AN ALGORITHM FOR SYSTEM IMMERSION INTO NONLINEAR OBSERVER FORM: FORCED SYSTEM

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Abstract: The class of nonlinear system that can be put into nonlinear observer form (linear system with output injection) can be broadened if we employ the system immersion. We provide an algorithm with which one can check whether a forced system with relative degree r can be transformed into nonlinear observer form or not. The proposed algorithm is an extension of the previous result and does not require the relative degree 1 assumption. In addition, it is seen that, the immersibility can always be checked via algebraic computations except one special case when the relative degree equals to the system dimension (in the case, only one first order differential equation appears in the algorithm).

Keywords: observer, system immersion, forced system.

1. INTRODUCTION

The observer error linearization problem is to transform the given system into a linear system with output injection. In this setting, the Luenberger-type observer yields a linear dynamics of the state estimation error. As far as the transformation is concerned, most results in the literature utilize the state diffeomorphism and output diffeomorphism (Krener and Isidori, 1983; Bestle and Zeitz, 1983; Krener and Respondek, 1985; Xia and Gao, 1989; Keller, 1987; Hou and Pugh, 1999; Glumineau *et al.*, 1996; Guay, 2001). There are some results which employee generalized transformations to enlarge the class of systems, for example, system immersions (Levine and Marino, 1986; Back and Seo, 2004*b*; Jouan, 2003) and smooth maps with continuous inverse (Xia and Zeitz, 1997). For further discussion on this topic, see the papers cited above and the references therein.

In this paper, we consider the system immersion of forced nonlinear system into nonlinear observer form. In particular, a constructive algorithm to check the immersibility is developed. Some algorithms for unforced systems are available in the literature (Back and Seo, 2004*b*; Back and Seo, 2004*a*; Jouan, 2003). However, these algorithms require some assumptions, for example, dimensional assumption or constant rank assumption. One of the main obstacle lies in the unforced case problem is that one should solve a differential equation with several unknowns. Fortunately, in the forced system case, all of the unknowns related to the system immersion can be solved via some algebraic computations except some very special case. In Jouan (2003), an algorithm was given for the forced system case under the condition that the system has relative degree 1, which has not been pointed out clearly in Jouan (2003). In this

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work, we will generalize this result to the case when the system admits a relative degree less than the system dimension.

In Section 2, we formulate the problem and characterize the class of forced nonlinear systems which can be immersed into nonlinear observer form. A straightforward algorithm to check the immersibility is derived in Section 3, and an illustrative example is given in Section 4.

2. PROBLEM FORMULATION AND BASIC RESULTS

Let us consider a nonlinear system of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{1}$$

where u is the control input belong to some set of measurable bounded functions $\mathcal{U} \in L^{\infty}(\mathbb{R}^{+}, \mathbb{R})$ and $f(x)$, $g(x)$ and $h(x)$ are smooth on an open connected subset $\mathcal{D}$ of $\mathbb{R}^n$. We assume the observability rank condition:

$$\dim \operatorname{span}\{dh, dL_f h, \cdots, dL_f^{n-1} h\} = n, \ \forall x \in \mathcal{D}. \tag{2}$$

The observability assumption enables us to define

$$\partial_i^j(a(x)) := \frac{\partial^j a(x)}{\partial (L_f^i h)^j}, \ 0 \le i \le n-1, \ j = 1, 2, \cdots.$$

We abbreviate $\partial_i^1 a(x)$ as $\partial_i a(x)$. In addition, $a'(h)$ denotes the derivative of $a(h)$ w.r.t. h. To obtain $\partial_i a(x)$, express the one form $da(x)$ as a linear combination of the one forms $dh, \cdots, dL_f^{n-1}h$ and then pick the coefficient of $dL_f^i h$.

