UNIT VECTOR CONTROL OF UNCERTAIN MULTIVARIABLE NONLINEAR SYSTEMS

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Abstract: A unit vector based output-feedback model-reference adaptive controller (UV-MRAC) for a class of uncertain multivariable nonlinear systems is proposed. This paper generalizes a previous controller which can be applied to plants with uniform relative degree one. Here, the relative degree can be arbitrary. The resulting sliding mode controller is applicable to plants with nonlinear state dependent disturbances which are possibly unmatched with respect to the plant input. The closed loop system has global or semi-global exponential stability with respect to some small residual set and the controller is free of peaking phenomena which could appear in high gain observer based schemes.

Keywords: Sliding mode control, multivariable systems, nonlinear systems, model reference adaptive control, output feedback.

1. INTRODUCTION

Variable structure control (VSC) using output information only has been the subject of several works in the recent past, e.g., (Emelyanov *et al.*, 1992; Oh and Khalil, 1995; Edwards and Spurgeon, 1996; Bartolini *et al.*, 2002). Particularly, the model-reference approach, which is followed in this paper, has been proposed for linear (Hsu *et al.*, 1997) and nonlinear (Min and Hsu, 2000) single-input-single-output (SISO) plants as well as for linear (Tao and Ioannou, 1989; Chien *et al.*, 1996) and nonlinear (Edwards and Spurgeon, 1996) multi-input-multi-output (MIMO) plants.

Recently, a model-reference output-feedback sliding mode control for MIMO nonlinear plants, relying on a nonlinear observer to estimate the plant state, was proposed in (Edwards and Spurgeon, 1996). In contrast, the approach of the present paper, likewise (Tao and Ioannou, 1989) and (Chien *et al.*, 1996), follows the model-reference adaptive control (MRAC) approach which does not require explicit state observers but rather input-output filters, and thus seems more natural for uncertain systems.

This paper utilizes the ideas introduced in (Min and Hsu, 2000) to deal with nonlinear disturbances in SISO systems and extend the results of the unit vector model-reference sliding mode controller (UV-MRAC) of (Hsu *et al.*, 2003) to the arbitrary relative degree case. This leads to a controller design for a class of uncertain MIMO plants with nonlinear state dependent disturbances which are not necessarily uniformly bounded. Due to the use of unit vector control instead of the vector $sign(.)$ function, a less restrictive prior knowledge of the plant high frequency gain is required. The proposed controller is free of the peaking phenomena, which is usual in controllers based on high gain observers, c.f., (Oh and Khalil, 1995).

The $\mathscr{L}_{\infty e}$ norm of the signal $x(t) \in \mathbb{R}^n$ is defined as $\|x_t\|_\infty := \sup_{\tau \leq t} \|x(\tau)\|$. The symbol "$s$" represents either the Laplace variable or the differential operator

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(d/dt), according to the context. The output signal y of a linear time-invariant system with transfer function matrix $H(s)$ and input u is denoted by $H(s)u$. Pure convolution $h(t)*u(t)$, where $h(t)$ is the impulse response of $H(s)$, is also denoted by $H(s)*u$.

2. PROBLEM STATEMENT

This paper considers the model-reference control of a nonlinear MIMO plant

$$\dot{x}_p = A_p x_p + \phi(x_p,t) + B_p u\,, \quad y = C_p x_p\,, \qquad (1)$$

where $x_p \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^m$ is the output, and ϕ is a state dependent nonlinear disturbance. The system matrices A_p, B_p and C_p are uncertain. For $\phi \equiv 0$ the plant (1) is assumed controllable and observable. The linear subsystem has transfer function matrix given by $G(s) = C_p(sI - A_p)^{-1}B_p$ and K_p is the plant high frequency gain (HFG). The following *assumptions* are made:

(A1) $G(s)$ is minimum phase and has full rank.
(A2) The observability index ν of $G(s)$ is known.
(A3) The interactor matrix $\xi(s)$ is diagonal and $G(s)$ has known uniform vector relative degree n^* (i.e., $\xi(s) = s^{n^*}I$).
(A4) A matrix S_p is known s.t. $-K_pS_p$ is Hurwitz.
(A5) The nonlinear disturbance term $\phi(x_p,t)$ is piecewise continuous in t and locally Lipschitz in x_p, $\forall x_p$.
(A6) The term ϕ satisfies $\|\phi(x_p,t)\| \le k_x\|x_p\| + \varphi(y,t)$, $\forall x_p, t$, where $k_x \ge 0$ is a scalar and $\varphi : \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^+$ is a known function piecewise continuous in t and continuous in y.

