GENERALIZED BLOCK CONTROL PRINCIPLE

S. Rao ** **H. Brandtstädter** * **V. Utkin** ** **M. Buss** *

* *Institute of Automatic Control Engineering (LSR), Technische Universität München, 80290 Munich, Germany*
** *The Ohio State University, Columbus, Ohio, USA*

Abstract: This article proposes a generalized block control principle. The algorithm, developed in the framework of sliding mode control, offers insensitivity to parameter variations and external disturbances and simplification of the control design. In contrast to the known Block Control Principle, decomposing of the system into blocks where the dimension of state and control input coincide is not required. This way the range of dynamic systems the principle can be applied to is enlarged and computational effort is reduced. In particular, the application to under-actuated mechanical systems and the problem of stabilization of systems with unstable zero dynamics are regarded.

Keywords: mechanical systems, sliding mode control, unstable zero dynamics, order reduction.

1. INTRODUCTION

Control algorithms that enforce sliding modes provide desired system dynamics with low sensitivity to variations in plant dynamics and disturbances. With recent developments in sliding mode control related to the so-called Block Control Principle (BCP) (Drakunov *et al.*, 1990), the problem of sliding mode control design for complex systems governed by highly non-linear differential equations can be broken down into a set of independend control subproblems of lower dimensions with equal dimensions of control and state. The idea here is to use the state of each block as a fictitious control in the preceding block. However, this approach may lead to a very large number of non-linear transformations needed for decomposing and furthermore, these transformations may not always exist.

In order to overcome these practical problems the requirement to maintain the equality of dimension of control and block state in each of the subproblems could be waived. This would reduce the complexity of the control design process and enlarge the range of dynamic systems to which the principle can be applied to. Additionally, for many systems with different dimensions of state and control, control design may be easily performed, for instance for non-linear systems given in canonical form.

This article offers design methods in the framework of sliding mode control associated with decomposing the system into reasonable number of blocks with reasonable dimensions. Particularly, under-actuated non-linear systems with blocks in canonical form, which are common for mechanical and electromechanical plants, will be addressed. The development of new methods is necessary since the direct application of previously published methods is not always possible, as they may lead to unstable internal dynamics.

The remainder of this article is organized as follows. Starting with an introduction to the BCP section 2 explains why the development of a new sliding mode control algorithm is required. The problem of unstable zero dynamics is regarded in section 3 and hereupon the design issues of the proposed methodology are formulated. Design methods for two second order non-linear systems, a class of arbitrary order systems and an infinite dimensional system are developed in section 4.

2. BLOCK CONTROL PRINCIPLE (BCP)

The design of sliding mode control for n-dimensional control affine systems with m-dimensional vector control can be easily performed for systems in the so-called Regular Form (Utkin, 1992). The BCP is a generalization of the concept of the Regular Form which leads to decomposition of the original design problem into a set of trivial ones with equal dimension of control input and state.

Any linear controllable system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \qquad (1)$$

($\boldsymbol{x} \in \Re^n,\ \boldsymbol{u} \in \Re^m,\ \mathrm{rank}(\boldsymbol{B}) = m$) can be transformed to the block control form

$$\begin{aligned}
\dot{\boldsymbol{x}}_r &= \boldsymbol{A}_{r,r}\boldsymbol{x}_r + \boldsymbol{B}_r\boldsymbol{x}_{r-1} \\
\dot{\boldsymbol{x}}_{r-1} &= \boldsymbol{A}_{r-1,r}\boldsymbol{x}_r + \boldsymbol{A}_{r-1,r-1}\boldsymbol{x}_{r-1} + \boldsymbol{B}_{r-1}\boldsymbol{x}_{r-2} \\
&\ \vdots \\
\dot{\boldsymbol{x}}_1 &= \boldsymbol{A}_{1,r}\boldsymbol{x}_r + \cdots + \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{B}_1\boldsymbol{u}
\end{aligned} \qquad (2)$$