We also consider the N dimensional observer form

$$\begin{aligned} \dot{z} &= Az + a(y, u), \quad z \in \mathbb{R}^N, \\ y &= Cz \end{aligned} \tag{3}$$

where

$$A := \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \ a(y,u) := \begin{bmatrix} a_1(y) + \tilde{a}_1(y)u \\ a_2(y) + \tilde{a}_2(y)u \\ \vdots \\ a_N(y) + \tilde{a}_N(y)u \end{bmatrix},$$

$$C := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Let $X_t(u, x_0)$ (resp., $Z_t(u, z_0)$) be the solution of system (1) (resp., (3)) starting from $x_0 \in \mathcal{D}$ (resp., $z_0 \in \Phi(\mathcal{D})$) and driven by u and let $\tau_{x_0}^u := \sup\{t \ge 0 \mid X_t(u, x_0) \in \mathcal{D}\}$

Given (1) and (3), we define the immersibility.

Definition 1. The system (1) is said to be immersible into $N = n + m$ dimensional observer form (3) in $\mathcal{D}$ if there exists a smooth function $\Phi : \mathcal{D} \to \Phi(\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^n$, $\Phi(\mathcal{D}) \subset \mathbb{R}^N$ such that if for every x_0 and every z_0 with $\Phi(x_0) = z_0$, then $h(X_t(u, x_0)) = CZ_t(u, z_0)$ for every t such that $0 \le t < \tau_{x_0}^u$.

The next result provides a condition for the system (1) to be immersible into observer form (3). The proof can be found in Jouan (2003).

Theorem 1. Suppose that the drift part of the system (1) is observable on $\mathcal{D}$. Then the system (1) is immersible into $N = n + m$ dimensional observer form in $\mathcal{D}$ if and only if, in $\mathcal{D}$:

(a) The drift part of the system (1) is immersible into observer form.
(b) The immersion $z = \Phi(x) = [\phi_1(x), \cdots, \phi_N(x)]^T$ satisfies

$$dL_g \phi_i \wedge dh = 0, \quad i = 1, \cdots, N. \tag{4}$$

Let us discuss on the conditions of the theorem. As far as the condition (a) is concerned, it is equivalent to find N functions $a_1(h), \cdots, a_N(h)$ of the characteristic equation

$$L_f^N h = a_1(h) + L_f a_2(h) + \cdots + L_f^{N-1} a_N(h) \tag{5}$$

of which proof can be found in Back and Seo (2004*b*), or in Jouan (2003). In addition, from these papers, we know that a solution $a_1(h), \cdots, a_N(h)$ to (5) provides us the immersion in a closed form and the components of Φ are given by

$$z_{N-k} := \phi_{N-k}(x) = L_f^k h - \sum_{i=0}^{k-1} L_f^{k-i-1} a_{N-i}(h), \quad k = 0, \cdots, N-1. \tag{6}$$

In general, it is not easy to solve the differential equation (5) since it contains N unknowns which should be solved simultaneously. Some algorithms are available under a dimensional assumption (Back and Seo, 2004*b*) or constant rank assumption (Back and Seo, 2004*a*). Although these algorithms are constructive, one should solve a differential equation at each step of the algorithm. We will show that in the forced system case, it is possible to check the immersibility via algebraic calculations except a very special case.

Let us recall a necessary and sufficient condition for condition (a) which has been proved in (Back and Seo, 2004*b*). We will use this result during the derivation of the algorithm.

Proposition 1. [Back and Seo (2004*b*)] Let $N = n + m$ and suppose the system is observable. The drift part of the system (1) is immersible into N dimensional observer form if and only if there exists a solution $a_1(h), \cdots, a_N(h)$ to the equations

$$\begin{aligned} \partial_i(L_f^N h) = &\sum_{k=i}^{N-1} \binom{k}{i} L_f^{k-i} a'_{k+1}(h) \\ &+ \sum_{k=0}^{m-1} \sum_{j=0}^{k} \binom{n+k}{n+j} L_f^{k-j} a'_{n+k+1}(h) \partial_i L_f^{n+j} h \end{aligned} \tag{7}$$

where $i = 0, \cdots, n-1$.