Assumptions (A1) is usual in MIMO adaptive control. Assumption (A2) can be relaxed to the knowledge of an upper bound for ν, however a high order controller is obtained. Some prior knowledge of the interactor is usually assumed in MIMO adaptive/VSC literature (Tao and Ioannou, 1988; Tao and Ioannou, 1989; Chien *et al.*, 1996). Assumption (A3) may look too strong, however, it can be argued that a diagonal interactor can be achieved by means of an appropriate precompensator. Indeed, in most cases (in a generic sense), Lemma 2.6 in (Tao and Ioannou, 1988) guarantees that there exists a precompensator $W_p(s)$ so that $G(s)W_p(s)$ has diagonal interactor matrix. Moreover, $W_p(s)$ does not depend on the plant parameters. Once the interactor is known to be diagonal and if the relative degree of each element of $G(s)$ (or of $G(s)W_p(s)$) is known, then $\xi(s)$ can be determined without any prior knowledge about the transfer function parameters (Wolovich and Falb, 1976). In order to achieve uniform vector relative degree, one can follow the approach of (Chien *et al.*, 1996) which employs a precompensator to render the relative degree uniform and equal to $n^* = \max_i\{n_i^*\}$. Assumption (A4) is a considerable reduction in the amount of *a priori* knowledge concerning the plant HFG matrix required. Indeed, in (Tao and Ioannou, 1988; Tao and Ioannou, 1989; Chien *et al.*, 1996) the more restrictive assumption of positive definiteness of K_pS_p (and also symmetry in some approaches) was needed. Assumption (A5) is made in order to allow us to develop a control law u that guarantees local existence and uniqueness (in positive time) of the solution of (1). Henceforth, locally Lipschitz in x implicitly assumes locally Lipschitz $\forall x$. According to Assumption (A6), no particular growth condition is imposed on φ. Thus, one could have, e.g., $\varphi(y) = \|y\|^2$. Finite-time escape is therefore not precluded, a priori.

The reference model is defined by

$$y_M = W_M(s)\,r\,, \quad r, y_M \in \mathbb{R}^m\,, \qquad (2)$$
$$W_M(s) = \mathrm{diag}\left\{(s+\gamma_j)^{-1}\right\} L^{-1}(s)\,, \qquad (3)$$
$$L(s) = L_1(s)L_2(s)\cdots L_N(s)\,, \qquad (4)$$
$$L_i(s) = (s+\alpha_i)\,, \qquad (5)$$

$\gamma_j > 0\,,\ (j = 1,\cdots,m)\,,\ \alpha_i > 0\,,\ (i = 1,\cdots,N)\,,$ and $N = n^* - 1$. The signal $r(t)$ is assumed piecewise continuous and uniformly bounded. $W_M(s)$ has the same n^* as $G(s)$ and its HFG is the identity matrix.

The control objective is to achieve asymptotic convergence of the output error $e(t) = y(t) - y_M(t)$ to zero, or to some small residual neighborhood of zero.

3. UNIT VECTOR CONTROL

The unit vector control law is given by

$$u = -\rho(x,t)\frac{v(x)}{\|v(x)\|}\,, \quad \|v\| \neq 0\,, \qquad (6)$$

where x is the state vector, $v(x)$ is a vector function of the state of the system. The modulation function $\rho(x,t) \ge 0$ $(\forall x,t)$ is designed to induce a sliding mode on the manifold $v(x) = 0$. We will henceforth set $u = 0$ if $v(x) = 0$, without loss of generality. Some lemmas regarding the application of the unit vector control within the MRAC framework are presented in (Hsu *et al.*, 2002*a*). These lemmas are instrumental for the controller synthesis and stability analysis.