where $\mathrm{rank}(\boldsymbol{B}_i) = \dim(\boldsymbol{x}_i)$ (Utkin *et al.*, 1984). Control design for the system (2) can be performed as follows: Starting from the top, the state $\boldsymbol{x}_{r-1}$ of the second block is handled as intermediate control input for the first block in order to obtain a desired dynamical behavior $\dot{\boldsymbol{x}}_r = \boldsymbol{\Lambda}_r\boldsymbol{x}_r$ with $\boldsymbol{\Lambda}_r = \mathrm{diag}(\lambda_{ri})$, $\lambda_{ri} < 0$. Assigning

$$\boldsymbol{x}_{r-1} = \boldsymbol{B}_r^+(\boldsymbol{\Lambda}_r\boldsymbol{x}_r - \boldsymbol{A}_{r,r}\boldsymbol{x}_r)\,, \qquad (3)$$

with $\boldsymbol{B}_r^+$ being the pseudo-inverse of $\boldsymbol{B}_r$, $\boldsymbol{x}_r$ decays at the desired rate. The difference between the desired and the real value of $\boldsymbol{x}_{r-1}$

$$\boldsymbol{s}_{r-1} = \boldsymbol{x}_{r-1} - \boldsymbol{B}_r^+(\boldsymbol{\Lambda}_r\boldsymbol{x}_r - \boldsymbol{A}_{r,r}\boldsymbol{x}_r) \qquad (4)$$

is governed by

$$\begin{aligned}
\dot{\boldsymbol{s}}_{r-1} = \boldsymbol{A}^*_{r-1,r}\boldsymbol{s}_r + \boldsymbol{A}^*_{r-1,r-1}\boldsymbol{s}_{r-1} \ldots \\
+ \boldsymbol{B}_{r-1}\boldsymbol{x}_{r-2}
\end{aligned} \qquad (5)$$

with $\boldsymbol{s}_r = \boldsymbol{x}_r$. Matrices $\boldsymbol{A}^*_{r-1,r}$ and $\boldsymbol{A}^*_{r-1,r-1}$ can be derived via (2) and (3). By handling $\boldsymbol{x}_{r-2}$ as fictitious control

$$\begin{aligned}
\boldsymbol{x}_{r-2} = \boldsymbol{B}^+_{r-1}(\boldsymbol{\Lambda}_{r-1}\boldsymbol{s}_{r-1} - \boldsymbol{A}^*_{r-1,r}\boldsymbol{s}_r \ldots \\
- \boldsymbol{A}^*_{r-1,r-1}\boldsymbol{s}_{r-1})
\end{aligned} \qquad (6)$$

with $\boldsymbol{\Lambda}_{r-1} = \mathrm{diag}(\lambda_{r-1,i})$, $\lambda_{r-1,i} < 0$, $\boldsymbol{s}_{r-1}$ can be reduced to zero. Based on (6) the time derivate $\boldsymbol{s}_{r-2}$ of the difference between real and desired value of $\boldsymbol{x}_{r-2}$ can be represented as function of $\boldsymbol{s}_{r-1}$ and $\boldsymbol{s}_{r-2}$.

Repeating this procedure $(r-3)$ times, the time derivate of $\boldsymbol{s}_1$ describing the desired dynamical behavior of the last block is found as a function of the real control input $\boldsymbol{u}$ and the system can be written in the form:

$$\begin{aligned}
\dot{\boldsymbol{s}}_r &= \boldsymbol{\Lambda}_r\boldsymbol{s}_r + \boldsymbol{B}_r\boldsymbol{s}_{r-1} \\
\dot{\boldsymbol{s}}_{r-1} &= \boldsymbol{\Lambda}_{r-1}\boldsymbol{s}_{r-1} + \boldsymbol{B}_{r-1}\boldsymbol{s}_{r-2} \\
&\ \vdots \\
\dot{\boldsymbol{s}}_1 &= \sum_{i=2}^{r}\boldsymbol{A}^*_{1,i}\boldsymbol{s}_i + \boldsymbol{B}_1\boldsymbol{u}
\end{aligned} \qquad (7)$$

Now, sliding mode is enforced on the surface $\boldsymbol{s}_1 = 0$ using the discontinuous control

$$\boldsymbol{u} = -\boldsymbol{B}_1^+\boldsymbol{M}\,\mathrm{sign}(\boldsymbol{s}_1)\,, \quad \boldsymbol{M} \in \Re^{m\times m} \qquad (8)$$

by selecting $\boldsymbol{M}$ based on the design methodology of sliding mode control. The resulting motion of the system is described by a linear system which consists of pre-selected eigenvalues of all matrices $\boldsymbol{\Lambda}_r \ldots \boldsymbol{\Lambda}_2$. Thus, using the above BCP the design problem is reduced to r subproblems which are trivial because the dimensions of control and state are equal in each block. The idea of the block control design principle is also applicable to nonlinear systems (Lukyanov, June, 1993).

