SYNTHESIS OF STABILIZING PID CONTROLLERS FOR BIOMECHANICAL MODELS

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Abstract: A framework for synthesis of stabilizing PID controllers for linear time-invariant systems using Hermite-Biehler Theorem is presented. The approach is based on the analytical characterization of the roots of the characteristic polynomial. Generalized Hermite-Biehler Theorem from functional analysis is used to derive stability results, leading to necessary and sufficient conditions for the existence of stabilizing PID controllers. An algorithm for the selection of stabilizing feedback gains using root locus techniques and Linear Matrix Inequalities (LMI) is presented.

Keywords: Biomedical systems, PID control, Stability analysis, Feedback systems, Hurwitz polynomial.

1. INTRODUCTION

Feedback control aims to design fixed-order controllers that would, taking into account the system dynamics, stabilize the closed-loop system and meet certain performance criteria. Among many of the issues that arise during the life cycle of system design, one of the first, and probably the most critical one, is stability of the feedback loop. The problem of stabilizing a given process by suitable low fixed-order compensator, and attempt to design optimal and robust controllers with constrained order, is a challenging one. Based on the methodology used to design the stabilizing controller and the manner in which the system is modeled, numerous approaches are available for stability analysis and system design; however, most lack complete analytical characterization.

Algebraic approaches to stability analysis and controller synthesis draw motivation from the Hermite-Biehler, Kharitonov, Edge, and Lipatov theorems (Bhattacharyya, *et al.*, 1995; Kharitonov, 1978; Bartlett *et al.*, 1988; Lipatov and Sokolov, 1978). Such methodologies use polynomials (or polynomial matrices) to provide conditions for the stability of linear feedback processes, or a family of processes (interval plants). Intelligent use of these theorems allows the simultaneous design of the characteristic polynomial of the closed-loop and synthesis of the controller. Furthermore, they offer the advantage of analytical characterization and fixed-order controller synthesis, which most classical control methods fail to address. For example, classical control techniques in frequency domain such as Evans root locus and Nyquist stability criteria are graphical in nature and fail to provide any analytical characterization of the stabilizing compensator parameters. The Routh-Hurwitz criterion on the other hand, does provide an analytical solution to the stability problem; however, the set of stabilizing compensators can only be determined by solving a set of nonlinear inequalities, a task that may become cumbersome for high-order processes. Similarly, other methods such as YJBK parameterization (Youla *at al.*, 1976) can be used to parameterize all proper feedback controllers that stabilize a given process, and minimize a specific ∞-norm, but the disadvantage of such an approach is that the controller order must be constrained. Nevertheless, the Hermite-Biehler framework can be used to design optimal constant gain controllers that minimize the ∞-norm to a value that is a reasonably good approximation to the unattainable infimum. Furthermore, when used in conjunction with the Kharitonov and Edge theorems, the set of all stabilizing controllers for a family of processes can be found, an issue that is central to the robust control theoretic framework.

In this paper, we present an analysis-synthesis framework for linear time-invariant systems based on the generalized Hermite-Biehler Theorem (Ho *et al.*, 1999; Ho *et al.*, 2000). The generalized Hermite-Biehler Theorem is used in functional analysis to study the stability properties of real polynomials defined over complex fields, and provides conditions for the Hurwitz stability of real polynomials. Since characteristic polynomials in linear systems with feedback are real polynomials, the Hermite-Biehler Theorem, in addition to providing information about stability also provides an elegant, easy, and analytical way of characterizing the set of all stabilizing controllers for a given process. The analytical framework developed in the paper is generic, and has been modified to apply to low order plants with feedback delays (Roy and Iqbal, 2003a). We may, however, point out that the analysis presented in this paper only concerns the stability and not the performance aspects of the system.

The stability of the closed-loop system is analyzed using generalized Hermite-Biehler Theorem in a manner that enables characterization of the stabilizing set of controller gains. It is shown how the generalized Hermite-Biehler Theorem can be used not only to derive conditions for the existence of the set of stabilizing compensators but also as a convenient and elegant analytical method to design compensators for linear control systems (Datta *et al.*, 1999). The stability problem is solved in (Datta *et al.*, 1999) for PID controllers for processes without time-delay, while in (Roy and Iqbal, 2003b) a modified and extended solution is presented for unstable processes with transport lags in the feedback path. The PID stabilization problem for first-order-plus-dead-time (FOPDT) and a fourth-order process are solved in (Roy and Iqbal, 2003a) and (Iqbal and Roy, 2002), respectively. The solution to the PI stabilizing problem can be found in (Datta *et al.*, 1999).

