$\mathcal{H}_\infty$ GUARANTEED COST COMPUTATION VIA POLYNOMIALLY PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS

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Abstract: Linear matrix inequality conditions for $\mathcal{H}_\infty$ guaranteed cost evaluation of uncertain linear systems in polytopic domains are presented in this paper. The conditions are based on homogeneous polynomially parameter-dependent Lyapunov functions of arbitrary degree. As the degree grows, tests of increasing precision are obtained providing more accurate $\mathcal{H}_\infty$ guaranteed costs. Both continuous and discrete-time uncertain systems are addressed, as illustrated by numerical examples.

Keywords: Robust stability, guaranteed cost, H-infinity norm, linear matrix inequality, homogeneous polynomially parameter-dependent Lyapunov functions, uncertain linear systems, polytopic domains.

1. INTRODUCTION

In the last years, affine linear parameter-dependent Lyapunov functions have been used to improve the results that use a fixed Lyapunov matrix (i.e. quadratic stability) for robust stability and performance analysis of uncertain systems. In (Feron *et al.*, 1996), linear matrix inequality (LMI) sufficient conditions for robust stability and $\mathcal{H}_\infty$ guaranteed cost for uncertain continuous-time systems, based on multiconvexity, are given. More recently, LMI conditions providing parameter-dependent Lyapunov functions to assess the robust stability of uncertain linear systems in polytopic domains have appeared, for both continuous and discrete-time systems (Peaucelle *et al.*, 2000). Using the Finsler's Lemma, these conditions are formulated with some extra variables, providing less conservative results. Extensions to compute $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs can be found in (Arzelier *et al.*, 2002) and, for the discrete-time case, robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control synthesis conditions are given in (de Oliveira *et al.*, 2002). However, those conditions are conservative in the sense that some matrices are fixed and must fulfill the entire set of LMIs.

A new approach to assess the robust stability of a polytope by means of affine parameter-dependent Lyapunov functions has been given in (Ramos and Peres, 2001) and (Ramos and Peres, 2002), where the product between parameter dependent system matrices and affine parameter dependent Lyapunov matrices has been dealt with in terms of LMIs. This idea was exploited in (Leite and Peres, 2003) to improve and extend the robust stability results, allowing the extra variables obtained from the Finsler's Lemma to be parameter-dependent as well. Extensions to compute $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs can be found in (de Oliveira *et al.*, 2004*a*) and (de Oliveira *et al.*, 2004*b*).

In this paper, homogeneous polynomially parameter dependent Lyapunov functions of arbitrary degree are used to obtain a systematic procedure to generate LMI conditions of increasing precision to compute $\mathcal{H}_\infty$ guaranteed costs of uncertain linear systems in polytopic domains. The conditions exploit the positivity of the uncertain parameters, being constructed such that: as the degree of the polynomial increases, the number of linear matrix inequalities and free variables increases and the tests become less conservative. Moreover, if a solution exists for a certain degree, the conditions will also be verified for larger degrees and the $\mathcal{H}_\infty$ guaranteed costs will be smaller or, at least, equal. For degree zero, quadratic stability based guaranteed cost computation (Palhares *et al.*, 1997), as well as, for degree one, affine parameter-dependent results (de Oliveira *et al.*, 2004*b*) are recuperated as special cases. Numerical examples illustrate the results.

The notation used throughout the paper is standard. The symbol $(')$ indicates transpose; $P > 0$ (≥ 0) means that P is symmetric positive (semi)definite. $\mathbb{N}$ denotes the natural numbers, $\mathbb{R}$ the real numbers and $N!$ denotes factorial. The indices c and d are used to denote the continuous and discrete-time cases, respectively, and the symbol $\star$ indicates symmetric blocks in the LMIs.