4. CONTROL PARAMETERIZATION

When the plant is perfectly known and free of nonlinear terms ($\phi \equiv 0$), a control law which achieves matching between the closed-loop transfer function matrix and $W_M(s)$ is given by

$$u^* = \theta^{*T}\omega + \theta_4^{*T} r\,, \qquad (7)$$

where the *parameter matrix* θ^* and the *regressor vector* $\omega(t)$ are given by

$$\theta^{*T} = [\theta_1^{*T}\ \theta_2^{*T}\ \theta_3^{*T}]\,,\ \ \omega = [\omega_1^T\ \omega_2^T\ y^T]^T\,, \qquad (8)$$
$$\omega_1 = A(s)\Lambda^{-1}(s)u\,,\ \ \omega_2 = A(s)\Lambda^{-1}(s)y\,, \qquad (9)$$
$$A(s) = [Is^{\nu-2}\ \ Is^{\nu-3}\ \cdots\ Is\ \ I]^T\,, \qquad (10)$$
$$\Lambda(s) = \lambda(s)I\,, \qquad (11)$$

$\omega_1, \omega_2 \in \mathbb{R}^{m(\nu-1)}$, $\theta_1^*, \theta_2^* \in \mathbb{R}^{m(\nu-1)\times m}$, $\theta_3^*, \theta_4^* \in \mathbb{R}^{m\times m}$ and $\lambda(s)$ is a monic Hurwitz polynomial of degree $\nu - 1$. Let $X = [x_p^T \;\; \omega_1^T \;\; \omega_2^T]^T$ be the state the open loop system composed by the plant (1) and the filters. Considering a nonminimal realization (A_c, B_c, Co) of $W_M(s)$, with state X_M, and defining $W_\phi(s) := K_p^{-1}[W_M(s)]^{-1}C_o(sI - A_c)^{-1}B_\phi$, the nonlinear term ϕ can be regarded as an input disturbance by including the disturbance cancellation term $W_\phi(s) * \phi$. Now, defining the error state as $X_e := X - X_M$, the error equation can be written as

$$\dot{X}_e = A_c X_e + B_c K_p \left[u - \bar{u}\right], \qquad e = C_o X_e, \qquad (12)$$

or in input-output form (Hsu *et al.*, 2002*a*)

$$e = W_M(s) K_p \left[u - \bar{u}\right] \qquad (13)$$

where $\bar{u} = \theta^{*T}\omega + \theta_4^{*T} r - W_\phi(s) * \phi$.

5. UV-MRAC DESIGN AND ANALYSIS

The proposed control law is

$$u = u^{nom} + S_p U_N \quad \left(U_N = -\rho_N \frac{\varepsilon_N}{\|\varepsilon_N\|}\right), \quad (14)$$

$$u^{nom} = \theta^{nomT}\omega + \theta_4^{nomT} r, \qquad (15)$$

where $S_p \in \mathbb{R}^{m\times m}$ is a design matrix which verifies Assumption (A4), $N := n^* - 1$ and θ^{nom} and θ_4^{nom} are nominal values for θ^* and θ_4^*. The control signal U_N and the auxiliary error ε_N are defined according to the controller scheme given in Figs. 1 and 2.

A key idea for the controller generalization is the introduction of the prediction error

$$\hat{e} = W_M(s) L(s) K^{\text{nom}} \left(U_0 - L^{-1}(s) U_N\right), \qquad (16)$$

where K^{nom} is a nominal value of $K = K_p S_p$ and the operator $L(s)$, as given by (5), is such that $G(s)L(s)$ and $W_M(s)L(s)$ have uniform vector relative degree one. The operator $L(s)$ is noncausal but can be approximated by the unit vector lead filter $\mathscr{L}$ shown in Fig. 2. The averaging filters $F_i^{-1}(\tau s)$ in Fig. 2

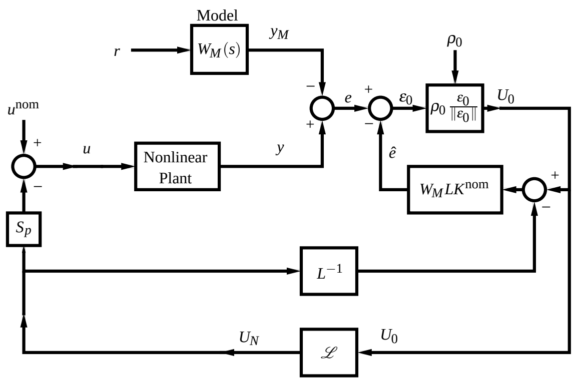

Fig. 1. UV-MRAC for $n^* \geq 2$

are low-pass filters with matrix transfer function given by $F_i^{-1}(\tau s) = [f_{\text{av}i}(\tau s) I]^{-1}$, with $f_{\text{av}i}(\tau s)$ being Hurwitz polynomials in τs such that the filter has unit