Now, let us investigate what the condition (4) means. If the system (1) is immersible into (3), then the immersion $\Phi(x)$ should satisfy the condition (4). For the immersion given by (6), consider the case $k = 0$:

$$dL_g\phi_N(x) \wedge dh = 0.$$

Since $\phi_N(x) = h$, the condition implies that L_gh should be a function of h only.

For $k = 1$, $dL_g\phi_{N-1}(x) \wedge dh = 0$. Using $\phi_{N-1}(h) = L_fh - a_N(h)$, we have

$$dL_g\phi_{N-1}(h) = dL_gL_fh - a'_N(h)dL_gh.$$

From $dL_gh \wedge dh = 0$, we obtain

$$dL_gL_fh \wedge dh = 0. \tag{8}$$

Thus, L_gL_fh should be a function of h only.

Let us consider the case $k = 2$. By definition, $\phi_{N-2}(h) = L_f^2h - L_fa_N(h) - a_{N-1}(h)$. Thus,

$$dL_g\phi_{N-2}(h) \wedge dh = 0$$

becomes $dL_gL_f^2h \wedge dh = a''_N(h)L_ghdL_fh \wedge dh$. Hence, if L_gh is not zero on $\mathcal{D}$ (i.e., $L_gh \neq 0, \forall x \in \mathcal{D}$), we can obtain a relation for $a''_N(h)$ as follows:

$$a''_N(h) = \frac{1}{L_gh}\frac{\partial L_gL_f^2h}{\partial L_fh}.$$

This relations implies that we can obtain the unknown using a_N condition (b) of Theorem 1.

The main purpose of this paper is to generalize this fact in order to derive the relations that the unknowns should satisfy. If it is possible, one can obtain N unknowns without solving the characteristic equation and check the immersibility via algebraic computations. To achieve this, we first note that the condition $L_gh \neq 0, \forall x \in \mathcal{D}$ implies that the system has relative degree 1. It will be shown that the relative degree plays an important role while we derive an algorithm to check the immersibility.

Definition 2. The system (1) is said to have relative degree r, $1 \le r \le n$, in $\mathcal{D}$ if

$$\begin{aligned} &L_gL_f^ih(x) = 0,\ 0 \le i \le r-2,\ \forall x \in \mathcal{D} \\ &L_gL_f^{r-1}h(x) \neq 0,\ \forall x \in \mathcal{D}. \end{aligned} \tag{9}$$

The rest of this section is devoted to develop some useful tools of which proofs are omitted for lack of space.

Lemma 1. For a smooth function $a(h)$, we have

$$L_f^ka(h) = \sum_{j=0}^{k-1}\binom{k-1}{j}L_f^{k-1-j}a'(h)L_f^{j+1}h. \tag{10}$$

Lemma 2. If the system (1) has relative degree r, then we have

$$L_gL_f^ka(h) = 0, \quad k = 0, \cdots, r-2, \tag{11}$$

where $a(h)$ is any smooth function.

Lemma 3. If the system (1) has relative degree r, then

$$dL_g\phi_{N-k} \wedge dh = 0, \quad 0 \le k \le r-2. \tag{12}$$

Lemma 4. If the system (1) has relative degree r, and immersible into $N = n + m$ dimensional observer form, then

$$dL_gL_f^{r-1}h \wedge dh = 0, \tag{13}$$

$$dL_gL_f^rh \wedge dh = 0. \tag{14}$$

3. IMMERSION ALGORITHM

In this section, an algorithm is presented for the immersion problem. As mentioned in the previous section, the condition (b) of Theorem 1 can be used to obtain the relations which the unknowns $a_i(h)$'s should satisfy. However, the condition does not provide everything we want. The number of unknowns can be obtained from the condition is closely related to the relative degree of the system. In fact, if the system has relative degree r, then $a_{r+2}(h), \cdots, a_N(h)$ can be obtained. Therefore, when $r = n$, the function $a_{n+1}(h)$ can not be obtained in a closed form from the condition. Thus, we first derive all relations that can be obtained from the condition (b) of Theorem 1 and provide a solution to the case $r = n$. Then, this result will be combined with Proposition 1 to construct the desired algorithm.

To begin with, we provide a necessary condition for the unknowns.