2. PRELIMINARIES

Consider the linear time-invariant uncertain system described by the following state-space equation

$$\begin{aligned} \delta[x(t)] &= A(\alpha)x(t) + B(\alpha)w(t) \\ y(t) &= C(\alpha)x(t) + D(\alpha)w(t) \end{aligned} \quad (1)$$

with $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. The symbol $\delta[\cdot]$ represents the derivative operator for continuous-time and the forward operator for discrete-time systems. The quadruple $(A,B,C,D)(\alpha)$ is not precisely known, but belongs to a convex bounded (polytope type) uncertain domain $\mathcal{S}$ given by

$$\mathcal{S} \triangleq \Big\{(A,B,C,D)(\alpha) \ : (A,B,C,D)(\alpha) = \sum_{i=1}^{N} \alpha_i(A_i,B_i,C_i,D_i); \quad \alpha \in \Omega\Big\}$$

$$\Omega = \Big\{\alpha_i \geq 0,\ i = 1,\dots,N;\ \sum_{i=1}^{N}\alpha_i = 1\Big\} \quad (2)$$

Any uncertain matrix quadruple $(A,B,C,D)(\alpha) \in \mathcal{S}$ can be written as a convex combination of the vertices (A_i,B_i,C_i,D_i), $i = 1,\dots,N$ of the polytope in terms of $\alpha \in \Omega$. Moreover, for any feasible α, $A(\alpha)$ is assumed to be asymptotically stable. For a fixed α, the transfer matrix from the input vector w to the output vector y is given by

$$H_{wy}(\alpha,s) = C(\alpha)\left(s\mathbf{I} - A(\alpha)\right)^{-1}B(\alpha) + D(\alpha) \quad (3)$$

with the frequency variable s replaced by the time-shift operator z in the discrete-time case.

The following lemmas are well known results that completely characterize bounds on the $\mathcal{H}_\infty$ norm for both continuous and discrete-time cases, respectively:

Lemma *1.* The inequality $\| H_{wy}(\alpha,s) \|_\infty^2 < \mu_c$ holds if and only if there exists a symmetric positive definite matrix $P(\alpha) \in \mathbb{R}^{n\times n}$ such that

$$M^c(\alpha) \triangleq \begin{bmatrix} A(\alpha)'P(\alpha) + P(\alpha)A(\alpha) & \star & \star \\ B(\alpha)'P(\alpha) & -\mathbf{I} & \star \\ C(\alpha) & D(\alpha) & -\mu_c\mathbf{I} \end{bmatrix} < 0 \quad (4)$$

is feasible for all $\alpha \in \Omega$.

Lemma *2.* The inequality $\| H_{wy}(\alpha,z) \|_\infty^2 < \mu_d$ holds if and only if there exists a symmetric positive definite matrix $P(\alpha) \in \mathbb{R}^{n\times n}$ such that

$$M^d(\alpha) \triangleq \begin{bmatrix} P(\alpha) & \star & \star & \star \\ P(\alpha)A(\alpha) & P(\alpha) & \star & \star \\ \mathbf{0} & B(\alpha)'P(\alpha) & \mathbf{I} & \star \\ C(\alpha) & \mathbf{0} & D(\alpha) & \mu_d\mathbf{I} \end{bmatrix} > 0 \quad (5)$$

is feasible for all $\alpha \in \Omega$.

Equivalent results including extra matrix variables can be obtained by using the Finsler's lemma.

Lemma *3.* The inequality $\| H_{wy}(\alpha,\cdot) \|_\infty^2 < \mu$ holds if and only if there exists a symmetric positive definite matrix $P(\alpha) \in \mathbb{R}^{n\times n}$ and matrices $\mathcal{X}(\alpha)(\alpha) \in \mathbb{R}^{(2n+m+p)\times(n+p)}$ such that

$$\Theta(\alpha) \triangleq \mathcal{Q}(\alpha) + \mathcal{X}(\alpha)\mathcal{B}(\alpha) + \mathcal{B}(\alpha)'\mathcal{X}(\alpha)' < 0 \quad (6)$$

is feasible for all $\alpha \in \Omega$, with

$$\mathcal{B}(\alpha) = \begin{bmatrix} \mathbf{I} & -A(\alpha) & \mathbf{0} & -B(\alpha) \\ \mathbf{0} & -C(\alpha) & \mathbf{I} & -D(\alpha) \end{bmatrix}$$

and $\mathcal{Q}(\alpha)$ given by

$$\mathcal{Q}^c(\alpha) = \begin{bmatrix} \mathbf{0} & \star & \star & \star \\ P(\alpha) & \mathbf{0} & \star & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mu_c\mathbf{I} \end{bmatrix};$$

$$\mathcal{Q}^d(\alpha) = \begin{bmatrix} P(\alpha) & \star & \star & \star \\ \mathbf{0} & -P(\alpha) & \star & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mu_d\mathbf{I} \end{bmatrix}$$

for the continuous and discrete-time cases, respectively.