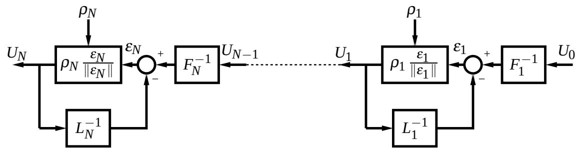

Fig. 2. Implementation of the operator $\mathscr{L}$

DC gain ($f_{\text{av}i}(0) = 1$), e.g., $f_{\text{av}i}(\tau s) = \tau s + 1$. If the time constant $\tau > 0$ is sufficiently small, the averaging filters give an approximation of the equivalent control signals (Utkin, 1992). According to the stability analysis, the time constant τ is chosen small enough to guarantee the maximum tracking error required and the stability region of interest. In addition, for discrete-time implementation, τ is chosen to be 1 or 2 decades above the sampling interval.

5.1 Error equations

The following expressions for the auxiliary error signals are convenient for the controller design and stability analysis (Hsu *et al.*, 1997). From (13) and (16), using $u = \theta^{\text{nom}T}\omega - S_p U_N$, $K = K_p S_p$ and

$$\bar{U} := (K^{\text{nom}})^{-1} K_p [(\theta^* - \theta^{\text{nom}})^T \omega + (\theta_4^* - \theta_4^{\text{nom}})^T r - \\ - W_\phi(s) * \phi] - \left[I - (K^{\text{nom}})^{-1} K\right] U_N, \qquad (17)$$

the auxiliary error $\varepsilon_0 = e - \hat{e}$ can be rewritten as

$$\varepsilon_0 = W_M(s) L(s) K^{\text{nom}} \left[-U_0 - L^{-1}(s)\bar{U}\right]. \qquad (18)$$

The auxiliary errors in the lead filters are given by

$$\varepsilon_i = F_i^{-1}(\tau s) U_{i-1} - L_i^{-1}(s) U_i. \qquad (19)$$

These auxiliary errors can be rewritten as

$$\varepsilon_i = L_i^{-1}(s)\left[-U_i - F_{1,i}^{-1}(\tau s) L_{i+1,N}^{-1}(s)\bar{U}\right] - \\ - \pi_{ei} - \pi_{0i}, \quad (i = 1, \ldots, N-1), \qquad (20)$$

$$\varepsilon_N = -L_N^{-1}(s)(K^{\text{nom}})^{-1} K \left[U_N + F_{1,N}^{-1}(\tau s) U_d\right] - \\ - \left[I - (K^{\text{nom}})^{-1} K\right] \beta_{uN} - \pi_{eN} - \pi_{0N}, \qquad (21)$$

where $L_{i,j}(s) = \prod_{k=i}^{j} L_k(s)$ ($L_{i,j}(s) = 1$ if $j < i$), $F_{i,j}(\tau s)$ is defined in similar way and (by convention, $\pi_{e1} \equiv 0$)

$$U_d = S_p^{-1}\left[(\theta^* - \theta^{\text{nom}})^T \omega + (\theta_4^* - \theta_4^{\text{nom}})^T r\right] - \\ - S_p^{-1}\left[W_\phi(s) * \phi\right], \qquad (22)$$

$$\beta_{uN} = \left[F_{1,N}(\tau s) - I\right] F_{1,N}^{-1}(\tau s) L_N^{-1}(s) U_N, \qquad (23)$$

$$\pi_{ei} = L_{i-1}(s) F_i^{-1}(\tau s)[\pi_{e,i-1} + \varepsilon_{i-1}], \qquad (24)$$

$$\pi_{0i} = \left[W_M(s) F_{1,i}(\tau s) L_{i,N}(s) K^{\text{nom}}\right]^{-1} \varepsilon_0. \qquad (25)$$

5.2 Error system stability

Consider the error system (12), (18), (20), and (21). Let X_ε denote the state vector of (18) and x_{FL}^0 denote the transient state (Hsu *et al.*, 1997) corresponding to the following operators: L^{-1} in (18), $F_{1,i}^{-1} L_{i+1,N}^{-1}$ in

(20) and all the remaining operators associated with $\beta_{uN}, \pi_{ei}, \pi_{0i}$ in (23)–(25). Since all these operators are linear and BIBO stable, there exist positive constants K_{FL} and a_{FL} such that $\|x^0_{FL}(t)\| \leq K_{FL} e^{-a_{FL}t} \|x^0_{FL}(0)\|$. In order to fully account for the initial conditions, the following state vector z is used

$$z^T = [(z^0)^T, \varepsilon_N^T, X_e^T],$$
$$(z^0)^T = [X_\varepsilon^T, \varepsilon_1^T, \varepsilon_2^T, \ldots, \varepsilon_{N-1}^T, (x^0_{FL})^T]. \qquad (26)$$

In what follows, all K's and a's denote generic positive constants, and "Π" and "Π^0" denote any term of the form $K\|z(0)\|e^{-at}$ and $K\|z^0(0)\|e^{-at}$, respectively. Since finite escape time cannot be excluded a priori, define $[0, t_M)$ as the maximum time interval of definition of a given solution, where t_M may be finite or infinite. Henceforth, $\forall t$ means $\forall t \in [0, t_M)$.