Lemma 5. If the system (1) has relative degree r, and immersible into $N = n + m$ dimensional observer form, the function $a_N(h)$ in (6) satisfies

$$a''_N(h) = \frac{1}{rL_gL_f^{r-1}h}\partial_1L_gL_f^{r+1}h, \tag{15}$$

and the functions $a_{N-(k-1)}(h)$ ($2 \le k \le N-r-1$) in (6) enjoy the relation:

$$\begin{aligned} &a''_{N-(k-1)}(h) \\ &= \frac{1}{rL_gL_f^{r-1}h}\Bigg[\partial_1L_gL_f^{r+k}h - \sum_{j=0}^{k-2}e_jL_gL_f^{e_j-1}a'_{N-j}(h) \\ &- \delta^+_{k,n}\sum_{i=n}^{k}\left(\sum_{j=0}^{k-i}\tbinom{e_j}{i}L_gL_f^{e_j-i}a'_{N-j}(h)\right)\partial_1L_f^ih \\ &- \sum_{i=2}^{k}\left(\sum_{j=0}^{k-i}\tbinom{e_j}{c_i}L_f^{k-i-j}a'_{N-j}(h)\right)\partial_1L_gL_f^{r-1+i}h\Bigg], \end{aligned} \tag{16}$$

where $e_j := r + k - 1 - j$, $c_i := r - 1 + i$, and the function $\delta^+_{k,n}$ is defined by

$$\delta^+_{k,n} = \begin{cases} 1, & \text{if } k \ge n \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Remark 1. The equation (16) has an interesting structure; the function $a_i(h)$ is dependent on the functions $a_{i+1}(h)$, $a_{i+2}(h)$, $\cdots$, $a_N(h)$. Thanks to this structure, the unknowns can be obtained one by one. Firstly, the function $a_N(h)$ is given by (15) directly. With the function $a_N(h)$ at hand, we can proceed to obtain $a_{N-1}(h)$ by (16) with $k = 2$. Similarly, the functions $a_{N-2}(h)$, $a_{N-3}(h)$, $\cdots$, $a_{r+2}(h)$ can be obtained by applying the equation (16) step by step.

It is worthwhile to point out that Lemma 5 is a necessary condition of the condition (b) of Theorem 1. In order to achieve our goal, we provide a necessary and sufficient condition for the condition as follows:

Lemma 6. Suppose the system (1) is observable and has relative degree r. If the drift part of the system (1) is immersible into observer form, then the condition (b) of Theorem 1 is equivalent to the following conditions

(a) $L_gL_f^{r-1}h \wedge dh = 0$, $L_gL_f^rh \wedge dh = 0$.
(b) The functions $a_{r+2}(h)$, $\cdots$, $a_N(h)$ satisfy the equation (15) and the equation (16) for all $2 \leq k \leq n+m-r-1$
(c) For all $s = 2, \cdots, n-1$ and $1 \leq k \leq n+m-r-1$, the functions $a_{r+2}(h)$, $\cdots$, $a_{n+m}(h)$ satisfy

$$
\begin{aligned}
&\partial_s L_g L_f^{r+k} h \\
&- \delta_{k,s}^{+} \sum_{j=0}^{k-s} \binom{e^\star(j)}{s} L_g L_f^{e^\star(s+j)} a'_{N-j}(h) \\
&- \delta_{k,n}^{+} \sum_{i=n}^{k} \left[\sum_{j=0}^{k-i} \binom{e^\star(j)}{i} L_g L_f^{e^\star(i+j)} a'_{N-j}(h) \right] \partial_s L_f^i h \\
&- \delta_{k,2}^{+} \sum_{i=2}^{k} \left[\sum_{j=0}^{k-i} \binom{e^\star(j)}{r-1+i} L_f^{k-i-j} a'_{N-j}(h) \right] \times \\
&\qquad \partial_s L_g L_f^{r-1+i} h = 0
\end{aligned}
$$

where $e^\star(q) := r+k-1-q$ (q is an integer).