The equivalence between Lemma 3 and Lemmas 1 and 2 can be proved by using the Finsler's Lemma (see for instance (de Oliveira and Skelton, 2001) for details). The exact $\mathcal{H}_\infty$ guaranteed cost could be obtained through the minimization of μ for all

$\alpha \in \Omega$ under the conditions of lemmas 1 to 3, by testing an infinite number of LMIs. Guaranteed costs can be obtained by choosing a special structure for the parameter dependent matrices $P(\alpha)$ and $\mathcal{X}(\alpha)$, as it has been done in (Arzelier *et al.*, 2002) and in (de Oliveira *et al.*, 2004*b*) for affine parameter-dependence, as well as in the context of homogeneous polynomially parameter-dependent Lyapunov functions, presented in the sequel.

3. HOMOGENEOUS POLYNOMIALLY PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS

In order to provide a systematic procedure to generate sufficient LMI conditions of increasing precision for Lemmas 1, 2 and 3, a quadratic Lyapunov function $v(x) = x'P_g(\alpha)x$ is defined, with $P_g(\alpha) \in \mathbb{R}^{n\times n}$ being a homogeneous form of arbitrary degree which depends polynomially on the uncertain parameters $\alpha_i, i = 1, \ldots, N$.

Define $\mathcal{K}(g)$ as the set of N-uples obtained as all possible combinations of $k_1k_2\cdots k_N$, $k_i \in \mathbb{N}$, $i = 1, \ldots, N$, such that $k_1 + k_2 + \cdots + k_N = g$. $\mathcal{K}_j(g)$ is the j-th N-uple of $\mathcal{K}(g)$ which is lexically ordered, $j = 1, \ldots, J(g)$. Since the number of vertices in the polytope $\mathcal{S}$ is equal to N, the number of elements in $\mathcal{K}(g)$ is given by $J(g) = (N+g-1)!/(g!(N-1)!)$. These elements define the subscripts $k_1k_2\cdots k_N$ of the Lyapunov constant matrices $P_{k_1k_2\cdots k_N} \triangleq P_{\mathcal{K}_j(g)}$ used to construct the homogeneous polynomially dependent Lyapunov matrices $P_g(\alpha)$ given by

$$P_g(\alpha) = \sum_{j=1}^{J(g)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_N^{k_N} P_{\mathcal{K}_j(g)} \ ; \qquad k_1k_2\cdots k_N = \mathcal{K}_j(g) \tag{7}$$

Note that, for $g = 0$, $P_0(\alpha) = P_0$ which leads to the standard fixed quadratic Lyapunov matrix. Note also that, since all coefficients α_i, $i = 1, \ldots, N$ are such that $\alpha \in \Omega$, a simple way to ensure $P_g(\alpha) > 0$ is to impose $P_{\mathcal{K}_j(g)} > 0$, $j = 1, \ldots, J(g)$.

For each set $\mathcal{K}(g)$, define also the set $\mathcal{I}(g)$ with elements $\mathcal{I}_j(g)$ given by subsets of i, $i \in \{1, 2, \ldots, N\}$, associated to the N-uples $\mathcal{K}_j(g)$ whose k_i's are nonzero. For each i, $i = 1, \ldots, N$ define the N-uples $\mathcal{K}_j^i(g)$ as being equal to $\mathcal{K}_j(g)$ but with $k_i > 0$ replaced by $k_i - 1$. Note that the N-uples $\mathcal{K}_j^i(g)$ are defined only in the cases where the corresponding k_i is positive. Note also that, when applied to the elements of $\mathcal{K}(g+1)$, the N-uples $\mathcal{K}_\ell^i(g+1)$ define subscripts $k_1k_2\cdots k_N$ of matrices $P_{k_1k_2\cdots k_N}$ associated to a homogeneous polynomially parameter dependent matrix of degree g. Finally, define the scalar constant coefficients $\beta_j^i(g+1) = g!/(k_1!k_2!\ldots k_N!)$, with $k_1k_2\ldots k_N \in \mathcal{K}_j^i(g+1)$.

Sufficient LMI conditions for the existence of $P_g(\alpha)$ given by (7) such that Lemmas 1, 2 and 3 hold are given in next section.