The organization of this paper is as follows: In Section 2, we present the stability analysis of the model in the Hermite-Biehler framework and develop an algorithm to synthesize stabilizing PID controllers. Applicability of the results is illustrated in Section 3 with a design example. Finally, conclusions are given in Section 4.

2. PROBLEM FORMULATION

In this section we develop a framework for stability analysis of a linear time-invariant system in unity gain feedback configuration based on the application of the generalized Hermite-Biehler Theorem. The system is shown in Fig. 1. Let $G_p(s) = W(s)/Q(s)$, where $W(s)$ and $Q(s)$ are relatively prime, and $G_c(s) = n_c(s)/d_c(s)$, where for simplicity we assume a PID controller given by $n_c(s) = K_d s^2 + K_p s + K_i$, $d_c(s) = s$; then, the closed-loop characteristic polynomial $\psi(s)$ is given as

$$\psi(s) = sQ(s) + (K_i + s^2 K_d)W(s) + sK_p W(s). \quad (1)$$

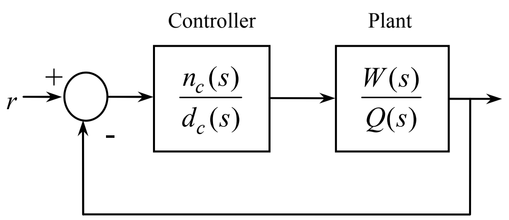

Fig. 1. Block diagram of a typical single-channel, unity feedback control system with a PID controller.

In the following, we first state the generalized Hermite-Biehler Theorem and then develop a framework for controller synthesis. For proof of the Theorem the reader is referred to (Ho *et al.*, 1999; Ho *et al.*, 2000).

2.1 The Generalized Hermite-Biehler Theorem

Generalized Hermite-Biehler Theorem (Ho *et al.*, 1999; Ho *et al.*, 2000): Let $\delta(s) = \sum_{i=0}^{n} \delta_i s^i$, $\delta_i \in \mathbb{R}\, \forall i$ with a root at the origin of multiplicity k. Writing $\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$, where $\delta_{e,o}(s^2)$ are the components $\delta(s)$ made up of even and odd powers of s, respectively. For every $\omega \in \mathbb{R}$, denote $\delta(\omega) = p(\omega) + jq(\omega)$ where $p(\omega) = \delta_e(-\omega^2)$, $q(\omega) = \omega\delta_o(-\omega^2)$, and let ω_{ej} denote the real, nonnegative, and distinct zeros of $\delta_e(-\omega^2)$ and let ω_{ok} denote the real, nonnegative, and distinct zeros of $\delta_o(-\omega^2)$ $\forall j,k$, both arranged on ascending order of magnitude. Let $0 < \omega_{o1} < \omega_{o2} < \cdots < \omega_{om-1}$ be the zeros of $q(\omega)$ that are real, distinct, and nonnegative. Also, define $\omega_0 = 0$, $\omega_{om} = \infty$, $p^{(k)}(\omega_0) = \left. \left(\frac{d^k}{d\omega^k} p(\omega) \right) \right|_{\omega=\omega_0}$. Then $\forall m \in \mathbb{Z}_+$

$$\begin{aligned} \sigma[\delta(s)] = (-1)^{m-1} \{ &\operatorname{sgn}[p^{(k)}(\omega_0)] \\ &+ 2\sum_{i=1}^{m-1} (-1)^i \operatorname{sgn}[p(\omega_{oi})] \\ &+ (-1)^m \operatorname{sgn}[p(\omega_{om})] \} \cdot \operatorname{sgn}[q(\infty)],\ n = 2m, \end{aligned}$$

$$\begin{aligned} \sigma[\delta(s)] = (-1)^{m-1} \{ &\operatorname{sgn}[p^{(k)}(\omega_0)] \\ &+ 2\sum_{i=1}^{m-1} (-1)^i \operatorname{sgn}[p(\omega_{oi})] \} \cdot \operatorname{sgn}[q(\infty)],\ n = 2m+1, \end{aligned}$$

where $\sigma[\delta(s)]$ denotes the signature of the polynomial defined as: $\sigma[\delta(s)] \triangleq n_\delta^{(L)} - n_\delta^{(R)}$, where $n_\delta^{(L)}$ and $n_\delta^{(R)}$ are the number of open left-half plane (LHP) and right-half plane (RHP) roots.