On the one hand, the BCP simplifies the design procedure. On the other hand, the principle complicates the control design for high-order systems, because the number of subproblems which have to be solved can be very large and correspondingly, a very large number of coordinate transformations is needed. In addition, for non-linear systems, these coordinate transformations may not always exist (Lukyanov and Utkin, 1981).

At the same time, in many cases designing control whose dimension is less than that of the system does not cause any problems. It is not efficient to decompose all subsystems because control may be found easily in terms of the original blocks. The set of m interconnected blocks of equations in the canonical form:

$$\begin{aligned}
\dot{x}_{i,1} &= x_{i,2} \\
\dot{x}_{i,2} &= x_{i,3} \\
&\ \vdots \\
\dot{x}_{i,n_i} &= f_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, \boldsymbol{u}) \qquad i = 1 \ldots m
\end{aligned} \qquad (9)$$

with $\boldsymbol{x}_i^T = [x_{i,1} \ldots x_{i,n_i}]$ and $\boldsymbol{u}^T = [u_1 \ldots u_m]$ may serve as an example of this situation. To assign desired right-hand sides of the last equation of each block (for example $-\sum_{j=1}^{n_i} k_{i,j}x_{i,j}\,,\ k_{i,j} < 0$), a system of m algebraic equations has to be solved with respect to control. Application of the BCP would be much more complicated. In consequence this article investigates a generalized block control principle which does not require decomposing until every block has the same dimension as its control input.

3. HIDDEN PROBLEMS

Following the idea shown before, additional problems have to be discussed. For this purpose, equation (2) is considered again. But, now in contrast to (2) it is assumed that $\dim(\boldsymbol{x}_i) > \dim(\boldsymbol{x}_{i-1})$ and each subproblem can be solved in such a way that there is no need for further decomposing. At first glance, the design appears to be similar to the case (9), which it is not, and serious problems arise for under-actuated systems. For illustration a rotational inverted pendulum system is regarded that is actuated by a DC motor (Fig. 1).

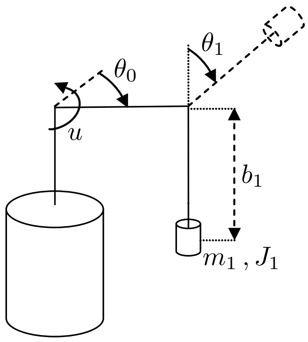

Fig. 1. Rotational inverted pendulum system.

The system dynamics are of fourth order (Utkin *et al.*, 2000):

$$\begin{aligned} \ddot{v} &= \frac{C_1}{K_1}\dot{\theta}_1 - \frac{m_1 g b_1}{K_1}\sin\theta_1 \\ \ddot{\theta}_1 &= -\frac{a_p K_1}{J_1}\dot{v} - \left(\frac{C_1}{J_1} + a_p\right)\dot{\theta}_1 \dots \\ &\quad + \frac{m_1 g b_1}{J_1}\sin\theta_1 + \frac{K_1 K_p}{J_1}u \end{aligned} \tag{10}$$

with

$$v = \theta_0 - \frac{J_1}{K_1}\theta_1 , \tag{11}$$

model parameters $a_p, K_p, m_1, g, b_1, C_1 > 0$ and a proportionality constant K_1. Following the BCP, the state of the second block $(\theta_1, \dot{\theta}_1)$ is handled as fictitious control for the first block in such a way that $\ddot{v} = -c_1 v - c_2 \dot{v}$ $(c_{1,2} > 0)$ is the desired equation for v. Now sliding mode can be enforced in the manifold

$$s = \frac{C_1}{K_1}\dot{\theta}_1 - \frac{m_1 g b_1}{K_1}\sin\theta_1 + c_1 v + c_2\dot{v} = 0 \tag{12}$$

using control $u = -M\mathrm{sign}(s)$ with M large enough. Although the state v decays at the desired rate, the equilibrium point $\theta_1 = 0$ of the system is unstable, since for $s = 0$ and $v = 0$

$$\dot{\theta}_1 = \frac{m_1 g b_1}{C_1}\sin\theta_1 \tag{13}$$

and the state θ_1 is diverging. With regard to the control objective $s = 0$ unstable zero dynamics appear and direct application of the proposed design methodology to an under-actuated system fails.