4. MAIN RESULTS

Theorem *1.* If there exist symmetric positive definite matrices $P_{\mathcal{K}_j(g)} \in \mathbb{R}^{n\times n}$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1, \ldots, J(g)$, such that the following LMIs hold for all $\mathcal{K}_\ell(g+1) \in \mathcal{K}(g+1)$, $\ell = 1, \ldots, J(g+1)$

$$\sum_{i\in\mathcal{I}_\ell(g+1)} \begin{bmatrix} A_i'P_{\mathcal{K}_\ell^i(g+1)} + P_{\mathcal{K}_\ell^i(g+1)}A_i & \star & \star \\ B_i'P_{\mathcal{K}_\ell^i(g+1)} & -\beta_l^i(g+1)\mathbf{I} & \star \\ \beta_l^i(g+1)C_i & \beta_l^i(g+1)D_i & -\beta_l^i(g+1)\mu_c\mathbf{I} \end{bmatrix} < 0 \tag{8}$$

then the homogeneous polynomially parameter-dependent Lyapunov function given by (7) assures $M^c(\alpha) < 0$ for all $\alpha \in \Omega$.

Moreover, if the LMIs of (8) are fulfilled for a given degree $\hat{g}$, then the LMIs corresponding to any degree $g > \hat{g}$ are also satisfied and smaller values of μ_c can be found.

Proof: Since $P_{\mathcal{K}_j(g)} = P_{\mathcal{K}_j(g)}' > 0$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1, \ldots, J(g)$, then $P_g(\alpha) > 0$ for all $\alpha \in \Omega$. Now, note that $M^c(\alpha)$ in (4) for $(A, B, C, D)(\alpha) \in \mathcal{S}$ and $P_g(\alpha)$ given by (7) is a homogeneous polynomial matrix equation of degree $g + 1$ that can be written as

$$M^c(\alpha) = \sum_{\ell=1}^{J(g+1)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_N^{k_N} \left\{ \sum_{i\in\mathcal{I}_\ell(g+1)} \begin{bmatrix} A_i'P_{\mathcal{K}_\ell^i(g+1)} + P_{\mathcal{K}_\ell^i(g+1)}A_i & \star & \star \\ B_i'P_{\mathcal{K}_\ell^i(g+1)} & -\beta_l^i(g+1)\mathbf{I} & \star \\ \beta_l^i(g+1)C_i & \beta_l^i(g+1)D_i & -\beta_l^i(g+1)\mu_c\mathbf{I} \end{bmatrix} \right\} \ ; \qquad k_1k_2\cdots k_N = \mathcal{K}_\ell(g+1) \tag{9}$$

Condition (8) imposed for all ℓ, $\ell = 1, \ldots, J(g+1)$ assures that $M^c(\alpha) < 0$ for all $\alpha \in \Omega$.

Suppose the LMIs of (8) are fulfilled for a certain $\hat{g}$, that is, there exist $J(\hat{g})$ symmetric positive definite matrices $P_{\mathcal{K}_j(\hat{g})}$, $j = 1, \ldots, J(\hat{g})$ such that $P_{\hat{g}}(\alpha)$ is a homogeneous polynomially parameter-dependent Lyapunov matrix assuring that $M^c(\alpha) < 0$. Then, the terms of the polynomial matrix $P_{\hat{g}+1}(\alpha) = (\alpha_1 + \cdots + \alpha_N)P_{\hat{g}}(\alpha)$ satisfy the LMIs of Theorem 1 corresponding to the degree $\hat{g}+1$, which can be obtained in this case by linear combination of the LMIs of Theorem 1 for $\hat{g}$. The smallest value of μ_c obtained with $\hat{g}$ is also feasible for $\hat{g}+1$ and, due to the extra variables, smaller values can be obtained. ■

The matrices composing the homogeneous polynomially parameter-dependent Lyapunov function $P_g(\alpha)$ as well as the LMIs of (8) can be generated from sets $\mathcal{K}(g)$ and $\mathcal{I}(g)$, which can be constructed from simple routines using, for instance, a recursive code. As the degree g of the polynomial increases, the conditions become less conservative since new free variables are added to the LMIs. Although the number of LMIs is also increased, each LMI becomes easier to be fulfilled due to the extra degrees of freedom provided by the new free variables and smaller values of $\mathcal{H}_\infty$ guaranteed costs can be obtained.

