k \leq -c_3 \left\|\breve{\boldsymbol{x}}_{\boldsymbol{k+s_k(m)+1}}\right\|^2 < 0,$$

for all $k \in \mathbb{N}^*$ such that $\breve{\boldsymbol{x}}_{\boldsymbol{k+s_k(m)+1}} \neq \mathbf{0}$;

2. the *homogenous* part – represented by the system (18) – of the estimator given by (3)–(5) and (8)–(12)associated to the model (1a)–(1b), as well as the homogenous part of the estimation error, $\hat{\boldsymbol{x}}_{\boldsymbol{k}} - \boldsymbol{x}^*_{\boldsymbol{k}}$, is *uniformly asymptotically stable* ;

ix. the volume and the lengths of all the axes of $\mathcal{E}(\hat{\boldsymbol{x}}_{\boldsymbol{k}}, \sigma_k^2 P_k)$ are bounded for all $k \in \mathbb{N}^*$. ■

Proof. For the lack of place, the proof of this theorem is omitted. ❑

Remark** 1.* The point ***vi. of the *Theorem* 1 means that after a sufficient number of iterations, the estimator's dynamics approaches those of the system (1a)–(1b). While the gain matrix becomes smaller and smaller, the estimator consists merely in its time update part, such that it can simply follow the system's evolution. ▼

Remark** 2.* One can notice that the aims ***i–iii of the *Section* 2 are indeed fulfilled : all the way through the paper, it seems obvious that the set containing all possible values of $\boldsymbol{x}^*_{\boldsymbol{k}}$ is quantified by the ellipsoid $\mathcal{E}(\hat{\boldsymbol{x}}_{\boldsymbol{k}}, \sigma_k^2 P_k)$ (aim ***i***) ; from the point ***iv***. of the *Theorem* 1, it is clear that the output error vector is acceptable (aim ***ii***) ; and the points ***v.–viii***. show that the estimator is stable and convergent (aim ***iii***). ▼

All the remarks made in (Becis-Aubry *et al.*, 2003) can also be applied here.

6. NUMERICAL EXAMPLE

The example studied here is the one presented in (Maksarov and Norton, 1996*b*) within the minimal volume state estimation in the presence of bounded noises framework. Consider the system (1a)–(1b) where $\Phi_k = \begin{pmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.18 & -0.96 & 1.5 \end{pmatrix}$ and $F_k = \begin{pmatrix} 1.2h_k & 1.8 & -0.3 \\ -1.0 & 0.6 & -2.0h_k \end{pmatrix}$ where $h_k = \cos(0.1k)$ and the noise vectors satisfy $\boldsymbol{v_k} \in \mathcal{E}(\mathbf{0}, V_k)$ and $\boldsymbol{w_k} \in \mathcal{E}(\mathbf{0}, W_k)$ with $V_k = \alpha \begin{pmatrix} 100 & -50 \\ -50 & 100 \end{pmatrix}$ and $W_k = \begin{pmatrix} 5.0 & 0.5 & -1.0 \\ 0.5 & 5.0 & 0.5 \\ -1.0 & 0.5 & 5.0 \end{pmatrix}$, V_k involves a factor $\alpha \in \{1.6,\ 10\}$. $\boldsymbol{w_k}$ is uniformly distributed in the ellipsoid $\mathcal{E}(\mathbf{0}, W_k)$, as for $\boldsymbol{v_k}$, it is generated according to two different configurations :

config. 1 : $\boldsymbol{v_k}$ is uniformly distributed in the ellipsoid $\mathcal{E}(\mathbf{0}, V_k)$;

config. 2 : $\boldsymbol{v_k}$ can be anywhere in the ellipsoid $\mathcal{E}(\mathbf{0}, V_k)$ but is more likely to draw near to its boundary.

We will compare two algorithms :

algo. 1 : the algorithm presented here and given by the equations (3)–(5) and (8)–(12), where ρ is chosen such that the volume of the ellipsoid $\mathcal{E}_{k/k-1}$ computed at the time update step is minimized.

algo. 2 : an other algorithm of the literature (Maksarov and Norton, 1996*a*) that minimizes the ellipsoid's volume at both steps : time and observation updates.

For each value of α, the simulations are run $M = 50$ times and the simulation horizon is $N = 200$ samples. The ellipsoid $\mathcal{E}_0$ containing the initial state estimate is $\mathcal{E}(\mathbf{0}, 100I_3)$ and $\boldsymbol{x}^*_{\mathbf{0}}$ is uniformly distributed in this ellipsoid.