2.2 *Stability Analysis*

In order to develop a stability framework based on the Hermite-Biehler Theorem, we proceed as follows: define the signature of $\psi(s)$ as $\sigma[\psi(s)] \triangleq n_\psi^{(L)} - n_\psi^{(R)}$ and the order of $\psi(s)$ as $n_\psi^{(L)} + n_\psi^{(R)} \triangleq \Theta[\psi(s)]$; then, from a stability perspective, if $\psi(s)$ is Hurwitz, then $n_\psi^{(R)} = 0$ (no RHP poles), or equivalently, $\sigma[\psi(s)] = \Theta[\psi(s)] = m_\psi$ (i.e., if the system is Hurwitz stable, then the signature equals the order of the characteristic polynomial). In order to apply the generalized Hermite-Biehler Theorem, we decompose $W(s)$ and $Q(s)$ into polynomials with even and odd powers of s. To this effect we let $W(s) = W_e(s^2) + sW_o(s^2)$, $Q(s) = Q_e(s^2) + sQ_o(s^2)$. Also, define $W^*(s) \triangleq W(-s) = W_e(s^2) - sW_o(s^2)$ and let

$$\delta(s) \triangleq \psi(s)W^*(s) = sQ(s)W^*(s) + (K_i + s^2 K_d)W(s)W^*(s) + sK_p W(s)W^*(s) \quad (3)$$

then, it can be verified that $\sigma[\delta(s)] = \sigma[\psi(s)W^*(s)] = \sigma[\psi(s)] - \sigma[W(s)]$.

Further, if $\psi(s)$ is Hurwitz stable, then $\sigma[\psi(s)W^*(s)] = m_\psi - \sigma[W(s)]$. Now, substituting $s = j\omega$ in $\delta(s)$ we obtain

$$\delta(\omega) = \psi(\omega)W^*(\omega) = p(\omega) + jq(\omega) \quad (4)$$

where

$$p(\omega) \triangleq p_1(\omega) + (K_i - K_d\omega^2)p_2(\omega) \quad (5)$$

$$q(\omega) \triangleq q_1(\omega) + K_p q_2(\omega) \quad (6)$$

and the polynomials $p_1(\omega), p_2(\omega), q_1(\omega)$, and $q_2(\omega)$ are given as:

$$p_1(\omega) \triangleq \omega^2[Q_e W_o - Q_o W_e],\ p_2(\omega) \triangleq [W_e^2 + \omega^2 W_o^2],$$

$$q_1(\omega) \triangleq \omega[W_e Q_e + \omega^2 W_o Q_o], q_2(\omega) \triangleq \omega p_2(\omega) \quad (6)$$

We note that Eqs. (5) and (6) provide a decoupling of the position gain, K_p, from the velocity and integral gains, K_d and K_i. This structure will be exploited to develop a synthesis procedure for PID controllers later in the section. For now, in order to develop stability characterization using the generalized Hermite-Biehler Theorem, let $m_q \triangleq \Theta[q(\omega)]$ and $\tilde{m}_q \le m_q$ be the number of real, nonnegative, and distinct zeros of $q(\omega)$ with odd multiplicities that satisfy the following condition for $K_p \in (-\infty, \infty)$:

$$0 = \omega_0 < \omega_{o1} < \omega_{o2} < \cdots < \omega_{o\tilde{m}_q - 1} < \omega_{o\tilde{m}_q} = \infty .$$

Furthermore, define $\rho \triangleq m_\psi - \sigma[W(s)]$, $\alpha \triangleq \frac{1}{2}[m_\psi + m_W]$, and $\gamma \triangleq (-1)^{\tilde{m}_q - 1}\operatorname{sgn}[q(\infty)]$, then application of the generalized Hermite-Biehler Theorem to Eq. (4) leads to the following:

$$\rho = \{\operatorname{sgn}[p(0)] + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \operatorname{sgn}[p(\omega_{oi})] + (-1)^{\tilde{m}_q}\operatorname{sgn}[p(\omega_{o\tilde{m}_q})]\}\cdot\gamma,\ \alpha \in \mathbb{Z}_+, \quad (7)$$