Theorem *2.* If there exist symmetric positive definite matrices $P_{\mathcal{K}_j(g)} \in \mathbb{R}^{n\times n}$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1,\dots,J(g)$, such that the following LMIs hold for all $\mathcal{K}_\ell(g+1) \in \mathcal{K}(g+1)$, $\ell = 1,\dots,J(g+1)$

$$\sum_{i\in\mathcal{I}_\ell(g+1)} \begin{bmatrix} P_{\mathcal{K}_\ell^i(g+1)} & \star & \star & \star \\ P_{\mathcal{K}_\ell^i(g+1)}A_i & P_{\mathcal{K}_\ell^i(g+1)} & \star & \star \\ \mathbf{0} & B_i' P_{\mathcal{K}_\ell^i(g+1)} & \beta_l^i(g+1)\mathbf{I} & \star \\ \beta_l^i(g+1)C_i & \mathbf{0} & \beta_l^i(g+1)D_i & \beta_l^i(g+1)\mu_d\mathbf{I} \end{bmatrix} > 0 \quad (10)$$

then the homogeneous polynomially parameter-dependent Lyapunov function given by (7) assures that $M^d(\alpha) < 0$ for all $\alpha \in \Omega$.

Moreover, if the LMIs of (10) are fulfilled for a given degree $\hat{g}$, then the LMIs corresponding to any degree $g > \hat{g}$ are also satisfied and smaller values of μ_c can be found.

Proof: Similar to the proof of Theorem 1, being thus omitted. ■

Theorem *3.* If there exist symmetric positive definite matrices $P_{\mathcal{K}_j(g)} \in \mathbb{R}^{n\times n}$, and matrices $\mathcal{X}_{\mathcal{K}_j(g)} \in \mathbb{R}^{(2n+m+p)\times(n+p)}$, $\mathcal{K}_j(g) \in \mathcal{K}(g)$, $j = 1,\dots,J(g)$ such that the following LMIs hold for all $\mathcal{K}_\ell(g+1) \in \mathcal{K}(g+1)$, $\ell = 1,\dots,J(g+1)$

$$\sum_{i\in\mathcal{I}_\ell(g+1)} \Big\{ \mathcal{Q}_{\mathcal{K}_\ell^i(g+1)} + \mathcal{X}_{\mathcal{K}_\ell^i(g+1)}\mathcal{B}_i + \mathcal{B}_i'\mathcal{X}_{\mathcal{K}_\ell^i(g+1)}' \Big\} < 0 \quad (11)$$

with

$$\mathcal{B}_i = \begin{bmatrix} \mathbf{I} & -A_i & \mathbf{0} & -B_i \\ \mathbf{0} & -C_i & \mathbf{I} & -D_i \end{bmatrix};$$

$$\mathcal{X}_g(\alpha) = \sum_{j=1}^{J(g)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_N^{k_N}\mathcal{X}_{\mathcal{K}_j(g)} \;; \quad (12)$$

and $\mathcal{Q}_{\mathcal{K}_\ell^i(g+1)}$ given by

$$\mathcal{Q}^c_{\mathcal{K}_\ell^i(g+1)} = \begin{bmatrix} \mathbf{0} & \star & \star & \star \\ P_{\mathcal{K}_\ell^i(g+1)} & \mathbf{0} & \star & \star \\ \mathbf{0} & \mathbf{0} & \beta_l^i(g+1)\mathbf{I} & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\beta_l^i(g+1)\mu\mathbf{I} \end{bmatrix};$$

$$\mathcal{Q}^d_{\mathcal{K}_\ell^i(g+1)} = \begin{bmatrix} P_{\mathcal{K}_\ell^i(g+1)} & \star & \star & \star \\ \mathbf{0} & -P_{\mathcal{K}_\ell^i(g+1)} & \star & \star \\ \mathbf{0} & \mathbf{0} & \beta_l^i(g+1)\mathbf{I} & \star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\beta_l^i(g+1)\mu\mathbf{I} \end{bmatrix} \quad (13)$$

for the continuous and discrete-time cases, respectively, then the homogeneous polynomially parameter-dependent Lyapunov function given by (7) assures that $\Theta(\alpha) < 0$ for all $\alpha \in \Omega$.

Moreover, if the LMIs of (11) are fulfilled for a given degree $\hat{g}$, then the LMIs corresponding to any degree $g > \hat{g}$ are also satisfied and smaller values of μ can be found.

