MOTION PLANNING FOR MULTIPLE SYSTEMS UNDER COORDINATED CONSTRAINTS

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Abstract: Two problems of motion planning for controlled systems which are required to attain a given target set under coordinated constraints are formulated and solved using dynamic optimization techniques. Constraint coordination arises when the state of each system is mapped onto state constraints for other systems. These problems are formulated in terms of backward reach sets which are the sub-zero level sets of appropriate value functions for non-standard cost functions. The value functions are the solutions of Hamilton-Jacobi-Bellman type PDEs. For linear dynamics and ellipsoidal constraints the value functions are calculated through duality techniques from convex analysis.

Keywords: Dynamic programming, path planning, reachability.

1. INTRODUCTION

The problem of motion planning and coordination for multiple systems has received significant attention in the literature. A significant body of this work deals with the problem of formation planning and control (Wang and Hadaegh, 1996; Spry, 2002; Tabuada *et al.*, 2001). However, there are requirements for motion planning and control other than keeping a formation (de Sousa *et al.*, 2002; de Sousa and Sengupta, 2001). Some of these requirements are more appropriately described by coordinated state constraints. Constraint coordination arises when the state of each system is mapped onto state-constraints for the other systems.

Here we address the problem of planning the motions of multiple systems to reach a certain number of targets under coordinated state constraints. The state constraints are modelled as set-valued maps mapping the state of each system onto constraints for the other systems. There is one target set for each system. The problem is solvable when the target sets are reached at **some** time θ within some prescribed time interval T. In this paper we address two versions of this problem: 1) the motion of one system is known in advance; 2) the motions of all systems are planned to take advantage of the coordinated constraints. We address these problems using backward reach set computation and dynamic optimization techniques (Kurzhanskii and Varaiya, 2001; Kurzhanskii and Varaiya, 2000). We do this for two coordinated systems. The solution methodology is directly applicable to a larger number of systems.

Dynamic optimization techniques are used in an efficient algorithm for globally optimal trajectories for systems given by $\dot{x} = u(t), \|u\| \leq 1$ subject to simple state constraints and a travelling

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cost that depends only on the state (Tsitsiklis, 1995). Ordered Upwind Methods have been used to solve Hamilton-Jacobi-Bellman-type equations describing path planning problems for systems modelled by an hybrid automaton with switching costs among different dynamics (Sethian and Vladimirsky, 2002). Techniques from optimal control and game theory are used in (Lygeros *et al.*, 1995; Tomlin *et al.*, 2000) to design controllers for safety specifications in hybrid systems.

The paper is organized as follows. In section 2 we introduce the mathematical preliminaries. In section 3 we state the problems under consideration. In section 4 we use dynamic optimization techniques to characterize the solution to these problems and for controller synthesis. In section 5 we find the solution for linear systems using duality techniques from non-linear analysis. In section 6 we draw the conclusions.

2. PRELIMINARIES

Consider the controlled motions of a dynamic system evolving in $\mathbb{R}^n$ described as:

$$\dot{x} = f(t, x, u), u(t) \in P(t) \subset \mathbb{R}^m \tag{1}$$

with the standard conditions for uniqueness and prolongability of the solutions for $t \geq t_0$ (see for example (Arnold, 1995)).

Definition 1. The backward reach set at time τ relative to target set $\mathcal{X}_f$ and time $t_f \geq \tau$, $W[\tau, t_f, \mathcal{X}_f]$, is the set of points $W[\tau, t_f, \mathcal{X}_f] = \bigcup \Big\{ x[\tau] | u(s) \in P(s), s \in [\tau, t_f), x[t_f] \in \mathcal{X}_f \Big\}$ where $x[\tau]$ is state of the system at time τ when driven by control $u(t)$.

The definition of backward reach set for the case where the target set $\mathcal{X}_f$ can be reached within some time interval $T = [t_\alpha, t_\beta]$ with $t_\alpha \geq t_0$ follows.

Definition 2. The backward reach set at time $\tau \leq t_\alpha$, $W[\tau, t_\alpha, t_\beta, \mathcal{X}_f]$, is the set of points $x \in \mathbb{R}^n$ such that there exists a control $u(t)$ that drives the trajectory of the system $x[t] = x(t, \tau, x)$ from state (τ, x) to the target set $\mathcal{X}_f$ at **some** time $\theta \in [t_\alpha, t_\beta]$.