Although Lemma 5 is a necessary condition for immersibility, it provides us a checkable method to find the unknowns. In fact, as pointed out in Remark 1, the unknowns $a_{r+2}(h)$, $\cdots$, $a_N(h)$ can be obtained recursively. Thus, when $r \leq n-1$, using Lemma 5, Lemma 6 and Proposition 1, it is possible to develop a necessary and sufficient condition which characterizes the immersibility and can be checked step by step. It should be mentioned that when $r = n$, it is required to derive a relation for $a_{n+1}(h)$, since in this case Lemma 5 works only for the unknowns $a_{n+2}(h)$, $\cdots$, $a_N(h)$. In order to solve this problem, we provide a relation for $a_{n+1}(h)$ as follows:

$$
\begin{aligned}
\partial_{n-1}^2 L_f^N h &= \overline{a}''_{n+1}(h) + \overline{a}'_{n+1}(h)\partial_{n-1}^2 L_f^n h \\
&+ \partial_{n-1}\left(\sum_{s=n+1}^{n+m-1} L_f^{s-n+1} \overline{a}'_{s+1}(h) \right) \\
&+ \partial_{n-1}\left(\sum_{s=1}^{m-1} \sum_{j=0}^{s} \binom{n+s}{n+j} L_f^{s-j} \overline{a}'_{n+s+1}(h) \partial_{n-1} L_f^{n+j} h \right)
\end{aligned}
$$

which is obtained by operating ∂_{n-1} to the equation (7) with $i = n-1$.

Now, we are in a position to state the main result of this paper. It is an immersion algorithm for forced system (IMALGOFS for short). Using IMALGOFS, one can check whether a nonlinear system can be immersed into observer form. Moreover, the algorithm is constructive; one does not need to solve the unknowns simultaneously.

Let us explain the algorithm in detail. At first, check if the control vector field satisfies Lemma 4 (Step 0). Secondly, at Step 1, Step $k(2 \leq k \leq N-r-1)$ and Step $N-r$, one calculates $\overline{a}_{r+2}(h)$, $\cdots$, $\overline{a}_N(h)$ (when $r < n$) or $\overline{a}_{n+1}(h)$, $\cdots$, $\overline{a}_N(h)$ (when $r = n$) and check the condition (c) of Lemma 6. Finally, at Step $N-r+1$ and Step $N-r+2$, one checks whether $a_1(h)$, $\cdots$, $a_n(h)$ can be obtained or not and verify that the functions $\overline{a}_{r+1}(h)$, $\cdots$, $\overline{a}_n(h)$ obtained from Lemma 5 are equivalent to $a_{r+1}(h)$, $\cdots$, $a_n(h)$. For simplicity, we define a function $M_k(x)$ for $1 \leq k \leq N-r-1$ as

$$
\begin{aligned}
M_k(x) := \frac{1}{rL_gL_f^{r-1}h} \Bigg[&\partial_1 L_g L_f^{r+k} h \\
&- \delta_{k,1}^{+} \sum_{j=0}^{k-2} (e^\star(j)) L_g L_f^{e^\star(j)-1} \overline{a}'_{N-j}(h) \\
&- \delta_{k,2}^{+} \delta_{k,n}^{+} \sum_{i=n}^{k} \left(\sum_{j=0}^{k-i} \binom{e^\star(j)}{i} L_g L_f^{e^\star(i+j)} a'_{N-j}(h) \right) \partial_1 L_f^i h \\
&- \delta_{k,2}^{+} \sum_{i=2}^{k} \left(\sum_{j=0}^{k-i} \binom{e^\star(j)}{r-1+i} L_f^{k-i-j} \overline{a}'_{N-j}(h) \right) \partial_1 L_g L_f^{r-1+i} h \Bigg].
\end{aligned}
\tag{18}
$$

IMALGOFS-the immersion algorithm for forced system:
Suppose the observable system (1) has relative degree $r \leq n$. Follow the steps to obtain the unknowns $a_1(h)$, $\cdots$, $a_N(h)$. If the procedure fails at any step, then the algorithm fails.