$$\rho = \{\operatorname{sgn}[p(0)] + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \operatorname{sgn}[p(\omega_{oi})]\}\cdot\gamma,\ \alpha \notin \mathbb{Z}_+, \quad (8)$$

where $\mathbb{Z}_+$ defines the set of positive integers. Substituting Eq. (5) in Eqs. (7) and (8) we obtain

$$\rho = \{\operatorname{sgn}[p_1(0) + K_p p_2(0)] + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \operatorname{sgn}[p_1(\omega_{oi}) + (K_i - K_d\omega_{oi}^2)p_2(\omega_{oi})] + (-1)^{\tilde{m}_q}\operatorname{sgn}[p_1(\omega_{o\tilde{m}_q}) + (K_i - K_d\omega_{o\tilde{m}_q}^2)p_2(\omega_{o\tilde{m}_q})]\}\cdot\gamma,\ \alpha \in \mathbb{Z}_+, \quad (9)$$

$$\rho = \{\operatorname{sgn}[p_1(0) + K_p p_2(0)] + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \operatorname{sgn}[p_1(\omega) + (K_i - K_d\omega^2)]\}\cdot\gamma, \quad \alpha \notin \mathbb{Z}_+ \quad (10)$$

We now define the following integer variables

$$\tilde{I}_0 \triangleq \operatorname{sgn}[p_1(0) + K_p p_2(0)], \quad (11a)$$

$$\tilde{I}_i \triangleq \operatorname{sgn}[p_1(\omega_{oi}) + (K_i - K_d\omega_{oi}^2)p_2(\omega_{oi})], \quad (11b)$$

where $\tilde{I}_0 \in \{-1, 0, 1\}$ and $\tilde{I}_i \in \{-1, 1\} \forall i \in [1, \tilde{m}_q]$; then the necessary and sufficient conditions for the existence of stabilizing PID controllers are given by the following theorem:

Theorem: The characteristic polynomial $\psi(s)$ is Hurwitz stable if and only if there exist a feasible non-empty solution $\{\tilde{I}_i\} \ne \varnothing$ to either of the following equations:

$$\rho = \left[\tilde{I}_0 + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \tilde{I}_i + \tilde{I}_{\tilde{m}_q}\right]\gamma,\ \alpha \in \mathbb{Z}_+ \quad (12)$$

$$\rho = \left[\tilde{I}_0 + 2\sum_{i=1}^{\tilde{m}_q - 1}(-1)^i \tilde{I}_i\right]\gamma,\ \alpha \notin \mathbb{Z}_+, \quad (13)$$

where $\tilde{I}_i, i = 0, 1, \ldots, \tilde{m}_q$ denote the unknown integer variables as defined in Eqs. (11a) and (11b).

Proof: (*Necessity*) If Eq. (12) or (13) has a feasible solution, then by using Eqs. (11a) and (11b) in Eqs. (9) and (10), and applying the generalized Hermite-Biehler Theorem, we can ensure that $\sigma[\delta(s)] = m_\psi - \sigma[W(s)]$, so that $\sigma[\psi(s)] = \Theta[\psi(s)]$ and $\psi(s)$ is Hurwitz stable.

(*Sufficiency*) If $\psi(s)$ is Hurwitz stable, then $\sigma[\psi(s)] = \Theta[\psi(s)]$ s.t. that $\sigma[\delta(s)] = m_\psi - \sigma[W(s)]$, and from the generalized Hermite-Biehler Theorem either Eq. (7) or (8) must be satisfied. The only way to satisfy it is if Eq. (12) or (13) has a feasible solution. ♣

Characterization of the stabilizing PID gains for the system is now provided by the following results:

Corollary 1: The range of K_p for which the root distribution of $q(\omega)$ is such that Eq. (12) or (13) is satisfied, can be identified from the root locus plot of $\{1+K_p\, q_2(\omega)/q_1(\omega)\}$. This range is denoted as $S(K_p)=(\underline{K}_p, \overline{K}_p)\subseteq(-\infty,\infty)$, where the under bar and the over bar represent the lower and upper limit, respectively.

