ON THE DISTURBANCE ATTENUATION FOR INPUT DELAY SYSTEMS

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Abstract: This paper addresses the disturbance attenuation problem for multivariable linear systems with a delayed input. To solve this problem, we use a static feedback based on a state prediction, which allows us to analyze an equivalent linear system without delay. Then, geometric conditions are given on this new equivalent system, and a numerical example illustrates our results.

Keywords: Delay systems, disturbance attenuation, almost disturbance decoupling, prediction, geometric approach, feedback.

1. INTRODUCTION

This work addresses the disturbance attenuation problem in linear multivariable systems with input delay. Time–delays appear frequently in industrial processes, economical, physiological and biological systems (Niculescu, 2001), and their presence is a consequence of delays in the process itself, or is caused by controllers (transport, communication, processing, ...).

Disturbance attenuation is a topic of recurrent interest. Among different methods well developed in the literature for solving this problem, geometric approach is an effective tool. Various versions of this problem have been solved (Conte & Perdon, 1995), (Conte & Perdon, 2000), (Willems & Commault, 1981), (Willems, 1981), (Wonham, 1985), (Basile & Marro, 1992). The solutions consist in necessary and sufficient conditions in terms of certain subspaces associated to the considered system. The computation of the subspaces effectively permits to check the solvability and to construct a solution controller.

For time-delays systems, necessary and sufficient conditions are also established for static or dynamic output feedback. The corresponding closed-loop systems have in general an infinite number of poles. Consider a linear multivariable system with delayed input

$$\dot{x}(t) = Ax(t) + Bu(t-h) + Ew(t),$$

where $h \in \mathbb{R}_+$ is the delay. The problem is to make $z = Gx$, with G of appropriate dimension, insensitive in closed–loop to the disturbance w which is not available by measurement.

Following Smith, Olbrot and Manitius (Olbrot, 1978), (Manitius & Olbrot, 1979), if all the state is measured, a prediction $x_p(t)$ of the state vector $x(t+h)$, is given by

$$x_p(t) = \mathrm{e}^{Ah}x(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau,$$

which is available at time t. This prediction is established without taking into account the disturbance. It is also natural to use a static feedback control law of the form

$$u(t) = Fx_p(t) + v(t),$$

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with F a real multivariable static gain and v a new input. The motivation to using such control laws is their simplicity and the induced properties of the closed–loop system (Mondié & Loiseau, 2001).

In this paper, we are interested to solve problems of disturbance attenuation by such a static state feedback coupled with a prediction equation, for linear input delay systems. We also address the dual question of making the estimation of a time-delay system insensitive to a perturbation. A decomposition of the closed–loop transfer function from w to z allows us to reduce this problem to a disturbance decoupling problem without delay. Geometric conditions are also given to solve them. Different versions of the disturbance decoupling problem by state feedback are considered, namely the so-called exact and almost disturbance decoupling. Similarly, two cases are considered when using static output injection and observer-predictor based structure to solve the question of estimating a system with input delays subject to disturbances.

The paper is organized as follows. In Section 2, we formulate the problem under consideration. Section 3 is devoted to the analysis of the closed–loop transfer function. A time–decomposition of impulse response in closed–loop is established in Section 4. Necessary and sufficient geometric conditions are given in Section 5, to solve problems under interest. Finally, a numerical example is provided in Section 6, to illustrate these geometric conditions.

Notations. We denote by $\hat{(\cdot)}(s)$ the Laplace transform of $(\cdot)$. $\mathcal{A}$ denotes the Wiener algebra (Callier & Desoer, 1978). L_p denotes the set of the complex-valued measurable functions $g(t)$ on the nonnegative real axis such that $\|g\|_{L_p}^p = \int_0^\infty |g(t)|^p \, \mathrm{d}t < \infty$, for $1 \leq p < \infty$, and such that $\|g\|_{L_\infty} = \operatorname{ess\,sup}_{t\in\mathbb{R}_+} |g(t)| < \infty$ in the case $p = \infty$.