The relation between dynamic optimization and reachability was first observed in (Leitmann, 1982). See also (Varaiya, 1998) for a description of reach set computation using optimal control. The key observation is that the reach set is the level set of an appropriate value function (Kurzhanskii and Varaiya, 2002). To illustrate this point consider the following value function:

$$V(\tau, x) = \min_{u(.)} \{ d^2(x(t_f), \mathcal{X}_f) | x(\tau) = x \}$$
$$V(t_f, x) = d^2(x, \mathcal{X}_f) \tag{2}$$

where u(.) is an admissible control function defined for $[\tau, t_f]$ and $d(x(t_f), \mathcal{X}_f)$ is the Euclidean distance between the state of the system at time t_f and target set $\mathcal{X}_f$ for a trajectory starting at (τ, x). Obviously, x belongs to the backward reach set if this distance is zero. But this also means that the backward reach set is the zero level set of the value function V:

$$W[\tau, t_f, \mathcal{X}_f] = \{ x | V(\tau, x) \leq 0 \} \tag{3}$$

If the value function satisfies the principle of optimality then it can be determined from the solution of the generalized Hamilton-Jacobi-Bellman (HJB) PDE associated with it. This is the case for V in equation (2). The corresponding HJB equation is:

$$V_t(t, x) + \max_{u \in P(t)} \langle V_x(t, x) \cdot f(t, x, u) \rangle = 0$$
$$V(t_f, x) = d^2(x, \mathcal{X}_f) \tag{4}$$

Definition 3. The ellipsoid $\mathcal{E}(a, Q)$ with center a and shape matrix $Q = Q' > 0$ is the set of points:

$$\mathcal{E}(a, Q) = \{ x : (x - a, Q^{-1}(x - a)) \leq 1 \} \tag{5}$$

Its support function is $\rho(l | \mathcal{E}(a, Q)) = \max\{ (l, x) | x \in \mathcal{E}(a, Q) \} = (l, p) + (l, Pl)^{1/2}$ (Rockafellar and Wets, 1998).

3. PROBLEM FORMULATION

Consider the motions of two controlled systems under the assumptions from section 2 for $t \geq t_0$ and given by

$$\dot{x}_1(t) = f_1(t, x_1, u_1), u_1(t) \in P_1(t) \tag{6}$$
$$\dot{x}_2(t) = f_2(t, x_2, u_2), u_2(t) \in P_2(t) \tag{7}$$

where $P_1(t), P_2(t) \in Comp_m$ – the variety of compact sets in $\mathbb{R}^m$. Moreover,

$$x_1(t_0) \in \mathcal{X}_1, \quad x_2(t_0) \in \mathcal{X}_2 \tag{8}$$

Let $\mathcal{M}_1, \mathcal{M}_2 \in Comp_n$ be convex target sets for the motions of system $(i = 1, 2)$.

Denote $u(.) = col\{u_1(.), u_2(.)\}$, $x = col\{x_1, x_2\}$ and $f(t, x, u) = col\{f_1(t, x_1, u_1), f_2(t, x_2, u_2)\}$, and $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. In what follows we will refer both to each system (i = 1, 2) separately, and to the composed system whose state x is driven by control $u(.)$.

Consider the time interval $T = [t_\alpha, t_\beta]$ with $t_\alpha \geq t_0$. Now consider that the motions of the two systems (i = 1, 2) are coupled through the following state constraints (convex and complementary-convex as in (Kurzhanskii *et al.*, 2004)):

$$x_1(t) \in F_2(x_2(t)), \quad x_2(t) \in F_1(x_1(t)) \tag{9}$$
$$x_1(t) \notin G_2(x_2(t)), \quad x_2(t) \notin G_1(x_1(t)) \tag{10}$$

where F_1, F_2, G_1 and G_2 are continuous convex set-valued maps with values in $Comp_n$ with non-empty interior. G_1 and G_2 are avoidance sets which represent safety regions to prevent collisions between the motions of the two systems. F_1 and F_2 are containment sets since the motions of x_2 and x_1 are restricted to stay inside F_1 and F_2 respectively.