Corollary 2: Assume that a non-empty solution to Eq. (12) or (13) has been found, i.e., let $\tilde{g}\triangleq\{\tilde{I}_i\}\neq\varnothing$, then the ranges of the stabilizing gains K_i and K_d, i.e., $(\underline{K}_i, \overline{K}_i)$ and $(\underline{K}_d, \overline{K}_d)$, can be solved from the following linear matrix inequalities (LMI):

$$\operatorname{sgn}\{[P_1]+[P_2][\tilde{\kappa}]\}=(\tilde{g})^T, \qquad (14)$$

where the matrices $[P_1]\in\mathbb{R}^{\tilde{m}_q,1}$, $[P_2]\in\mathbb{R}^{\tilde{m}_q,2}$, and $[\tilde{\kappa}]\in\mathbb{R}^{2,1}$ are defined as:

$$[P_1]\triangleq[p_1(0)\quad p_1(\omega_{o1})\quad p_1(\omega_{o2})\quad\cdots\quad p_1(\omega_{o\tilde{m}_q-1})]^T \quad (15)$$

$$[P_2]\triangleq\begin{bmatrix} p_2(\omega_0) & 0 \\ p_2(\omega_{o1}) & -\omega_{o1}^2 p_2(\omega_{o1}) \\ p_2(\omega_{o2}) & -\omega_{o2}^2 p_2(\omega_{o2}) \\ \vdots & \vdots \\ p_2(\omega_{o\tilde{m}_q-1}) & -\omega_{o\tilde{m}_q-1}^2 p_2(\omega_{o\tilde{m}_q-1}) \end{bmatrix} \quad (16)$$

$$[\tilde{\kappa}]\triangleq[K_i\quad K_d]^T \quad (17)$$

thus, for a given $K_p\in S(K_p)$, the stabilizing PID controller set is: $S_{\tilde{\kappa}}\triangleq\{K_p, (\underline{K}_i, \overline{K}_i), (\underline{K}_d, \overline{K}_d)\}$.

We note that the solution of the LMI given by Eq. (14) is either a convex polygon or a half-plane in the K_i-K_d space. We also note that the procedure leading to Eq. (14) and its solution can be easily coded on the computer. We further note that by repeating this process for different values of $K_p\in S(K_p)$, one can obtain a picture of stability in the 3-D space. The stability characterization in the controller parameter space is shown in Fig. 2. Finally, the above analysis can be modified to restrict $K_p, K_i, K_d\in\mathbb{R}_+$ in order to force non-negative solution to the problem, if so desired.

Based on the above discussion, we provide the following algorithm for PID controller synthesis for the single-channel unity feedback system (Fig. 1).

2.3 Algorithm for Controller Synthesis

Step 1: Given $W(s)$ and $Q(s)$, use Eq. (6b) to obtain the polynomials $q_1(\omega)$, $q_2(\omega)$; and plot the root locus for $\{1+K_p\, q_1(\omega)/q_1(\omega)\}$. Also, compute ρ.

Step 2: Select some $S(K_p)\subseteq(-\infty,\infty)$ from the root locus such that $\forall K_p\in S(K_p)$, $\tilde{m}_q$ has the potential to satisfy Eq. (12) or (13), i.e., $2\tilde{m}_q-1\geq\rho$ on the root locus plot.

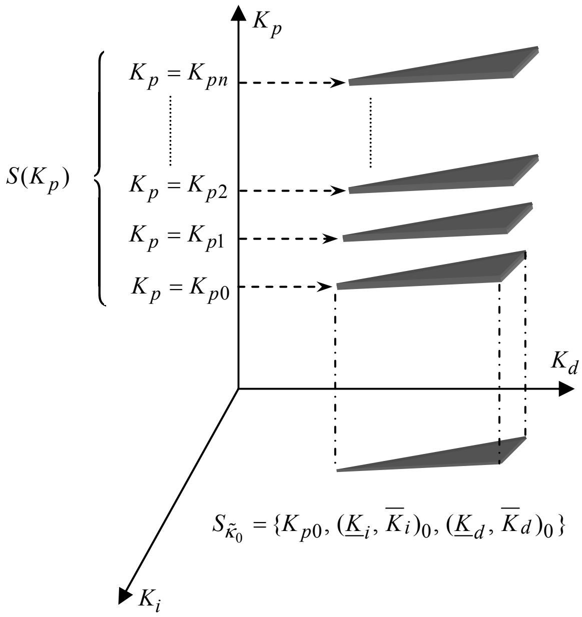

Fig. 2. The design methodology illustrated in the $K_p-K_i-K_d$ space.