Let us define a measure of each component of the estimation error vector :

$$e_i = \frac{1}{M}\sum_{j=1}^{M}\frac{1}{N}\sum_{k=1}^{N}\left({}^j x^*_{i_k} - {}^j\hat{x}_{i_k}\right)^2,\ i \in \{1,2,3\}.$$

where ${}^j x^*_{i_k}$ (resp. ${}^j\hat{x}_{i_k}$) is the i^{th} component of the true (resp. estimated) state vector for the k^{th} iteration (among N) of the j^{th} simulation (among M). Let us also define the mean $\mathcal{E}_k$'s volumes :

$$\nu_k = \frac{1}{M}\sum_{j=1}^{M}\text{vol}\left(\mathcal{E}({}^j\hat{\boldsymbol{x}}_{\boldsymbol{k}}, {}^j\sigma_k^2\, {}^jP_k)\right),\ \bar{\nu} = \frac{1}{N}\sum_{k=1}^{N}\nu_k,$$

where, similarly, ${}^j\hat{\boldsymbol{x}}_{\boldsymbol{k}}$ and ${}^j\sigma_k^2\, {}^jP_k$ are the center and the shape matrix of the ellipsoid for the k^{th} iteration (among N) of the j^{th} simulation (among M). We define the mean number of steps for which the observation update of the algorithm is frozen, *i.e.*,

$$\begin{aligned}\bar{r} &= \frac{1}{M}\textstyle\sum_{j=1}^{M}\text{Card}\left(k \in \{1,\ldots,N\} | {}^j\omega^*_k = 0\right) \\ &= \frac{1}{M}\textstyle\sum\text{Card}\left(k \in \{1,\ldots,N\} | {}^j\hat{\boldsymbol{x}}_{\boldsymbol{k}} = {}^j\hat{\boldsymbol{x}}_{\boldsymbol{k/k-1}},\right. \\ &\qquad\left. {}^jP_k = {}^jP_{k/k-1},\ {}^j\sigma_k^2 = {}^j\sigma^2_{k/k-1}\right),\end{aligned}$$

and finally, define the mean computation time, T, in seconds, (which corresponds to the mean duration of one simulation for the M simulations made by the software of technical computing, MATLAB). The values of all these measures are given in the tables 1 and 2.

After examining these tables, one can be struck by the difference between the ellipsoid's volumes obtained by the two methods. Indeed, it is reasonable that the algorithm (algo. 2) that minimizes the volume of the ellipsoid produces ellipsoids with volumes smaller than the ones of the algo. 1. This is the main drawback of our algorithm. We can however notice that the difference between the ellipsoid's volumes decreases when the ellipsoid $\mathcal{E}(\mathbf{0}, V_k)$ becomes bigger.

On the other hand, the values of the ratio $\bar{r}/N$ and the values of the mean computation time T show that the number of observation updates for alog. 1 are less than those for algo. 2. Roughly speaking, out of 200 iterations of a simulation, more than 100 observation updates are frozen for the algo 1, where only the observation updates remain active. For algo. 2, however, both time and observation updates are almost always active, except when the noise $\boldsymbol{v_k}$ is very important.

Despite of the small number of observation updates, the *Table* 1 shows that the estimation errors given by the algo. 1 is acceptable and is even better than those obtained by the algo. 2 : the estimation errors for algo. 1 are in all cases smaller than those given for algo. 2 (except, perhaps, when the noise $\boldsymbol{v_k}$ is very small). The more the noise $\boldsymbol{v_k}$ is important, the smaller is the estimation error for the algo. 1 and the bigger it is for the algo. 2.

	$\alpha = 1.6$			$\alpha = 10$		
	e_1	e_2	e_3	e_1	e_2	e_3
algo. 2, config. 1	5.3	4.9	6.1	7.6	7.4	7.6
algo. 1, config. 1	7.0	6.9	8.8	12.9	11.0	10.5
$\frac{e_i(\text{algo. 2})}{e_i(\text{algo. 1})}$	1.3	1.4	1.4	1.6	1.5	1.4
algo. 2, config. 2	4.0	3.6	4.6	5.2	4.9	5.8
algo. 1, config. 2	7.5	7.5	9.2	16.7	14.9	13.1
$\frac{e_i(\text{algo. 2})}{e_i(\text{algo. 1})}$	1.9	2.1	2.0	3.9	2.9	2.3

Table 1. Estimation errors

	$\alpha = 1.6$			$\alpha = 10$		
algo., config.	$\bar{\nu}$	$\bar{r}/N$	T	$\bar{\nu}$	$\bar{r}/N$	T
2, 1	10101	0.68	38.5	13429	0.83	38.8
1, 1	1498	0	74.4	6477	0.14	68.4
2, 2	7615	0.53	41.5	10680	0.60	39.7
1, 2	1294	0	73.7	4629	0.77	70.4

Table 2. Mean volume, update ratio and computation time

7. CONCLUSION

A recursive state estimation technique for linear discrete-time systems corrupted by unknown but bounded noises has been presented. An ellipsoid that encloses all the possible values of the state vector and which is consistent with the bounds of the noises was determined at each sampling time. As the Kalman filter, the algorithm has been decomposed into *time update* and *observation update* steps. The observation update stage is skipped as soon as the *a priori* output error (innovation) vector is acceptable, that is, when $(\boldsymbol{y_k} - F_k\hat{\boldsymbol{x}}_{\boldsymbol{k/k-1}}) \in \mathcal{E}(\mathbf{0}, V_k)$, what makes the algorithm faster. It was proved that the estimate provided by this algorithm assures the acceptability of the *a posteriori* output error and that this estimate converges to the state vector. Finally, a comparison study showed the advantages and the drawbacks of this method with respect to an other method of literature.

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