Proof: The proof is quite similar to the proof of Theorem 1. With $P_g(\alpha) > 0$ given by (7), $\mathcal{X}_g(\alpha)$ by (12) and $(A, B, C, D)(\alpha) \in \mathcal{S}$, $\Theta(\alpha)$ in (6) can be written as a homogeneous polynomial form with positive coefficients involving the left-hand side of the LMIs (11). By imposing that each LMI is negative definite, a sufficient condition for $\Theta(\alpha) < 0$ is obtained. Again, if the conditions (11) are fulfilled for a given $\hat{g}$, then $P_{\hat{g}+1}(\alpha) = (\sum_{i=1}^N \alpha_i)P_{\hat{g}}(\alpha)$ and $\mathcal{X}_{\hat{g}+1}(\alpha) = (\sum_{i=1}^N \alpha_i)\mathcal{X}_{\hat{g}}(\alpha)$ are a feasible solution to the LMIs (11) for $g = \hat{g}+1$ and smaller values of μ can be obtained. ■

By minimizing the value of μ such that the conditions of theorems 1-3 hold, $\mathcal{H}_\infty$ guaranteed costs can be computed by means of convex optimization procedures. Due to the additional variable $\mathcal{X}(\alpha)$, the results of Theorem 3 encompass both Theorems 1 and 2. Moreover, for the same degree g, Theorem 3 provides smaller (or equal, at least) $\mathcal{H}_\infty$ guaranteed costs than Theorems 1 and 2.

The numerical complexity associated to the LMI conditions can be estimated from the number K of scalar variables and the number L of LMI rows. For instance, the complexity associated to the interior point method from LMI Control Toolbox (Gahinet *et al.*, 1995) is proportional to K^3L whereas the solver SeDuMi (Sturm, 1999) yields $K^2L^{2.5}+L^{3.5}$. Table 1 shows the values of K and L as a function of n (states), m (inputs), p (outputs), N (number of vertices) and $J(g)$ for theorems 1 (T1), 2 (T2) and 3 (T3).

Table 1. Values of K (number of scalar variables) and L (number of LMI rows) as a function of n (states), m (inputs), p (outputs), N (number of vertices) and $J(g)$ for theorems 1 (T1), 2 (T2) and 3 (T3).

	K
T1 & T2	$1 + n(n+1)J(g)/2$
T3	$1 + n(5n+1)J(g)/2$
	L
T1	$(n + m + p)J(g + 1) + nJ(g)$
T2 & T3	$(2n + m + p)J(g + 1) + nJ(g)$

5. NUMERICAL EXPERIMENTS

As a first example, consider the fourth order mass-spring system presented in (Iwasaki, 1996), where a linear fractional formulation of the uncertainty was adopted to provide an estimate of the robust $\mathcal{H}_2$ performance. The same transfer function is considered here, i.e. from the input force d applied to mass m_1 to the error signal $e = x_2$ (position of mass m_2). The nominal parameters are: masses $m_1 = 1$ and $m_2 = 1.5$; stiffness of the springs $k_1 = 1$ and $k_2 = 1$ and viscous friction coefficient $c_0 = 2$. It is assumed that the uncertainties affects the system in the following way: $k_2 + \delta_{k_2}$ and $c_0 + \delta_{c_0}$, with $|\delta_{k_2}| \leq 0.99$ and $|\delta_{c_0}| \leq 1.97$, resulting in a polytope of $N = 4$ vertices. The $\mathcal{H}_\infty$ guaranteed costs obtained from quadratic stability based computation (Palhares *et al.*, 1997) (QS), the parameter-dependent based approach from (Arzelier *et al.*, 2002, Theorem 6.12.2) (Ar), Lemmas 1, 2 and 3 (dO_{L1},dO_{L2},dO_{L3}) from (de Oliveira *et al.*, 2004*b*), Theorem 1 and 3 (T1,T3) are shown in Table 2, for both quadruples (A, B, C, D) (primal) and (A', C', B', D') (dual). The worst case $\mathcal{H}_\infty$ norm has been computed through a fine grid on the parameter space, yielding $\|H\|_{\infty w.c.} = 27.76$. Note that Theorem 3 reached the worst case with $g = 2$ while Theorem 1 demanded $g = 4$ to attain the same guaranteed cost.