4. DESIGN METHODS

Based on the above example the main issues of this article can be formulated:

A control algorithm for under-actuated, interconnected systems in canonical form

$$\begin{aligned} \dot{x}_{i,1} &= x_{i,2} \\ \dot{x}_{i,2} &= x_{i,3} \\ &\vdots \\ \dot{x}_{i,n_i} &= f_i(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m, \boldsymbol{u}) \qquad i = 1 \dots k \end{aligned} \tag{14}$$

with $\boldsymbol{x}_i^T = [x_{i,1} \dots x_{i,n_i}]$ and $\boldsymbol{u}^T = [u_1 \dots u_m]$ and $m < k$ will be developed. Many mechanical, electrical and electromechanical systems can be represented as systems of class (14), system (10) is an example.

Following the idea of the BCP, the design problem will be decomposed into independent subproblems of lower dimension. However, the design simplicity for systems in canonical form will be preserved in order to avoid additional blocks. The design for the class of systems given by (14) with blocks where the dimension of intermediate control is less than dimension of the state will be referred to as the generalized block control principle.

At the moment, a mathematically well formalized universal design procedure based on the generalized block control principle for non-linear dynamic systems exhibiting unstable zero dynamics' behavior does not exit. But for numerous examples of non-linear electromechanical systems the generalized block control principle proved to be fruitful. The ideas underlying this design principle will be presented in the next section.

4.1 Direct application of the generalized block control principle

Special cases allow the modified BCP to be applied directly. As an example the linearized model (equilibrium point $(q, \theta_1, \theta_2, \dot{q}, \dot{\theta}_1, \dot{\theta}_2) = \mathbf{0}$) of the double pendulum on a cart without viscous friction (Fig. 2) is considered

$$\begin{aligned} \ddot{q} &= -A_1\theta_1 - A_2\theta_2 + \frac{1}{M_1}u \\ \ddot{\theta}_1 &= B_1\theta_1 + B_2\theta_2 - \frac{1}{l_1 M}u \\ \ddot{\theta}_2 &= -C_1\theta_1 + C_2\theta_2 \end{aligned} \tag{15}$$

where A_1, A_2, B_1, B_2, C_1 and C_2 represent system constants.

System (15) is of class (14). Similar to section 2 it can be reduced to the form (2) with respect to second derivatives and the described design method is applicable practically without any changes. This

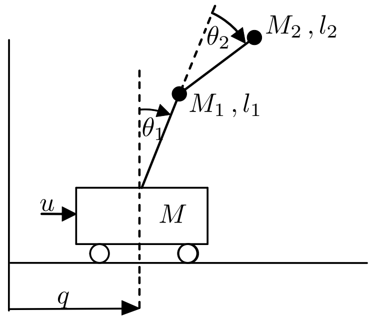

Fig. 2. Double inverted pendulum on a movable cart.

happens because right-hand sides do not depend on derivatives.

4.2 Stabilization of systems with unstable zero dynamics

Below, some design options demonstrating how to avoid the difficulties caused by unstable zero dynamics will be discussed. At first, the example given by (10) will be considered again.