Step 0: Check the relations:$dL_gL_f^{r-1}h \wedge dh = 0$, $dL_gL_f^rh \wedge dh = 0$.

Step 1: Check if $M_1(x)$ is a function of h only and check if

$$\partial_s L_g L_f^{r+1} h = 0, \quad s = 2, \cdots, n-1.$$

If this is true, set $\overline{a}''_N(h) = M_1(x)$ and proceed to Step 2.

Step $k(2 \leq k \leq N-r-1)$: Let $e^\star(q) := r+k-1-q$ (q is an integer). Using the functions

$\overline{a}''_{N-k}(h), \cdots, \overline{a}''_N(h)$ obtained at previous steps, check if $M_k(x)$ is a function of h only and check if the following equations for $s = 2, \cdots, n-1$ hold.

$$\partial_s L_g L_f^{r+k} h - \delta^+_{k,s} \sum_{j=0}^{k-s} \binom{e^\star(j)}{s} L_g L_f^{e^\star(s+j)} \overline{a}'_{N-j}(h)$$
$$- \delta^+_{k,n} \sum_{i=n}^{k} \left[\sum_{j=0}^{k-i} \binom{e^\star(j)}{i} L_g L_f^{e^\star(i+j)} \overline{a}'_{N-j}(h) \right] \partial_s L_f^i h$$
$$- \sum_{i=2}^{k} \left[\sum_{j=0}^{k-i} \binom{e^\star(j)}{r-1+i} L_f^{k-i-j} \overline{a}'_{N-j}(h) \right] \times$$
$$\partial_s L_g L_f^{r-1+i} h = 0.$$

If this is true, then set $\overline{a}''_{N-(k-1)}(h) = M_k(x)$ and proceed to Step $k+1$.

Step $N - r$: If $r \neq n$, jump to next step. If $r = n$, solve (19) (shown below) for $\overline{a}'_{n+1}(h)$ using the functions $\overline{a}_{n+2}(h), \cdots, \overline{a}_N(h)$.

$$\begin{aligned}
&\partial^2_{n-1} L_f^N h \\
=&\overline{a}''_{n+1}(h) + \overline{a}'_{n+1}(h) \partial^2_{n-1} L_f^n h \\
&+ \partial_{n-1} \left(\sum_{s=n+1}^{N-1} L_f^{s-n+1} \overline{a}'_{s+1}(h) \right) \\
&+ \partial_{n-1} \left(\sum_{s=1}^{m-1} \sum_{j=0}^{s} \binom{n+s}{n+j} L_f^{s-j} \overline{a}'_{n+s+1}(h) \partial_{n-1} L_f^{n+j} h \right).
\end{aligned} \tag{19}$$

If $\overline{a}'_{n+1}(h)$ can be obtained, proceed to next step.

Step $N - r + 1$: Let $a_{n+i}(h) = \overline{a}_{n+i}(h)$, $1 \leq i \leq m$, where $\overline{a}_{n+i}(h)$ are the functions obtained at previous steps. Using the functions $a_{n+i}(h)$, solve the equation (20) (shown below) recursively for $a_1(h), \cdots, a_n(h)$. Precisely, let $i = n-1$ and solve (20) for $a_n(h)$. Then, let $i = n-2$. Using the functions $a_{n+1}(h), \cdots, a_N(h)$ from previous steps, and the function $a_n(h)$ obtained at the operation $i = n-1$, solve (20) for $a_{n-1}(h)$, and so on.

$$\begin{aligned}
&a'_{i+1}(h) \\
=&\partial_i L_f^N h - \sum_{s=i+1}^{N-1} \binom{s}{i} L_f^{s-i} a'_{s+1}(h) \\
&- \sum_{s=0}^{m-1} \sum_{j=0}^{s} \binom{n+s}{n+j} L_f^{s-j} a'_{n+s+1}(h) \partial_i L_f^{n+j} h.
\end{aligned} \tag{20}$$

Step $N - r + 2$: Decide the integral constants of $\overline{a}_i(h)$ and $a_i(h)$, $i = 1, \cdots, n$, in order that $\overline{a}_i(h) = a_i(h)$.