Theorem 1. For $N = n^* - 1 \geq 1$, consider the auxiliary errors (18), (20) and (21). If $-K^{\text{nom}}$ and $-(K^{\text{nom}})^{-1}K$ are Hurwitz matrices, and the relay modulation functions satisfy ($\forall t$)

$$\rho_0(t) \geq (1 + c_{d0})\|L^{-1} * \bar{U}\| + c_{\varepsilon 0}\|\varepsilon_0\|,$$
$$\rho_i(t) \geq (1 + c_{di})\|(F^{-1}_{1,i} L^{-1}_{i+1,N}) * (\bar{U})\|, \qquad (27)$$
$$\rho_N(t) \geq (1 + c_{dN})\|F^{-1}_{1,N} * U_d\|,$$

$(i = 1, \cdots, N-1)$, and

$$\rho_N(t) \leq C(t) := M_\omega \|\omega_t\|_\infty + M_{\text{red}}, \qquad (28)$$

with some $M_\omega, M_{\text{red}} > 0$, and some appropriate constants $c_{\varepsilon 0}, c_{dN} \geq 0$ and $c_{di} \geq 0$, then the auxiliary errors ε_i, $(i = 0, \cdots, N-1)$, tend to zero at least exponentially. Moreover,

$$\|\varepsilon_i(t)\|, \|X_\varepsilon(t)\| \leq \Pi^0, \qquad (29)$$
$$\|\varepsilon_N(t)\| \leq \tau \left\|I - (K^{\text{nom}})^{-1}K\right\| K_{eN} C(t) + \Pi, \qquad (30)$$

$$\|\pi_{ei}(t)\|, \|\pi_{0i}(t)\| \leq \Pi^0, \quad i = 1, \ldots, N, \qquad (31)$$
$$\|\beta_{uN}(t)\| \leq \tau K_{\beta N} C(t) + \Pi^0. \qquad (32)$$

PROOF. The proof is identical to that of Theorem 4 of (Hsu *et al.*, 2002*a*, p. 300) except that, here, $\forall t$ means $\forall t \in [0, t_M)$. □

Theorem 2. For $N = n^* - 1 \geq 1$, assume that (A1)–(A7) hold, $-K^{\text{nom}}$ and $-(K^{\text{nom}})^{-1}K$ are Hurwitz matrices, and that the modulation functions satisfy (27) and (28). Then, $\exists k_x^* > 0$, such that for $k_x < k_x^*$ and for sufficiently small $\tau > 0$, the error system (12), (18), (20) and (21) with state z as defined in (26) is semi-globally exponentially stable with respect to a residual set of order τ, i.e., $\forall R_0$, $\exists a, k > 0$ such that $\|z(t)\| \leq ke^{-at}\|z(0)\| + O(\tau)$, $\forall t$ provided $\|z(0)\| \leq R_0$. The constant R_0 can be made arbitrarily large as $\tau \to +0$ and the constants k and a are independent of τ.

PROOF. The proof follows Theorem 5 given in (Hsu *et al.*, 2002*a*), but here the nonlinearities that appear in the input disturbance terms must be taken into account generating a semi-global stability result. See Appendix A for a concise proof. □