As a second example, consider a SISO discrete-time system with two states and two vertices given by

$$A_1 = \begin{bmatrix} -0.05 & 0.97 \\ -1.00 & 0.01 \end{bmatrix}; \quad A_2 = \begin{bmatrix} -1.68 & -1.44 \\ 0.94 & 0.22 \end{bmatrix}$$

$$B_i = \begin{bmatrix} 1 & 0 \end{bmatrix}'; \; C_i = \begin{bmatrix} 0 & 1 \end{bmatrix}; \; D_i = 0; \; i = 1, 2$$

The conditions of (de Oliveira *et al.*, 2004*b*) and (Arzelier *et al.*, 2002) did not provide a feasible solution, whereas the conditions of theorems 2 and 3 provided $\mathcal{H}_\infty$ guaranteed costs. Table 3 shows the guaranteed costs computed and the worst case $\mathcal{H}_\infty$ norm for this example. Figure 1 shows the singular values plot for the uncertain system of Example 2 as well as the $\mathcal{H}_\infty$ guaranteed costs provided by theorems 2 and 3.

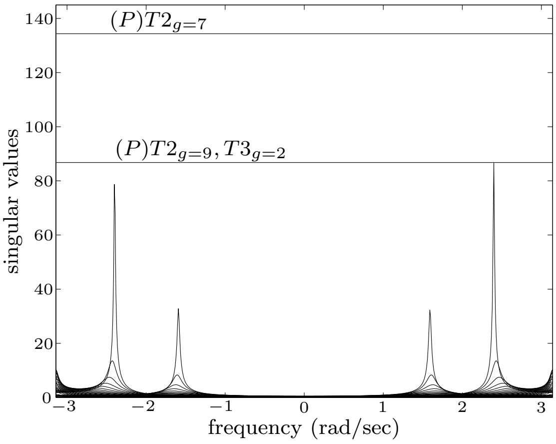

Fig. 1. Diagram of singular values for the uncertain system of Example 2. The guaranteed costs computed through $T2_{g=7,g=9}$ and $T3_{g=2}$ are also plotted.

This example illustrates the fact that sometimes an affine parameter-dependent Lyapunov function cannot be used to compute $\mathcal{H}_\infty$ guaranteed costs, but a higher degree parameter-dependent Lyapunov function is able to assess the robust stability of the uncertainty domain and can provide more accurate results. In this example, due to the extra variables, Theorem 3 reaches the $\mathcal{H}_\infty$ worst case for $g = 2$ whereas Theorem 2 needed $g = 9$ to achieve the same result.

6. CONCLUSION

A systematic procedure to construct homogeneous polynomially parameter-dependent Lyapunov functions of increasing degree used to compute $\mathcal{H}_\infty$ guaranteed cost of linear uncertain systems in polytopic domains has been given. As the degree of the polynomial increases, the conditions obtained become increasingly less conservative and more accurate, providing a simple and efficient test for evaluating $\mathcal{H}_\infty$ guaranteed costs.

7. ACKNOWLEDGMENTS

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Table 2. Guaranteed costs for the uncertain continuous-time system of Example 1, computed using quadratic stability (QS), (Ar), (dO_{L1},dO_{L2},dO_{L3}), T1 and T3. for (A, B, C) (primal - P) and (A', C', B') (dual - D). The worst case $\mathcal{H}_\infty$ norm is $\|H\|_{\infty w.c.} = 27.76$. The symbol '–' means that no feasible solution has been obtained.

	QS	(Ar)	dO_{L1}	dO_{L2}	dO_{L3}	$T1_{g=1}$	$T1_{g=2}$	$T1_{g=3}$	$T1_{g=4}$	$T3_{g=1}$	$T3_{g=2}$
(P)	–	–	190.13	196.69	178.57	190.13	141.46	114.86	96.88	178.57	126.97
(D)	–	196.69	39.35	–	36.70	39.35	27.93	27.77	27.76	36.70	27.76

Table 3. Guaranteed costs for the uncertain discrete-time system of Example 2, computed using T2 and T3, for (A, B, C) (primal - P) and (A', C', B') (dual - D). The worst case $\mathcal{H}_\infty$ norm is $\|H\|_{\infty w.c.} = 86.77$. The symbol '–' means that no feasible solution has been obtained.

	$T2_{g<7}$	$T2_{g=7}$	$T2_{g=8}$	$T2_{g=9}$	$T3_{g=1}$	$T3_{g=2}$
(P)	–	134.36	86.79	86.77	–	86.77
(D)	–	302.35	86.77	86.77	–	86.77

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