Problem 1. (Motion planning). Find the set of all initial conditions $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ such that there exist controls $u_1(t), u_2(t)$ which starting at time t_0 steer the trajectories of both systems to reach $\mathcal{M}_1 \times \mathcal{M}_2$ at **some** time $\theta \in T$ under constraints (9, 10).

The following assumptions ensure that: 1) the problem is well-posed; 2) at most two constraints are active at a time; and 3) the problem has non-empty solution sets.

Assumption 1. $\forall x \in \mathbb{R}^n : G_i(x) \subset F_i(x), i = 1, 2.$

Assumption 2. $\forall x_1, x_2 \in \mathbb{R}^n : \exists y \in \mathbb{R}^n, G_1(x_1) \cap G_2(x_2) = y$ we have $G_1(x_1) \cup G_2(x_2) \subset F_i(x_i), i = 1, 2.$

Assumption 3. $\exists (x_1, x_2) \in \mathcal{M}_1 \times \mathcal{M}_2 : x_1 \in F_2(x_2) \wedge x_2 \in F_1(x_1) \wedge x_1 \notin G_2(x_2) \wedge x_2 \notin G_1(x_1)$

The solution to this problem is given in two steps.

Step 1 Find the backward reach set relative to target set $\mathcal{M}_1 \times \mathcal{M}_2$ and time interval T under state constraints given by equations (9) and (10). This is the reach-evasion set (Tomlin *et al.*, 2000). Next we consider two versions of this problem.

Problem 2. [Given feasible motion x_2^f] Calculate the backward reach set $W_1^g[\tau, t_\alpha, t_\beta, \mathcal{M}_1]$ under constraints (9,10) when a feasible motion $x_2^f(.)$ is known in advance.

A feasible motion of $x_2^f(.)$ is a trajectory $x_2^f[t] = x_2^f(t, \tau, x_2), x_2^f(t_0) \in \mathcal{X}_2$ defined on $[t_0, t_\beta]$ such that $x_2^f(t) \in \mathcal{M}_2$ for some $t \in [t_\alpha, t_\beta]$.

Problem 3. [Coordinated controls] Calculate the backward reach set $W^c[\tau, t_\alpha, t_\beta, \mathcal{M}_1 \times \mathcal{M}_2]$ under constraints (9,10) and coordinated controls.

A pair of controls (u_1, u_2) is said to be **coordinated** when both controls are responsible for both constraints.

Step 2 The solutions to the motion planning problem (1) for the two versions of the backward reach set problem (2, 3) are given respectively by the following sets:

$$S_1^a(t_0) = W_1^g[t_0, t_\alpha, t_\beta, \mathcal{M}_1] \cap \mathcal{X}_1$$
$$S^c(t_0) = W^c[t_0, t_\alpha, t_\beta, \mathcal{M}_1 \times \mathcal{M}_2] \cap \mathcal{X}_1 \times \mathcal{X}_2$$

4. DYNAMIC PROGRAMMING APPROACH

We follow the approach described in (Kurzhanskii and Varaiya, 2004) to calculate the solutions to problems 2 and 3.

4.1 Value functions

First we consider problem (2). Let a feasible trajectory $x_2^f[t] = x_2^f(t, \tau, x_2)$ satisfying assumption 3 be given. Let $T_g = [t_{\alpha_g}, t_{\beta_g}]$, where $t_{\alpha_g}, t_{\beta_g}$ are the first entry and first exit times of this trajectory in $\mathcal{M}_2$. From assumption (3) and the fact that $x_2^f[t]$ is a feasible trajectory we conclude that $S = T \cap T_g \neq \emptyset$.

Let:

$$\begin{aligned}
\varphi_0^1(x_1) &= d^2(x_1, \mathcal{M}_1) \\
\varphi_0^2(x_2) &= d^2(x_2, \mathcal{M}_2) \\
\varphi_1(t, x_1, x_2) &= d^2(x_1, F_2(x_2)) \\
\varphi_2(t, x_1, x_2) &= d^2(x_2, F_1(x_1)) \\
\varphi_3(t, x_1, x_2) &= -d^2(x_1, G_2(x_2)) \\
\varphi_4(t, x_1, x_2) &= -d^2(x_2, G_1(x_1))
\end{aligned} \tag{11}$$

The continuity of functions φ_1^0, φ_2^0 and $\varphi_i (i = 1, \dots, 4)$ results from the continuity and convexity of the set-valued maps G_1, G_2, F_1 and F_2, the convexity of both $\mathcal{M}_1$ and $\mathcal{M}_2$, and the fact that d is the Euclidean distance function.