2. PROBLEM FORMULATION

Consider a linear input delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t-h) + Ew(t) \\ y(t) = Cx(t) \\ z(t) = Gx(t) \end{cases}, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $w \in \mathbb{R}^d$ is an unknown disturbance, $h \in \mathbb{R}_+$ is the delay, $y \in \mathbb{R}^p$ is the measure, and $z \in \mathbb{R}^c$ is the output to be controlled. Matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{n\times d}$, $C \in \mathbb{R}^{p\times n}$ and $G \in \mathbb{R}^{c\times n}$ have real entries. The disturbance w is not available by measurement.
We are interested in the synthesis of a control law guarantying the attenuation of the disturbance effect on the output. For this aim, we will distinguish two problems. The first one is the case where all the state is measured, *i.e.* $C = I_n$. We will use a *static state feedback distributed control law*, of the form

$$u(t) = Fx_p(t) + v(t), \quad (2)$$

where $F \in \mathbb{R}^{m\times n}$, v is an eventually new input for the closed–loop system, and $x_p(t)$ is a prediction of $x(t+h)$ given by

$$x_p(t) = \mathrm{e}^{Ah}x(t) + \int_{t-h}^{t} \mathrm{e}^{(t-\tau)A}Bu(\tau)\,\mathrm{d}\tau. \quad (3)$$

This problem is reduced to make z insensitive in closed–loop to the disturbance w.
The second one is the case of static output injection, where we will use an injection with an observer–predictor based structure (Mirkin, 2003), *i.e.*

$$\dot{x}_o(t) = Ax_o(t) + Bu(t-h) - i(t), \quad (4)$$

with $i(t) = L(y(t) - Cx_o(t))$, is an observer of (1), and a state prediction based on this estimation is given by

$$x_{op}(t) = \mathrm{e}^{Ah}x_o(t) + \int_{t-h}^{t} \mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau, \quad (5)$$

where $x_o(t)$ is an estimate of $x(t)$, $x_{op}(t)$ is a prediction of $x(t+h)$, and $L \in \mathbb{R}^{n\times p}$. The problem is to make $e_p(t) = z(t) - Gx_{op}(t-h)$ insensitive in closed–loop to the disturbance.
We will suppose that the pair (A, B) of system (1) is stabilizable, and the pair (C, A) is detectable (Olbrot, 1978).

3. PRELIMINARY RESULTS

This section is devoted to characterize the input–output evolution in closed–loop from the disturbance and the controlled output. We take also $v = 0$ in (2).

Lemma 1. Consider the input delay system described by (1).
(*i*) The closed-loop transfer matrix (1)-(2)-(3) from the disturbance $w(t)$ and the controlled output $z(t)$ is

$$T_{wz}(s) = T_1(s) + \mathrm{e}^{-sh}T_2(s), \quad (6)$$

with

$$\begin{aligned} T_1(s) &= G(sI - A)^{-1}(I - \mathrm{e}^{-sh}\mathrm{e}^{Ah})E, \\ T_2(s) &= G(sI - A - BF)^{-1}\mathrm{e}^{Ah}E. \end{aligned}$$

(*ii*) The closed-loop transfer matrix (1)-(4)-(5) from $w(t)$ to $e_p(t) = z(t) - Gx_{op}(t-h)$ is

$$T_{we_p}(s) = T_1(s) + \mathrm{e}^{-sh}T_3(s), \quad (7)$$

where $T_1(s)$ is as above, and

$$T_3(s) = G\mathrm{e}^{Ah}(sI - A - LC)^{-1}E.$$

Proof. (i) Denote

$$\varphi(t) = \int_{t-h}^{t} \mathrm{e}^{A(t-\theta)} E w(\theta)\,\mathrm{d}\theta,$$

the error of prediction $x_p(t-h)$ of $x(t)$, so that $x(t) = \varphi(t) + x_p(t-h)$. Furthermore, in closed–loop, we have

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BFx_p(t-h) + Ew(t)\\ &= (A+BF)x(t) + Ew(t) - BF\varphi(t).\end{aligned}$$

By subtracting $x(t) - \varphi(t)$, we obtain

$$\dot{x}_p(t) = (A+BF)x_p(t) + \mathrm{e}^{Ah}Ew(t).$$

Whereas

$$\hat{\varphi}(s) = (I - \mathrm{e}^{-sh}\mathrm{e}^{Ah})(sI-A)^{-1}E\hat{w}(s),$$

and $z(t) = G(\varphi(t) + x_p(t-h))$, the result follows.