The algorithm developed so far provides a necessary and sufficient condition stated below.

Theorem 2. Suppose the system (1) has relative degree $r \leq n$. Then, the systems is immersible into $N = n + m$ dimensional observer form if and only if there exist a solution $\{a_1(h), \cdots, a_N(h)\}$ to IMALGOFS.

4. EXAMPLE

Let us consider 3 dimensional nonlinear system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + \frac{1}{2} x_3^2 + \frac{1}{3} x_3^3 + \alpha(x_1) + g_2(x) u \\
\dot{x}_3 &= x_3 \\
y &= h(x) = x_1,
\end{aligned} \tag{21}$$

where $\alpha(\cdot)$ is a smooth function, and $g_2(x)$ is a smooth function such that $g_2(x) \neq 0$ $\forall x \in \mathbb{R}^3$. The system has relative degree 2 and globally observable. Simple calculation yields:

$$\begin{aligned}
L_f h =& x_2, \quad L_f^2 h = x_3 + \frac{1}{2} x_3^2 + \frac{1}{3} x_3^3 + \alpha(x_1) \\
L_f^3 h =& x_3 + x_3^2 + x_3^3 + \alpha'(x_1) x_2 \\
L_f^4 h =& [1 + \alpha'(x_1)] x_3 + [2 + \frac{1}{2} \alpha'(x_1)] x_3^2 \\
&+ [3 + \frac{1}{3} \alpha'(x_1)] x_3^3 + \alpha'(x_1) \alpha(x_1) \\
L_f^5 h =& [1 + 3\alpha''(x_1) x_2 + \alpha'(x_1)] x_3 \\
&+ [4 + \frac{3}{2} \alpha''(x_1) x_2 + \alpha'(x_1)] x_3^2 \\
&+ [9 + \alpha''(x_1) x_2 + \alpha'(x_1)] x_3^3 + \alpha^{(3)}(x_1) x_2^2 \\
&+ 3\alpha''(x_1) \alpha(x_1) x_2 + \alpha'(x_1) \alpha'(x_1) x_2.
\end{aligned}$$

Let $F(x_1, x_2, x_3)$ be a smooth function. The identity:

$$\frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 + \frac{\partial F}{\partial x_3} dx_3 = \partial_0 F dh + \partial_1 d L_f h + \partial_2 d L_f^2 h,$$

yields

$$\begin{aligned}
\partial_0 F &= \frac{\partial F}{\partial x_1} - \frac{1}{1 + x_3 + x_3^2} \frac{\partial F}{\partial x_3} \cdot \alpha'(x_1), \\
\partial_1 F &= \frac{\partial F}{\partial x_2}, \ \partial_2 F = \frac{1}{1 + x_3 + x_3^2} \frac{\partial F}{\partial x_3}.
\end{aligned}$$

The system is not diffeomorphic to 3 dimensional observer form and IMALGOFS tell us that the system is not immersible into 4 dimensional observer form. Now, let us try to immerse this system into 5 dimensional observer form ($n = 3$ and $m = 2$). From Step 0, we set $g_2(x) = g_2(x_1)$. At Step 1, we have $\overline{a}''_5(x_1) = 0$. Let $\overline{a}'_5(x_1) = c_5$ (c_5 is a constant). At Step 2, since

$$\begin{aligned}
L_g L_f^4 h &= 2x_2 \cdot \alpha''(x_1) g_2(x_1), \\
\partial_2 L_g L_f^4 h &= 2\alpha''(x_1) g_2(x_1), \quad \partial_2 L_g L_f^4 h = 0,
\end{aligned}$$

we have

$$\begin{aligned}
M_2(x) = \frac{1}{2 g_2(x_1)} [&\partial_1 L_g L_f^4 h - 3 L_g L_f^2 \overline{a}'_5(x_1) \\
&- \overline{a}'_5(x_1) \partial_1 L_g L_f^3 h] := \alpha''(x_1),
\end{aligned}$$

and

$$\partial_2 L_g L_f^4 h - 3 L_g L_f \overline{a}'_5(x_1) - \overline{a}'_5(x_1) \partial_2 L_g L_f^3 h = 0.$$