5.3 Modulation functions

In order to design modulation functions that satisfy (27), a norm bound for $W_\phi(s) * \phi$ must be estimated, according to the definition of the signals $\bar{U}$ (17) and U_d (22). However, ϕ is a function of the unmeasured system state x_p, thus an estimative of $\|x_p\|$ is also needed. According to assumption (A6) and (Hsu *et al.*, 2003, Lemma 3), it is possible to find a constant $k_x^* > 0$ such that, for $k_x \in [0, k_x^*]$ an norm bound for x_p can be obtained by using first order approximation filters, deriving the following inequality

$$\|x_p(t)\| \leq \rho_X(t) + \Pi(t) \qquad (33)$$

where

$$\rho_X(t) = c_1 \tau_{av}\|U_{av}\| + \frac{1}{s + \lambda_x}[(c_2 + \tau_{av} c_3)\|U_{av}\|]$$
$$+ \frac{1}{s + \lambda_x}[c_4 \varphi + c_5\|\omega\| + c_6\|r\|], \qquad (34)$$

with c_i $(i = 1, \cdots, 6)$ and λ_x being appropriate positive constants, see (Hsu *et al.*, 2003). An upper bound for $\|\phi\|$ can be derived from ρ_X through the application of the inequality in assumption (A6). If $n^* = 1$, then $W_\phi(s)$ is proper and stable, thus the development of an upper bound for the disturbance term $W_\phi(s) * \phi$ is straightforward (Hsu *et al.*, 2003).

For $n^* > 1$, $W_\phi(s)$ could be improper. In this case, we write $W_\phi = W_N s^N + \cdots + W_1 s + W_0 + \bar{W}_\phi(s)$, where $W_i \in \mathbb{R}^{m \times n}$ and $\bar{W}_\phi(s)$ is strictly proper and BIBO stable. Now, considering the following additional assumption

(A7) The terms $W_i \phi(x_p, t)$ are continuous with respect x_p and its partial derivatives of order up to N exist and are locally Lipschitz and $\|W_i \phi^{(i)}\|$ can be bounded by some class-$\mathcal{K}$ function of $\|x_p\|$, $\forall t$ and $\forall i \in \{1, \cdots, N\}$.

Then, one has

$$\|(W_N s^N + \cdots + W_1 s) * \phi\| \leq \hat{\phi}(\rho_X + \Pi), \qquad (35)$$

where $\hat{\phi}$ is a class-$\mathcal{K}$ function. The following property is useful to separate the upper bound (35) $\hat{\phi}(\rho_X + \Pi)$ as a sum of a known term, depending on ρ_X, and an unknown one decaying term, depending on Π.

Property 3. (*Separability property for class-$\mathcal{K}$ functions*) Let f be a class-$\mathcal{K}$ function and a, b be arbitrary positive constants. Then, the inequality

$$f(a + b) \leq f((\alpha + 1)a) + f((\alpha^{-1} + 1)b).$$

is verified.

PROOF. Let α be any arbitrary positive constant. Since f is an increasing function then $f(a + b) \leq f((\alpha^{-1} + 1)b)$ when $a < b/\alpha$. In addition, $f(a + b) \leq f((\alpha^{-1} + 1)b) + f((\alpha + 1)a)$, since f assumes positive values only. Using the same argument for the $a \geq b/\alpha$ case, the same inequality results thus proving the stated property. $\square$

By using the above property, $W_\phi * \phi$ can be bounded by

$$\|W_\phi * \phi\| \leq \hat{\phi}((\alpha+1)\rho_X) + \|W_0\| k_x \rho_X + \|W_0\| \varphi(y,t) + \frac{c_\phi}{s+\gamma_\phi} * [k_x \rho_X + \varphi(y,t)] + v_\pi\,, \qquad (36)$$

with constants $c_\phi > 0$ and $0 < \gamma_\phi < \lambda_o$, where λ_o is the stability margin of A_c, in (12), and $v_\pi := \hat{\phi}((\alpha^{-1}+1)\Pi) + \Pi$. Thus, a modulation function that satisfies (27), *modulo* decaying terms, can be implemented using only the available signals ρ_X, ω and r. The additional transient terms due to the filters used to implement the modulation functions can be easily included in the "Π" terms, which are considered in the stability analysis.

6. SIMULATION RESULTS

In (Hsu *et al.*, 2003), the application of the UV-MRAC was illustrated considering a fault tolerant velocity control of a chain of three trailers as depicted in Fig. 3. In that case the plant had relative degree one. Here, taking the control input as the actuator forces $\bar{u} \in \mathbb{R}^3$ and the output as the *positions* $y = [y_1, y_2]^T$, the plant has relative degree $n^* = 2$. The state vector is $x_p = [v_1\ v_2\ v_3\ y_1\ y_2]^T$, where $v_i (i = 1,2,3)$ are the velocities. The trailers are connected by dampers, two of them are linear with damping coefficients $B_{31} = B_{23} = 1\,Ns/m$ while the two others are nonlinear with velocity $\times$ force characteristics ($F_d(v)$) given in (Hsu *et al.*, 2003). The masses of the trailers are $m_1 = 1\,kg$, $m_2 = 2\,kg$ and $m_3 = 0.5\,kg$. The vector of resultant forces $F_r \in \mathbb{R}^2$ is given by $F_r = S\mathscr{F}\bar{u}$ where the matrix $S = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $\mathscr{F} = \text{diag}\{\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3\}$, where the fault index $\mathscr{F}_i = 1$ if the *i-th* actuator is working correctly, $0 < \mathscr{F}_i < 1$ if its performance is degraded or $\mathscr{F}_i = 0$ if it is completely lost.