The first option is to find a coordinate transformation

$$x = \dot{v} - \frac{C_1}{K_1}\theta_1 \qquad (16)$$

such that the first block after decomposition does not depend on the derivative as offered in (Utkin *et al.*, 2000):

$$\begin{aligned}\dot{x} &= -\frac{m_1 g b_1}{K_1}\sin\theta_1 \\ \ddot{\theta}_1 &= -\left(\frac{C_1}{J_1} + a_p\right)\dot{\theta}_1 + \frac{m_1 g b_1}{J_1}\sin\theta_1 \cdots \\ &\quad -\frac{C_1 a_p}{J_1}\theta_1 - \frac{K_1 a_p}{J_1}x + \frac{K_1 K_p}{J_1}u\end{aligned} \qquad (17)$$

It is evident that if

$$s = \frac{m_1 g b_1}{K_1}\sin\theta_1 + \alpha_1 x = 0\,, \qquad (18)$$

x, $\dot{x} \to 0$ for $\alpha_1 > 0$, since $\dot{x} = -\alpha_1 x$. Discontinuous control can be selected such that sliding mode is enforced on the surface $s_1 = \dot{s} + \alpha s = 0$, $(\alpha > 0)$. Note that $\dot{s}$ may be found as a function of state variables. At last $s \to 0$ means $x \to 0$ and $\theta_1 \to 0$ and according to (11) and (16) $\dot{v} \to 0$ and $\dot{\theta}_0 \to 0$. The offered solution suppresses instability, but it cannot be taken as the final one, because the coordinate θ_0 is not diverging, but tends to some final value, which will generally be different from zero. Asymptotic stability can be achieved by slightly modifying the function s on the switching surface $s = \sin\theta_1 + \alpha_1(x + v) = 0$.

The second option to avoid instability is to select the variable θ_1 in (10) only as a function of v and $\dot{v}$ in the form $\theta_1 = l_1(v + \alpha_1\dot{v})$ with $l_1, \alpha_1 > 0$, then

$$\ddot{v} = -a\dot{v} + b\sin\left(l_1(v + \alpha_1\dot{v})\right) \qquad (19)$$

with $a = \dfrac{C_1 l_1}{K_1 + C_1 l_1 \alpha_1}$ and $b = \dfrac{m_1 g b_1}{K_1 + C_1 l_1 \alpha_1}$.

The solution of (19) is asymptotically stable for $-\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}$ and $-\pi < l_1(v + \alpha_1\dot{v}) < \pi$, which can be shown by using the Lyapunov function

$$V = \frac{1}{2}\dot{v}^2 + \frac{1}{2}(v + m\dot{v})^2\,, \quad m > 0\,. \qquad (20)$$

Sliding mode is enforced on the surface

$$s = \dot{s}_1 + \alpha s_1 = 0\,, \quad \alpha > 0 \qquad (21)$$

where

$$s_1 = \theta_1 - l_1(v + \alpha_1\dot{v}) = 0 \qquad (22)$$

using a discontinuous control $u = -M\mathrm{sign}(s)$.

The introduced approach for the sample system can be generalized in three rather simple situations for under-actuated mechanical systems of an arbitrary order with vector control

$$\begin{aligned}\dot{\boldsymbol{z}} &= \boldsymbol{f}_1(\boldsymbol{z}, \boldsymbol{y}, \dot{\boldsymbol{z}}, \dot{\boldsymbol{y}})\,, \quad \boldsymbol{z} \in \Re^{n-m},\ \boldsymbol{y} \in \Re^m \\ \dot{\boldsymbol{y}} &= \boldsymbol{f}_2(\boldsymbol{z}, \boldsymbol{y}, \dot{\boldsymbol{z}}, \dot{\boldsymbol{y}}) + \boldsymbol{B}_2(\boldsymbol{z}, \boldsymbol{y})\boldsymbol{u}\,.\end{aligned} \qquad (23)$$

It is assumed that $\dim(\boldsymbol{z}) \le \dim(\boldsymbol{y})$ and $\boldsymbol{f}_1$ is an affine function with respect to $\dot{\boldsymbol{y}}$, that means

$$\boldsymbol{f}_1 = \boldsymbol{f}_{11}(\boldsymbol{y})\dot{\boldsymbol{y}} + \boldsymbol{f}_{12}(\boldsymbol{z}, \boldsymbol{y}, \dot{\boldsymbol{z}})\,. \qquad (24)$$