Corresponding to this problem we introduce the value function:

$$\begin{aligned}
V^g(\tau, x_1, S) = \min_{u_1(.)} \min_{t_f \in S} \Big\{ max \Big\{ & \varphi_0^1(x_1(t_f)), \\
& \{\max_t \phi_1(t, x_1(t)) | t \in [\tau, t_f]\}, \\
& \{\max_t \phi_2(t, x_1(t)) | t \in [\tau, t_f]\}, \\
& \{\max_t \phi_3(t, x_1(t)) | t \in [\tau, t_f]\}, \\
& \{\max_t \phi_4(t, x_1(t)) | t \in [\tau, t_f]\} \Big\} \\
& | x_1(\tau) = x_1 \Big\}
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\phi_1(t,x_1) &= \varphi_1(t,x_1,x_2^f(t)) \\
\phi_2(t,x_1) &= \varphi_2(t,x_1,x_2^f(t)) \\
\phi_3(t,x_1) &= \varphi_3(t,x_1,x_2^f(t)) \\
\phi_4(t,x_1) &= \varphi_4(t,x_1,x_2^f(t))
\end{aligned} \tag{13}$$

The functions $\phi_i, (i = 1,\ldots,4)$ are continuous since $x_2^f(t)$ is continuous in t.

Lemma 1. The following relation is true:

$$W_1^a[\tau, t_{\alpha_g}, t_{\beta_g}, \mathcal{M}_1] = \{x_1 : V^g(\tau, x_1, S) \le 0\}$$

Proceeding similarly for problem (3)

$$\begin{aligned}
V^c(\tau,x,T) = \min_{u(.)}\min_{t_f\in T}\Big\{ & max\Big\{\varphi_0^1(x_1(t_f)), \\
& \varphi_0^2(x_2(t_f)) \\
& \{\max_t \varphi_1(x_1(t),x_2(t))|t\in[\tau,t_f]\}, \\
& \{\max_t \varphi_2(x_1(t),x_2(t))|t\in[\tau,t_f]\}, \\
& \{\max_t \varphi_3(x_1(t),x_2(t))|t\in[\tau,t_f]\}, \\
& \{\max_t \varphi_4(x_1(t),x_2(t))|t\in[\tau,t_f]\}\Big\} \\
& |x_1(\tau)=x_1, x_2(\tau)=x_2\Big\}
\end{aligned} \tag{14}$$

Lemma 2. The following relation is true:

$$W^c[\tau, t_\alpha, t_\beta, \mathcal{M}] = \{x : V^c(\tau, x, T) \le 0\}$$

4.2 Solution approach

Here, we consider the following assumption.

Assumption 4. The functions V^c, V^g, φ_0^1, φ_0^2, $\varphi_i, (i = 1,\ldots,4)$, and $\phi_i, (i = 1,\ldots,3)$ are differentiable.

Next we describe how to calculate $V^c(\tau, x_1, x_2, T)$ (the calculation of $V^g(\tau, x_1, S)$ is identical).

First we consider the case where $t_f = t_\alpha = t_\beta$ and denote $V^c(\tau, x, T) = V^c(\tau, x, t_f) = V^c(\tau, x) = V^c(\tau, x|V^c(t_f, .))$ where

$$\begin{aligned}
V^c(t_f,x_1,x_2) = \max\{&\varphi_0^1(x_1),\varphi_0^2(x_2), \\
&\varphi_1(x_1,x_2),\varphi_2(x_1,x_2),\varphi_3(x_1,x_2), \\
&\varphi_4(x_1,x_2)\}
\end{aligned} \tag{15}$$

The following lemma states the Principle of Optimality for this problem.