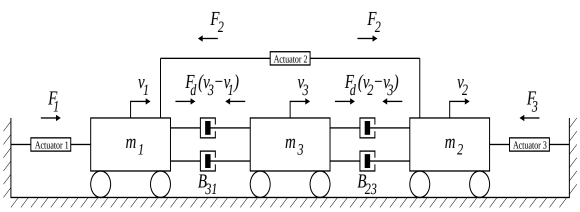

Fig. 3. Chain of three trailers.

Since the system has two outputs, only two control signals are needed in view of the proposed control objective, thus a mixer matrix is used to combine the actuators (Hsu *et al.*, 2003). Let $u \in \mathbb{R}^2$ be the controller output vector. In contrast to the usual active scheme to deal with actuators failure, here one has a passive fault tolerant control approach based on the UV-MRAC and an appropriate constant mixer matrix S^T such that the closed loop system stability and tracking performance are immune to some actuator faults. After applying the mixer ($\bar{u} = S^T u$), the dynamics of the trailers and mixer can be represented by (1) with

$$A_p = \begin{bmatrix} -B_{31}m_1^{-1} & 0 & B_{31}m_1^{-1} & 0 & 0 \\ 0 & -B_{23}m_2^{-1} & B_{23}m_2^{-1} & 0 & 0 \\ B_{31}m_3^{-1} & B_{23}m_3^{-1} & -(B_{31}+B_{23})m_3^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_p = \begin{bmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} S\mathscr{F}S^T\,, \quad C_p = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\phi(x_p) = \begin{bmatrix} F_d(v_3 - v_1)m_1^{-1} \\ -F_d(v_2 - v_3)m_2^{-1} \\ [F_d(v_2 - v_3) - F_d(v_3 - v_1)]\,m_3^{-1} \\ 0 \\ 0 \end{bmatrix}.$$

Note that the Hurwitz condition required to apply the UV-MRAC is satisfied for $S_p = I$. It is noteworthy that the uncertain matrix K_p may not be symmetric and sign definite, thus precluding the application of algorithms which require such properties. From the definition of $F_d(v)$, the plant nonlinearity can be bounded by $\|\phi(x_p)\| \leq 0.96\|x_p\| + 2.1$. The reference model is given by $W_M(s) = (s+4)^{-2}I$ and the unit vector lead filter is such that $F_1^{-1}(\tau s) = (\tau s + 1)I$, with $\tau = 0.003$ and $L_1(s) = s + 4$. The state is chosen with $\lambda(s) = s^2 + 20s + 100$. The nominal parameter matrices θ^{nom} and θ_4^{nom} were computed for the case of perfectly operating actuators and $K^{nom} = 0.25I$. The controller parameters were chosen such that the closed loop system performance is maintained if at least two actuators operate correctly. The modulation functions ρ_0 and ρ_1 were developed following (Hsu *et al.*, 2003) and (Hsu *et al.*, 2002*b*). The complete loss of actuator 1 is simulated. The convergence of the output signals is observed in Fig. 4 where the reference signals are a square wave and a sine wave, respectively, with amplitude 10 and frequency $4\,Hz$.

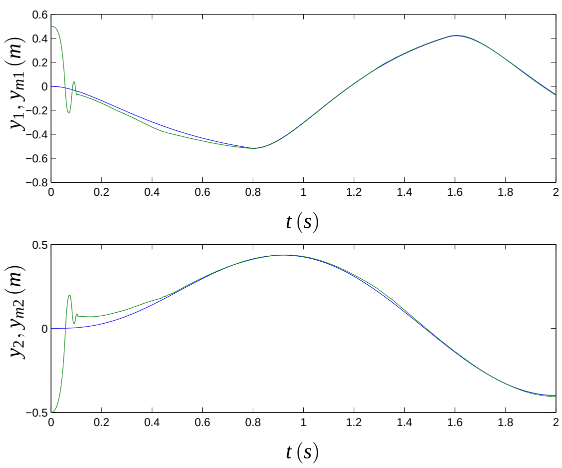

Fig. 4. Trailers positions and reference model outputs.