Step 3: Choose a trial value $K_{p0}\in S(K_p)$; accordingly, calculate the values of γ and α.

Step 4: For the selected $K_{p0}\in S(K_p)$, use Eqs. (12) or (13) to obtain $\tilde{g}$. If $\tilde{g}\neq\varnothing$, proceed to Step 5; otherwise repeat Steps 2-3 with different ranges of $S(K_p)$ until $\tilde{g}\neq\varnothing$ is obtained.

Step 5: Compute $[P_1]$ and $[P_2]$ for $K_{p0}\in S(K_p)$. Use the set of LMI given by Eq. (14) to obtain the non-empty stabilizing set $S_{\tilde{\kappa}}$. If the solution to Eq. (14) is empty, then repeat Steps 3-5 with a different $K_{p0}\in S(K_p)$.

The algorithm and the stability characterization is illustrated in the form of a Venn diagram in Fig. 3.

3. DESIGN EXAMPLE

As an example of the stability analysis and controller synthesis, we apply the synthesis algorithm to a single-link biomechanical system with position, velocity, and force feedback, and with physiological latencies in he feedback loops (Iqbal and Roy, 2004).

Using [1/1] Padé approximation to represent the delays, the closed-loop transfer function is given as

$$\frac{y(s)}{r(s)}=\frac{n_G(s)}{sQ(s)+n_c(s)W(s)},$$

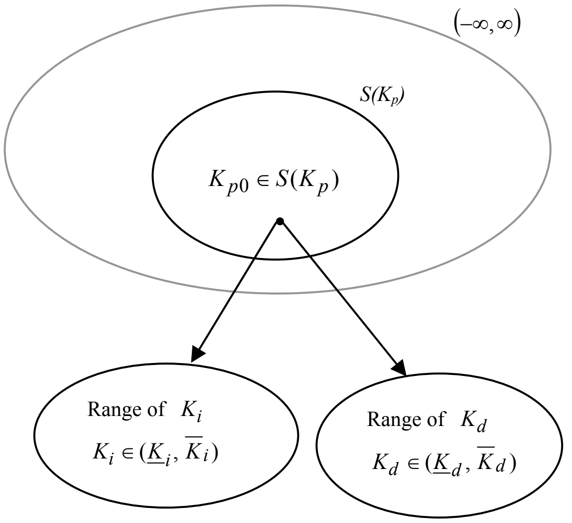

Fig. 3. Venn diagram illustrating the controller synthesis algorithm based on the Hermite-Biehler Theorem.

where

$$n_G(s) = 0.112s^3 + 13.03s^2 + 305.08s + 596.17,$$

$$W(s) = -0.2s^6 - 28.78s^5 - 383.69s^4 + 9.92\times10^4 s^3 + 4.89\times10^6 s^2 + 8.28\times10^7 s + 4.23\times10^8,$$

$$Q(s) = -4\times10^{-4} s^8 + 0.14s^7 + 22.17s^6 + 1681.63s^5 + 6.78\times10^4 s^4 + 1.36\times10^6 s^3 + 1.05\times10^7 s^2 - 6.23\times10^6 s - 5.47\times10^7,$$

The even and odd parts of $W(s)$ and $Q(s)$ are calculated as:

$$W_e(s) = -0.2s^6 - 383.69s^4 + 4.89\times10^6 s^2 + 4.23\times10^8,$$

$$W_o(s) = -28.78s^4 + 9.92\times10^4 s^2 + 8.28\times10^7,$$

$$Q_e(s) = -4\times10^{-4} s^8 + 22.17s^6 + 6.78\times10^4 s^4 + 1.05\times10^7 s^2 - 5.47\times10^7,$$

$$Q_o(s) = 0.14s^6 + 1681.63s^4 + 1.36\times10^6 s^2 - 6.23\times10^6.$$

Then, using Steps 1-4 of the controller synthesis algorithm we obtain: $\rho = 5$, $\alpha = 7.5$, and the stabilizing range of K_p is given as: $S(K_p) = (0.129,\ 13.3)$. For illustration we choose $K_{p0} = 5$; then $\operatorname{sgn}[q(\infty)] = 1$ and the real, nonnegative, and distinct zeros of $q(\omega)$ with odd multiplicities are: $\omega_0 = 0$, $\omega_{o1} = 23.35$, and $\omega_{o2} = 161.19$ rad/sec. Therefore, $\tilde{m}_q = 3$, $\gamma = 1$, and from Eq. (12) we obtain $\{\tilde{I}_0 - 2\tilde{I}_1 + 2\tilde{I}_2\} = 5$, which is solved as $\tilde{g} = \{\tilde{I}_0, \tilde{I}_1, \tilde{I}_2\} = \{1, -1, 1\}$. Use of Eqs. (13)-(16) then results in the following linear inequalities:

$$\begin{cases} K_i < 0 \\ K_i + 545.31K_d + 40.95 < 0 \\ K_i + 13502.1K_d - 4193.79 > 0 \end{cases} \qquad (17)$$

The shaded area in Fig. 4 shows the bounded feasible region in the $K_i - K_d$ space for the active constraints given by Eq. (17). Therefore, the set of all stabilizing PID controllers for $K_p = 5$ is: $S_{\tilde{K}} = \{5,\ (0,\ 220),\ (-0.075,\ 0.325)\}$. For illustration, we select $\{K_i = 10,\ K_d = 0.16\}$; then, the closed-loop poles of $G(s)$ are given as: $s_{1,2} = -34.83 \pm j80.27$, $s_3 = -85.16$, $s_4 = -66.67$, $s_5 = -28.9$, $s_6 = -20.64$, $s_{7,8} = -9.45 \pm j10$, and $s_9 = -2.15$. Since $\mathrm{Re}[s_i] < 0$, the closed-loop system is Hurwitz stable. The closed-loop step response for $K_p = 5,\ K_i = 10,\ K_d = 0.16$ is shown in Fig. 5.

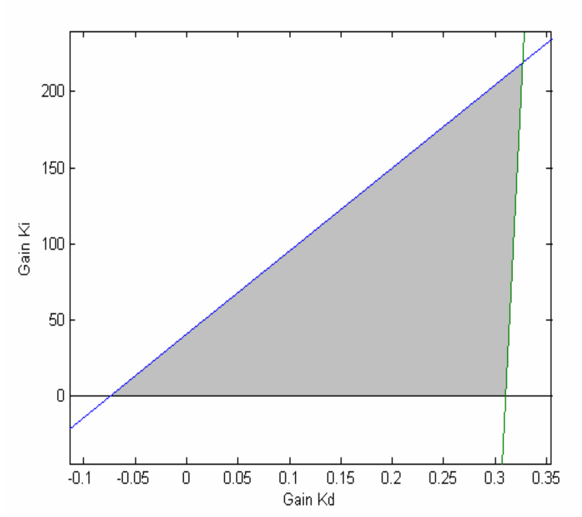

Fig. 4. Feasible $K_d - K_i$ region obtained from active constraints. The feasible region is either a convex or a hyper plane.

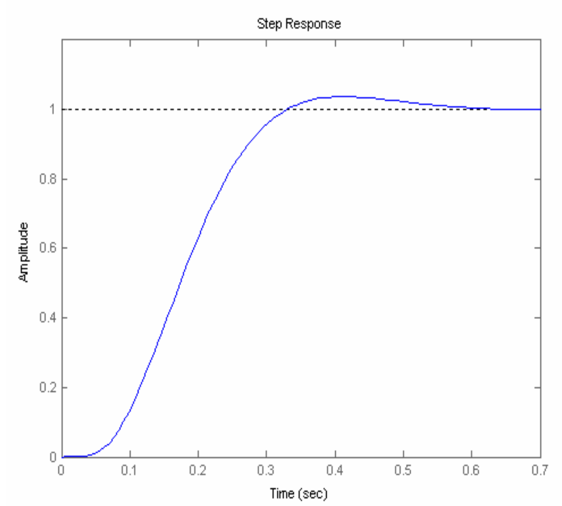

Fig. 5. Stable closed-loop response to a step input with $K_p = 5$, $K_d = 0.16$, and $K_i = 10$.

4. CONCLUSIONS

In this paper rigorous mathematical stability of linear time-invariant systems of arbitrary order is established using Hermite-Biehler framework from functional analysis. Stabilizing PID gains are chosen through a combination of root-locus techniques and linear matrix inequalities that result from application of the generalized Hermite-Biehler Theorem. The resultant controller synthesis algorithm is, in general, programmable[1] for linear systems of arbitrary order.

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