Lemma 3. $V^c(\tau, x)$ satisfies a semi-group property, namely:

$$\begin{aligned}
&V^c(\tau,x|V^c(t_f,.)) = \\
&V^c\Big(\tau,x|V^c\big(t,.|V^c(t_f,.)\big)\Big), \tau\le t\le t_f
\end{aligned} \tag{16}$$

The proof of the lemma is based on a standard technique from (Fleming and Soner, 1993). Basically, this means that the value function inherits the semi-group property from the reach set. The infinitesimal form of the Principle of Optimality yields a generalized Hamilton-Jacobi-Bellman PDE for $V^c(\tau, x)$.

Observe that:

$$\begin{aligned}
V^c(\tau,x_1,x_2|V^c(t_f,.)) &\le \varphi_0^2(x_2), \\
V^c(\tau,x_1,x_2|V^c(t_f,.)) &\le \varphi_0^1(x_1), \\
V^c(\tau,x|V^c(t_f,.)) &\le \varphi_i(x), i=1,\ldots,4
\end{aligned} \tag{17}$$

Let:

$$\mathcal{H}(t,x,V^c,u) = V_t^c + (V_x^c(t,x), f(t,x,u)) \tag{18}$$

Following (Kurzhanskii *et al.*, 2004) we conclude that the HJB equation for $V^c(\tau, x_1, x_2)$ is

Case 1) all the inequalities in equation (17) are strict:

$$V_t^c + \min_u \langle V_x^c(t,x), f(t,x,u)\rangle = 0 \tag{19}$$

Case 2) assume there is only one equality relation in equation (17), for example $V^c(\tau, x) = \varphi_i(\tau, x)$. Consider $(x^0(t), u^0(t))$ to be an optimal solution of problem (3) that goes through point x at time t (under the usual assumptions these exist). Then

$$\begin{aligned}
\max\{\mathcal{H}(t,x^0(t),V^c,u),\mathcal{H}(t,x^0(t),\varphi_i,u)\} &\ge \\
\mathcal{H}(t,x^0(t),V^c,u^0(t)) &= \\
\mathcal{H}(t,x^0(t),\varphi_i,u^0(t)) &= 0
\end{aligned} \tag{20}$$

Now we turn to $V^c(\tau, x, T)$.

Lemma 4. The following relation is true:

$$V^c(\tau,x,T) = \min_{t_f\in T} V^c(\tau,x,t_f)$$

In general value functions are not differentiable and assumption (4) does not hold. However, the above derivations are still valid if we use some generalized concept of derivative. In this case, the solutions to the HJB equation have to treated in a generalized ("viscosity" or "minmax") sense (Bardi and Capuzzo-Dolcetta, 1997; Fleming and Soner, 1993; Subbotin, 1995; Lions, 1992; G.Crandall *et al.*, 1984).

4.3 Controller synthesis

The motion planning problem (1) under **coordinated controls** (given feasible trajectory x_2^f) is solvable if $S^c(t_0) \ne \emptyset$ ($S_1^a(t_0) \ne \emptyset$).

Let $t_0 \in \mathbb{R}$ be such that the problem (1) under coordinated controls is solvable. Consider $(x_1^0, x_2^0) \in S^c(t_0)$ and let $\theta = argmin_{t_f \in T} V^c(t_0, x_1^0, x_2^0, t_f)$. Pick the value function $V^c(t_0, x_1^0, x_2^0, \theta)$. Starting at time t_0 the control strategy which solves problem 1 under coordinated controls has a feedback form $u(t, x_1, x_2) \in \mathcal{U}(t, x_1, x_2)$, where the feasible controls $\mathcal{U}(t, x_1, x_2)$ are the minimizers in the HJB equation (19, 20) for $V^c(., ., ., \theta)$. The same type of calculations yield the control strategy for problem 1 under a given feasible trajectory x_2^f.

It may happen that the feedback law $u(t, x_1, x_2)$ is discontinuous in the state. This requires another notion of solution for differential equations (6, 7). One possible approach is to define the solution as a "constructive" motion introduced in (Krasovskii and Subbotin, 1988).

5. LINEAR SYSTEMS

The solution approach described above involves solving a HJB equation for the value functions V^g and V^c. This is not a trivial matter for non-linear systems and general constraints. However, for systems with linear structure and complementary convex constraints the value function can be found through techniques of convex analysis and minimax theory (Gusev and Kurzhanskii, 1971*a*; Gusev and Kurzhanskii, 1971*b*). We illustrate these techniques to find the value function for problem 2 with linear structure and convex and complementary ellipsoidal convex constraints.