For the coordinate transformation $\boldsymbol{z}_1 = \boldsymbol{z} - \boldsymbol{\phi}(\boldsymbol{y})$, $\boldsymbol{z}_2 = \boldsymbol{z}$, $\boldsymbol{z}_{1,2} \in \Re^{n-m}$ the system equations are:

$$\begin{aligned}\dot{\boldsymbol{z}}_1 &= \boldsymbol{F}_{11}(\boldsymbol{y})\dot{\boldsymbol{y}} + \boldsymbol{f}_{12}\left(\boldsymbol{z}_2, \boldsymbol{y}, \boldsymbol{z}_1 + \boldsymbol{\phi}(\boldsymbol{y})\right)\ldots \\ &\qquad - \frac{\partial\boldsymbol{\phi}(\boldsymbol{y})}{\partial\boldsymbol{y}}\dot{\boldsymbol{y}} \\ \dot{\boldsymbol{z}}_2 &= \boldsymbol{z}_1 + \boldsymbol{\phi}(\boldsymbol{y}) \\ \dot{\boldsymbol{y}} &= \boldsymbol{f}_2(\boldsymbol{z}, \boldsymbol{y}, \dot{\boldsymbol{z}}, \dot{\boldsymbol{y}}) + \boldsymbol{B}_2(\boldsymbol{z}, \boldsymbol{y})\boldsymbol{u}\end{aligned} \qquad (25)$$

where $\dfrac{\partial\boldsymbol{\phi}(\boldsymbol{y})}{\partial\boldsymbol{y}} = \dfrac{\partial\phi_i}{\partial y_j}$, $(i = 1\ldots(n-m),\ j = 1\ldots m)$.

Choosing $\boldsymbol{\phi}(\boldsymbol{y})$ such that $\dfrac{\partial\boldsymbol{\phi}(\boldsymbol{y})}{\partial\boldsymbol{y}} = \boldsymbol{F}_{11}(\boldsymbol{y})$, the generalized block control principle is applicable. The state $\boldsymbol{y}$ can be handled as an intermediate control $\boldsymbol{y} = -\boldsymbol{f}(\boldsymbol{z}_1, \boldsymbol{z}_2)$ to obtain desired dynamics and sliding mode is enforced in the manifold $\boldsymbol{s} = \boldsymbol{y} + \boldsymbol{f}(\boldsymbol{z}_1, \boldsymbol{z}_2) = \boldsymbol{0}$.

Case 1:
Zero dynamics of system (23) with vector $\boldsymbol{y}$ as an output, given by $\dot{\boldsymbol{z}} = \boldsymbol{f}_1(\boldsymbol{z}, \boldsymbol{0}, \dot{\boldsymbol{z}}, \boldsymbol{0})$, are stable. The system is stabilized by enforcing sliding mode in the manifold $\boldsymbol{s} = \dot{\boldsymbol{y}} + c\boldsymbol{y} = \boldsymbol{0}$ with a scalar parameter $c > 0$.

Case 2:
Zero dynamics of system (23) with vector $\boldsymbol{z}$ as

an output, given by $\dot{\boldsymbol{z}} = \boldsymbol{f}_1(\boldsymbol{0}, \boldsymbol{y}, \boldsymbol{0}, \dot{\boldsymbol{y}}) = \boldsymbol{0}$, are stable. The system is stabilized by enforcing sliding mode in the manifold

$$\boldsymbol{s} = \boldsymbol{f}_1 + c_1\boldsymbol{z} + c_2\dot{\boldsymbol{z}} = \boldsymbol{0}\,, \quad c_{1,2} > 0 \tag{26}$$

assuming that $\mathrm{rank}(\frac{\partial \boldsymbol{f}_1}{\partial \dot{\boldsymbol{y}}}\boldsymbol{B}_2) \geq \dim(\boldsymbol{z})$.