7. CONCLUSION

An output-feedback model-reference sliding mode controller (UV-MRAC) design for a class of uncertain multivariable nonlinear systems has been proposed. This represents an extension of the controller introduced in (Hsu *et al.*, 2003) for systems of arbitrary relative degree. The proposed controller is shown to be semi-globally exponentially stable with respect to a small residual set. Simulations results illustrate the performance of the proposed scheme in the presence of actuator failure. This suggests the potential of the UV-MRAC as a fault tolerant controller.

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Appendix A. PROOF OF THEOREM 2

Given $R > 0$ and $0 < R_0 < R$, then for some t^*, $t^* \in (0, t_M)$ and $\|z(0)\| < R_0$ one has $\|z(t)\| < R$ for $t \in [0, t^*)$. Then, while $t \in [0, t^*)$, if ρ_N satisfies (28) then $\bar{U}$, given in (17), can be bounded by

$$\|\bar{U}(t)\| \leq C(t) + \Pi(t), \tag{A.1}$$

see (28). Indeed, consider the upper bound (36) for $W_\phi * \phi$ obtained in terms of the class-$\mathcal{K}$ function $\hat{\phi}$. According to assumptions (A5) and (A7), the function $\hat{\phi}$ can be chosen locally Lipschitz and thus, $\exists k_{\hat{\phi}} > 0$, depending on R, such that $|\hat{\phi}(\zeta)| \leq k_{\hat{\phi}}\zeta$, for $0 \leq \zeta \leq R$. Since the bound (34) is a function of the signals ω and r, where $r(t)$ is uniformly bounded by assumption, then the terms $\hat{\phi}((\alpha+1)\rho_X)$ and $\hat{\phi}((\alpha^{-1}+1)\Pi)$ in (36) can be bounded affinely in $\|\omega_t\|_\infty$. Then, from (17) and (28) follows (A.1).

Now, according to Theorem 5, given in (Hsu *et al.*, 2002*a*), applied to $t \in [0, t^*)$, one has

$$\|z^0(t)\| \leq k_{zR} e^{-a_z t} \|z^0(0)\|, \tag{A.2}$$

$$\|z_e(t)\| \leq \tau k_{2R}\big(\|z_e(0)\| + \|z^0(0)\|\big) + O(\tau) + \Pi. \tag{A.3}$$

which are valid if ρ_N satisfies (27) and if $\tau < \bar{k}_{1R}^{-1}$, where $a_z > 0$, $\bar{k}_{1R}, k_{2R}, k_{zR} > 0$ are constants depending on R and $O(\tau)$ is independent of the initial conditions. From (A.2) and (A.3) $\exists N_R > 1$ such that $\|z(t)\| \leq N_R e^{-a_m t}\big(\|z_e(0)\| + \|z^0(0)\|\big) + \tau k_{2R}\big(\|z_e(0)\| + \|z^0(0)\|\big) + O(\tau)$, for $t \in [0, t^*)$, where $a_m = \min(a, a_z)$.

Now, since $z(t)$ is absolutely continuous, then, for sufficiently small τ there exists a constant k_{z_0}, depending on R, such that $\big(\|z_e(0)\| + \|z^0(0)\|\big) \leq k_{z_0}$ implies $\|z(t)\|$ to be bounded away from R as $t \to t^*$. If we assume that t^* is finite then $\|z(t)\| < R - \varepsilon_R$, $\forall t < t^*$ and some constant $\varepsilon_R > 0$. Therefore, one cannot reach the boundary of $B_R = \{z : \|z(t)\| < R\}$ in finite time. Thus $z(t) \in B_R, \forall t \in [0, t_M)$. This implies that $z(t)$ is uniformly bounded and cannot escape in finite time, i.e., $t_M = +\infty$. Furthermore, the constant k_{z_0} can be made arbitrarily large when $\tau \to +0$. Again, following the steps in the proof of the (Hsu *et al.*, 2002*a*, Theorem 5) and noting that the initial time is irrelevant in deriving the above expressions, linear recursive inequalities can be derived leading to the conclusion that, for τ small enough, the error system is semi-globally exponentially stable with respect to a residual set of order τ which is independent of the initial conditions.