(ii) The dynamic of the estimate $x_o(t)$ is governed by

$$\dot{x}_o(t) = (A+LC)x_o(t) + Bu(t-h) - LCx(t).$$

The estimation error $e_o(t) = x(t) - x_o(t)$ verifies

$$\dot{e}_o(t) = (A+LC)e_o(t) + Ew(t),$$

Then, the prediction error $e_p(t) = x(t) - x_p(t-h)$ is described by $e_p(t) = \mathrm{e}^{Ah}e_o(t-h) + \varphi(t)$, and the result directly follows. □

This decomposition of the transfer function from the disturbance and the controlled output is also presented in (Mirkin, 2003), and it is shown that it allows to characterize all stabilizing controllers of the delayed system (1). The reader is also referred to (Zhong, 2003), where the same idea is used.

4. TIME–DOMAIN ANALYSIS

The decomposition of the input–output transfer function described in Section 3 has an easy interpretation in the time domain. This section is devoted to describe it.

Consider a generalized function $f(t) \in \mathcal{A}$, of the form

$$f(t) = \begin{cases} 0 & , \ t < 0 \\ f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t - t_i) & , \ t \geq 0 \end{cases}, \quad (8)$$

where $f_a \in L_1$, *i.e.* $\|f_a\|_{L_1} = \int_0^\infty |f_a(t)|\,\mathrm{d}t < \infty$, $f_i \in \mathbb{R}$ for $i \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots$, $\delta(t)$ stands for the Dirac delta function, and $\sum_{i=0}^{\infty} |f_i| < \infty$.

$\mathcal{A}$ is closed under addition, multiplication, and convolution, and is a commutative Banach algebra, with unit $\delta(t)$, for the norm defined by

$$\|f\|_{\mathcal{A}} = \|f_a\|_{L_1} + \sum_{i=0}^{\infty} |f_i|.$$

Similarly, the set $\widehat{\mathcal{A}}$ of Laplace transforms of elements of $\mathcal{A}$ is a commutative Banach algebra, with unit 1, for the induced topology.

We consider the class of causal linear systems described by a convolution

$$y(t) = \int_0^t f(t-\tau)u(\tau)\,\mathrm{d}\tau \doteq (f * u)(t), \quad (9)$$

or equivalently $\hat{y}(s) = \hat{f}(s)\hat{u}(s)$, where the kernel f and the input u are assumed Laplace transformable, in the sense of distributions. One says that (9) is BIBO stable if $f \in \mathcal{A}$, or equivalently if $\hat{f} \in \widehat{\mathcal{A}}$, *i.e.* $\|f\|_{\mathcal{A}} < \infty$. The BIBO stability is also equivalent to an input-output stability, *i.e.* every bounded input $u \in L_\infty$ produces a bounded output $y \in L_\infty$.

Consider the closed-loop transfer matrix $T_{wz}(s)$ from w to z. The closed-loop system is described by a convolution, as in (9). Then, denoting by $h(t)$ the impulse response of the closed-loop transfer matrix $T_{wz}(s)$, we have $T_{wz}(s) = \hat{h}(s)$.

For $1 \leq p \leq \infty$, the L_p–induced norm of the L_p–norm of $T_{wz}(s)$, denoted by $\|T_{wz}\|_p$, is defined by (Desoer & Vidyasagar, 1975)

$$\|T_{wz}\|_p = \sup_{w \in L_p, w \neq 0} \frac{\|h * w\|_{L_p}}{\|w\|_{L_p}}.$$

It is well known that the following equality holds (Desoer & Vidyasagar, 1975), (Callier & Desoer, 1978)

$$\|T_{wz}\|_1 = \|T_{wz}\|_\infty = \|h\|_{\mathcal{A}},$$

which is well defined if and only if the closed-loop system is BIBO stable.

For all $1 < p < \infty$, an upper bound of $\|T_{wz}\|_p$ is also given by

$$\|T_{wz}\|_p \leq \|h\|_{\mathcal{A}}, \ \ 1 < p < \infty.$$

By the decomposition of the input–output transfer function $T_{wz}(s)$ established in Lemma 1, we have the following result.