The equations of motion are

$$\dot{x}_1(t) = A(t)x_1 + B(t)u_1, u_1(t) \in \mathcal{P}_1(t) \quad (21)$$

where $A(t)$ has continuous coefficients, $\mathcal{P}_1(t) = \mathcal{E}(0, P_1(t))$, P_1 is continuous in t and $P_1 > 0$. It is assumed that the system is completely controllable.

The ellipsoidal and the complementary ellipsoidal convex constraints are given by the set valued-maps F_2 and G_2 which map points to ellipsoids in $Comp_n$ with non-empty interior. For example $x_1 \in F_2(x_2^f)$ is given as $((x_1 - x_2^f), F_2^e(x_1 - x_2^f)) \leq 1)$. The target sets are also non-degenerate ellipsoids $(M_1 > 0, M_2 > 0)$ $\mathcal{M}_1 = \mathcal{E}(m_1, M_1)$, and $\mathcal{M}_2 = \mathcal{E}(m_2, M_2)$.

In order to calculate the backward reach set $W_1^g[t_0, t_\alpha, t_\beta, \mathcal{M}_1]$ through $V^g(\tau, x_1, S)$ we need to consider a constraint qualification from (Kurzhanskii and Varaiya, 2004):

Assumption 5. There exists a control $u_1(t) \in \mathcal{P}_1, t \in [t_0, t_{\beta_g}]$, a point $x_1^0 \in \mathcal{X}_1$, and a number $\epsilon > 0$ such that the trajectory $x_1[t] = x_1(t, t_0, x_1^0 | u_1(.))$ generated by $u_1(t)$ produces a tube

$$x_1(t, t_0, x_1^0) + \epsilon \mathcal{B}_n(0) \subseteq F_2(x_2^f(t)), t \in [t_0, t_{\beta_g}]$$

where $\mathcal{B}_n$ is the unit ball in $\mathbb{R}^n$.

As in (Gusev and Kurzhanskii, 1971*a*) we find a solvability condition for $V^g(\tau, x_1, t_f)$ of the system of inequalities

$$\begin{aligned}(x_1[t] - x_2^f[t]), F_2^e(t)(x_1[t] - x_2^f[t]) \leq \mu^2 \\ (x_1[t_f] - m_1), M_1(t_f)(x_1[t_f] - m_1) \leq \mu^2\end{aligned} \quad (22)$$

and find the smallest μ that ensures solvability.

Furthermore, we consider that assumption 3 holds.

Now let $s[t]$ be a row-vector solution to the adjoint equation

$$ds = -sAdt - q'(t)\Lambda(t), s(t_f) = l' \quad (23)$$

where $q(t)$ is continuous and Λ is nondecreasing of finite variation, then

Theorem 1. $V^g(\tau, x_1, t_f)$ is given by the formula

$$V^g(\tau, x_1, t_f) = \max_{q(.)} \max_{\Lambda(.)} \max_{l} \{(s[\tau], x_1) + \int_\tau^{t_f} (s[t]B(t)P_1(t)B'(t)s'[t])^{1/2} dt\} = \mu^0(\tau, x_1) \quad (24)$$

where the maximums are taken over all functions $(q(t), N^{-1}q(t)) \leq 1, t \in [\tau, t_f], N = F_2^e$ and all elements $(l, M_1^{-1} l^{1/2}) + \int_\tau^{t_f} d\Lambda(t) \leq 1$.

From this theorem we obtain as a corollary that the backward reach set is convex and compact.

6. CONCLUSIONS

We have described motion planning problems under coordinated constraints and used dynamic programming techniques to characterize the solution and to synthesize controllers. The solution method involves solving a HJB equation. This is not a trivial matter. However, for systems with linear structure and ellipsoidal constraints we can use the techniques from (Kurzhanskii and Valyi, 1997) to obtain numerical solutions to the HJB equation. We have not yet explored the geometry of coordinated constraints so as to obtain a better characterization of the solution properties which could lead to more efficient solution methods.

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REFERENCES

Arnold, V. I. (1995). *Ordinary differential equations*. MIT press. Ninth printing.