Case 3:
If $\boldsymbol{f}_1$ does not depend on $\dot{\boldsymbol{y}}$, sliding mode can be enforced in the manifold $\boldsymbol{s}_1 = \dot{\boldsymbol{s}} + \alpha\boldsymbol{s} = \boldsymbol{0}$, $\alpha > 0$, where $\boldsymbol{s}$ and assumption are similar to case 2. For this case, as seen before stability of zero dynamics is not required. Indeed, if $\boldsymbol{s}_1 = \boldsymbol{0}$, then $\boldsymbol{s}$ and $\boldsymbol{z}$ tend to zero. As a result, $\boldsymbol{y}$ tends to zero as a solution to the equation

$$\boldsymbol{s} = \boldsymbol{f}_1(\boldsymbol{z}, \boldsymbol{y}, \dot{\boldsymbol{z}}) + c_1\boldsymbol{z} + c_2\dot{\boldsymbol{z}} = \boldsymbol{0}\,. \tag{27}$$

4.3 Infinite dimensional systems

Finally, an application example with scalar control for which the equations of motion cannot be transformed into a set of blocks governed by first order differential equations as needed by the traditional BCP will be considered.

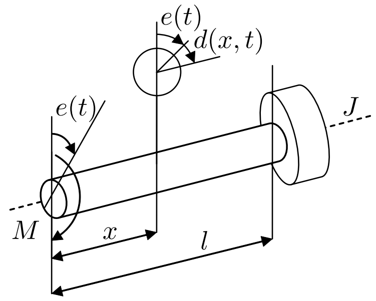

Fig. 3. Flexible shaft with load and DC motor.

The objective is to control the position of the load (Fig. 3) with the moment of inertia J connected to a DC motor (not shown in figure) by means of a flexible shaft of length l. The control input u is the voltage of the DC motor developing torque M at the left end of the shaft.

$$L\frac{di}{dt} = -iR - K\dot{e} + u\,, \tag{28}$$

L, R and K are motor constants, i is the motor current.

The angle $Q(x,t)$ in the cross section at the point x is equal to the sum of the angle of the shaft's left end $e(t)$ and the deformation angle $d(x,t)$ and is governed by the partial differential equation (Utkin, 1993)

$$\frac{\partial^2 Q(x,t)}{\partial t^2} = a^2\frac{\partial^2 Q(x,t)}{\partial x^2}, \quad a = \mathrm{const.} \tag{29}$$

with boundary conditions

$$\begin{aligned} M &= -a^2\frac{\partial^2 Q(0,t)}{\partial x} \\ J\frac{\partial^2 Q(l,t)}{\partial t^2} &= -a^2\frac{\partial Q(l,t)}{\partial x} \end{aligned} \tag{30}$$

and arbitrary initial conditions. The control u should be designed to reduce the load position $Q(l,t)$ to zero. In these situations, the commonly used control methodology is based on the modal forms of motion and as a result deals with an approximation by ordinary differential equations. In this article the generalized block control principle will be demonstrated by application on the accurate model.

First, the equations of motion will be represented in a different form using the Laplace transform:

$$\begin{aligned} s^2\tilde{Q}(s,x) &= a^2\tilde{Q}''(s,x) \\ a^2\tilde{Q}'(s,0) &= -\tilde{M}(s) \\ a^2\tilde{Q}'(s,l) &= -Js^2\tilde{Q}(s,l) \end{aligned} \tag{31}$$

$\tilde{Q}$ and $\tilde{M}$ denote Laplace transforms, Q' and Q'' denote derivatives with respect to x.

The second order differential equation with given boundary conditions can be solved and the Laplace transform of the load position can be found by substituting l for x, ($\tau = \frac{l}{a}$):

$$\tilde{Q}(s,l) = \frac{2e^{-\tau s}}{as\left(1 + \frac{Js}{a}\right) + \left(1 - \frac{Js}{a}\right)e^{-2\tau s}}\tilde{M}(s) \tag{32}$$

The corresponding differential-difference equation can be found in the form

$$\begin{aligned} &J\ddot{Q}(t) + J\ddot{Q}(t-\tau)\cdots \\ &\quad + a\dot{Q}(t) - a\dot{Q}(t-2\tau) = 2M(t-\tau) \end{aligned} \tag{33}$$

Denoting $q = Q(t), z(t) = J\ddot{Q}(t) + a\dot{Q}(t)$, the system can be represented in the form of three blocks:

$$\ddot{q}(t) = -\frac{a\dot{q}(t) + z(t)}{J} \tag{34}$$

$$z(t) = 2a\dot{x}(t-2\tau) - z(t-2\tau) + 2M(t-\tau) \tag{35}$$

$$L\frac{di}{dt} = -iR - K\dot{e} + u\ , M = \alpha i \tag{36}$$

Only one of the three blocks (36) is of first order while the first one (34) is of second order and the second block (35), an equation with delay, is of infinite order. Nevertheless, the design of the intermediate controls z, M and the real control u in each block is a rather simple problem. The methodology offered in this article is applied.