Thus, we set $\overline{a}'_4(x_1) = \alpha'(x_1) + c_4$, where c_4 is a constant. As $r \neq n$, we pass Step 3. For Step 4, let $a'_5(x_1) = c_5$ and $a'_4(x_1) = \alpha'(x_1) + c_4$. The equation (20) for $i = 2$ becomes

$$a'_3(x_1) = \partial_2 L_f^5 h - 2L_f a'_4(x_1) - 6L_f^2 a'_5(x_1)$$
$$- a'_4(x_1)\partial_x L_f^3 h - 4L_f a'_5(x_1)\partial_2 L_f^3 h - a'_5(x_1)\partial_2 L_f^4 h$$

Using the relations

$$\begin{aligned}
\partial_2 L_f^3 h &= \frac{1 + 2x_3 + 3x_3^2}{1 + x_3 + x_3^2}, \\
\partial_2 L_f^4 h &= \alpha'(x_1) + \frac{1 + 4x_3 + 9x_3^2}{1 + x_3 + x_3^2}, \\
\partial_2 L_f^5 h &= 3\alpha''(x_1)x_2 + \frac{1}{1 + x_3 + x_3^2} \cdot [1 + \alpha(x_1) \\
&\qquad + [8 + 2\alpha'(x_1)]x_3 + [27 + 3\alpha'(x_1)]x_3^2],
\end{aligned}$$

we have

$$a'_3(x_1) = -c_5\alpha'(x_1)$$
$$+ \frac{1 - c_4 - c_5 + [8 - 2c_4 - 4c_5]x_3 + [27 - 3c_4 - 9c_5]x_3^2}{1 + x_3 + x_3^2}.$$

Since $a'_3(x_1)$ is a function of x_1 only, it is required that $c_4 = -11$ and $c_5 = 6$. Similarly, from (20) with $i = 1$, we obtain $a'_2(x_1) = 11\alpha'(x_1)$. From (5) $a_1(x_1)$ can be computed as $a_1(x_1) = -6\alpha(x_1) + c_1$. Therefore, the system is immersible into 5-dimensional observer form. The transformed system and the immersion are shown below:

$$\begin{aligned}
\dot{z}_1 &= -6\alpha(y) - 6g_2(y)u \\
\dot{z}_2 &= z_1 + 11\alpha(y) + 11g_2(y)u \\
\dot{z}_3 &= z_2 - 6\alpha(y) + 6y - 6g_2(y)u \\
\dot{z}_4 &= z_3 + \alpha(y) - 11y + g_2(y)u \\
\dot{z}_5 &= z_4 + 6y \\
y &= z_5
\end{aligned}$$

$$z = \begin{pmatrix} -6x_2 + 6x_3 + \frac{3}{2}x_3^2 + \frac{2}{3}x_3^3 \\ -6x_1 + 11x_2 - 5x_3 - 2x_3^2 - x_3^3 \\ 11x_1 - 6x_2 + x_3 + \frac{1}{2}x_3^2 + \frac{1}{3}x_3^3 \\ -6x_1 + x_2 \\ x_1 \end{pmatrix}.$$

5. CONCLUSIONS

In order to characterize the immersibility of an unforced n dimensional nonlinear system into $n + m$ dimensional observer form, one should solve the characteristic equation, which is in general a hard problem because it is required to find $n + m$ unknowns simultaneously from one differential equation. Although the algorithms developed previously enable us to find the unknowns one by one, it is required to solve a differential equation at each steps.

In this paper, we observed the close relation between the relative degree and the system immersion in the forced systems case and developed an algorithm with which not only the the immersibility of a system into observer form can be checked but also the unknowns for the immersion can be found one by one. We note that the unknowns can always be computed through a closed form algebraic computation except a very special case when $r = n$; in this case only one first order differential equation appears. Extensions to MIMO case with vector relative degree will be of interest.

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