Bardi, Martino and I. Capuzzo-Dolcetta (1997). *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Birkhauser.

de Sousa, J. Borges, Aníbal C. Matos and F. Lobo Pereira (2002). Dynamic optimization in the coordination and control of autonomous underwater vehicles. In: *Proceedings of Decision and Control Conference*. IEEE. pp. 2087–2092.

de Sousa, J. Borges and Raja Sengupta (2001). Tutorial on autonomous and semi-autonomous networked multi-vehicle systems, decision and control conference.

Fleming, Wendell Helms and H. Mete Soner (1993). *Controlled Markov processes and viscosity solutions*. Springer-Verlag.

G.Crandall, M., L. C. Evans and P. L. Lions (1984). Some properties of solutions of hamilton jacobi equations. *Transactions of the American Mathematical Society* **282**(2), 487–502.

Gusev, M. I. and A. B. Kurzhanskii (1971*a*). Optimization of controlled systems with bounds on the control and the phase coordinates. *Differents. Uravneniya* **7**(9), 1210–1218.

Gusev, M. I. and A. B. Kurzhanskii (1971*b*). Optimization of controlled systems with restrictions. ii. *Differents. Uravneniya* **7**(10), 1789–1800.

Krasovskii, N.N. and A.I. Subbotin (1988). *Game-theoretical control problems*. Springer-Verlag.

Kurzhanskii, A. B. and I. Valyi (1997). *Ellipsoidal calculus for estimation and control*. Birkhauser.

Kurzhanskii, A. B. and P. Varaiya (2000). Ellipsoidal techniques for reachability analysis. In: *Computation and control* (N. Lynch and B. Krogh, Eds.). pp. 202–214. Lecture Notes in Computer Science. Springer-Verlag.

Kurzhanskii, A. B. and P. Varaiya (2001). Dynamic optimization for reachability problems. *Journal of Optimization Theory & Applications* **108**(2), 227–51.

Kurzhanskii, A. B. and P. Varaiya (2002). Optimization methods for target problems of control. In: *Proceedings of Mathematical Theory of Networks and Systems Conference*.

Kurzhanskii, A. B. and P. Varaiya (2004). Ellipsoidal techniques for reachability under state constraints. *submitted to SIAM Journal of Control and Optimization*.

Kurzhanskii, A. B., I. M. Mitchell and P. Varaiya (2004). Control synthesis for state constrained systems and obstacle problems. In: *Proceedings of Nolcos Conference*. IEEE.

Leitmann, G. (1982). Optimality and reachability with feedback controls. In: *Dynamic Systems and Microphysics* (G. Leitmann A. Blaquiere, Ed.). Academic Press.

Lions, P. L. (1992). Viscosity solutions and optimal control. In: *Proceedings of ICIAM 91*. pp. 182–195.

Lygeros, J., Datta N. Godbole and Shankar Sastry (1995). A game theoretic approach to hybrid system design. Technical Report UCB/ERL M95/77. University of California, Berkeley. Electronics Research Laboratory.

Rockafellar, R. and R. J. B. Wets (1998). *Variational analysis*. Springer.

Sethian, J. and A. Vladimirsky (2002). Ordered upwind methods for hybrid control. In: *Proceedings of the hybrid systems workshop*. Springer-Verlag.

Spry, Stephen (2002). Modeling and control of vehicle formations. PhD thesis. University of California at Berkeley.

Subbotin, A.I. (1995). *Generalized solutions of first-order PDEs : the dynamical optimization perspective*. Birkhauser.

Tabuada, P., G. J. Pappas and P. Lima (2001). Feasible formations of multi-agent systems. In: *Proceedings of the American Control Conference*. IEEE.

Tomlin, C. J., J. Lygeros and S. Shankar Sastry (2000). A game theoretic approach to controller design for hybrid systems. *Proceedings of the IEEE* **88**(7), 949–70.

Tsitsiklis, J. (1995). Efficient algorithms for globally optimal trajectories. *IEEE Transactions on Automatic Control* **409**(9), 1528–38.

Varaiya, P. (1998). Reach set computation using optimal control. In: *Proceedings of the KIT Workshop on Verification of Hybrid Systems*. Verimag, Grenoble, France.

Wang, P. and F. Y. Hadaegh (1996). Coordination and control of multiple microspacecraft moving in formation. *The Journal of the Astronautical Sciences* **44**(July-september), 315–355.