Step 1:
The fictitious control z is a selected as linear function

$$z = -k_1 q - k_2\dot{q}\,, \quad k_{1,2} > 0 \tag{37}$$

to provide a desired dynamics in the first block.

Step 2:
The second intermediate control $M(t-\tau)$ is selected such that

$$s_1 = z(t) + k_1 q(t) + k_2 \dot{q}(t) = 0 \tag{38}$$

or according to (35)

$$\begin{aligned} s_1 = k_1 q(t) + k_2 \dot{q}(t) + 2a\dot{q}(t-2\tau) \ldots \\ - z(t-2\tau) + 2M(t-\tau) = 0 \end{aligned} \tag{39}$$

The function $M(t)$ should satisfy the condition

$$\begin{aligned} M(t) = -\frac{1}{2}(k_1 q(t+\tau) + k_2 \dot{q}(t+\tau) + \cdots \\ 2a\dot{q}(t-\tau) - z(t-\tau)) \end{aligned} \tag{40}$$

To get the vector $y(t+\tau) = [q(t+\tau)\ \dot{q}(t+\tau)]^T$, the first block (34) is represented in the form

$$\begin{aligned} \dot{y} &= Ay + bz\,, \\ A &= \begin{pmatrix} 0 & 1 \\ 0 & -\dfrac{a}{J} \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ -\dfrac{1}{J} \end{pmatrix} \end{aligned} \tag{41}$$

to find its solution

$$\begin{aligned} y(t+\tau) &= \begin{pmatrix} q(t+\tau) \\ \dot{q}(t+\tau) \end{pmatrix} \\ &= e^{A\tau} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} + \ldots \\ &\int_t^{(t+\tau)} e^{A(t+\tau-y)} bz(\gamma) d\gamma \end{aligned} \tag{42}$$

The values of $z(\gamma)$ for the time interval $[t, t+\tau]$ can be found from (35) since the values of the right-hand side are known. Accordingly, the second intermediate control $M(t)$ given by (40) is known.

Step 3:
Finally, the real control, voltage u, has to be designed such that the motor current

$$\begin{aligned} i = \frac{1}{\alpha} M = \frac{1}{2\alpha}(z(t-\tau) - k_1 q(t+\tau) \cdots \\ - k_2 \dot{q}(t+\tau) - 2a\dot{q}(t-\tau)) \end{aligned} \tag{43}$$

Selecting control as a discontinuous state function

$$u = -u_0 \text{sign}(s) \tag{44}$$

$$\begin{aligned} s = i - \frac{1}{2\alpha}(z(t-\tau) - k_1 q(t+\tau) \cdots \\ - k_2 \dot{q}(t+\tau) - 2a\dot{q}(t-\tau)) \end{aligned} \tag{45}$$

sliding mode can be enforced on the surface $s = 0$. This means that in sliding mode, the condition (43) holds. As a result, condition (40) for intermediate control and condition (38) for the first intermediate control are satisfied. These conditions ensure stabilization of the position of the load as a solution to the time-invariant second order differential equation with the desired dynamics.

This example illustrates that the proposed generalized block control principle is a promising design method for infinite dimensional systems with finite dimensional control where the traditional approach fails.

5. CONCLUSION

In this article different design methods for second order non-linear systems, particular cases of arbitrary order systems and infinite dimensional systems were developed. Solutions for the problem of unstable zero dynamics were given based on the example of a rotational inverted pendulum and the example of the linearized model of a double inverted pendulum on a movable cart. The example of the flexible shaft with load and DC motor confirmed the applicability of the proposed design principle to a larger numbers of control problems than the traditional Block Control Principle.

It was demonstrated that the requirement of equal dimension of block states and intermediate controls is superfluous and that the idea of partitioning the system into blocks with a trivial design procedure (mainly blocks in the canonical form) called the generalized block control principle is promising for a wide range of dynamic systems.

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