Lemma 2. Let $T_{wz}(s) = \hat{h}(s)$ be the closed–loop transfer matrix from w to z. Then,

$$h(t) = h_1(t) + h_2(t),$$

where $h_1(t)$ and $h_2(t)$ are generalized functions with non overlapping supports, and are respectively the impulse response of $T_1(s)$, with bounded support $[0, h]$, and the impulse response of $\mathrm{e}^{-sh}T_2(s)$ given in (6), with support contained in $[h, \infty[$. Moreover, if the matrix $(A+BF)$ is stable, then $\|h\|_{\mathcal{A}} < \infty$ and

$$\|h\|_{\mathcal{A}} = \|h_1\|_{\mathcal{A}} + \|h_2\|_{\mathcal{A}}. \quad (10)$$

Proof. In (6), the transfer matrix $T_1(s)$ defined by

$$G(sI-A)^{-1}(I - \mathrm{e}^{-sh}\mathrm{e}^{Ah})E,$$

admits a finite impulse response $h_1(t)$ given by

$$h_1(t) = \begin{cases} G\mathrm{e}^{At}E \ , \ t \in [0, h] \\ 0 \qquad\quad , \ t > h \end{cases} , \qquad (11)$$

which lies in $L_p^{c\times d}$, for all $1 \leq p \leq \infty$. Then $h_1 \in \mathcal{A}$. The impulse response $h_2(t)$ of $\mathrm{e}^{-sh}T_2(s)$ has a support on $[h, \infty[$. If the matrix $(A + BF)$ is stable, by (6), it is clear that $h_2 \in \mathcal{A}$. Since h_1 and h_2 have non overlapping supports, the norm decomposition (10) directly follows. □

It is worth noting that h_1 does not depend on the control law applied to the system. One can evaluate $\|h_1\|_{\mathcal{A}}$ from the knowledge of A, E, G, by integration of (11). This norm gives a lower bound for the closed–loop transfer between the output z and the disturbance w

$$\|T_{wz}\|_1 \geq \|h_1\|_{\mathcal{A}}.$$

The same remarks can be made for the estimation problem described in the claim (ii) of Lemma 1. One obtains the following.

Lemma 3. The impulse responses $h_1(t)$ and $h_3(t)$ of the transfer $T_1(s)$ and $\mathrm{e}^{-sh}T_3(s)$ respectively have non overlapping supports. Moreover, if $(A + LC)$ is stable, then $\|T_{we_p}\|_1 < \infty$ and

$$\|T_{we_p}\|_1 = \|h_1\|_{\mathcal{A}} + \|h_3\|_{\mathcal{A}}. \qquad (12)$$

A lower bound of the closed–loop transfer between the disturbance w and the predicted estimate e_p is thus obtained. One has

$$\|T_{we_p}\|_1 \geq \|h_1\|_{\mathcal{A}}.$$

In the following, we shall be interested in the case where the lower bounds are reached, *i.e.*

$$\inf_F \|T_{wz}\|_1 = \|h_1\|_{\mathcal{A}}, \qquad (13)$$

and

$$\inf_L \|T_{we_p}\|_1 = \|h_1\|_{\mathcal{A}}.$$

In this case, note that $\|h_1\|_{\mathcal{A}}$ also provides an upper bound of $\|T_{wz}\|_p$ or $\|T_{we_p}\|_p$ for the other values of p, and one has

$$\inf_F \|T_{wz}\|_p \leq \|h_1\|_{\mathcal{A}}, \ 1 \leq p \leq \infty,$$

and

$$\inf_L \|T_{we_p}\|_p \leq \|h_1\|_{\mathcal{A}}, \ 1 \leq p \leq \infty.$$

5. INTERPRETATION IN GEOMETRIC TERMS

In this section, necessary and sufficient geometric conditions are given to solve various problems of disturbance attenuation or disturbed estimation for the time-delay system (1), taking into account the stability of the closed-loop system or not.

For the case of linear systems without delays, conditions to solve this problem are given in the literature, the reader is referred to (Willems & Commault, 1981), (Basile & Marro, 1992), (Wonham, 1985) and references therein. We recall here only definitions of (A, B) and (C, A) invariance.

Definition 1. A subspace $\mathcal{V}$ of $\mathcal{X}$ is called (A, B)–invariant if

$$A\mathcal{V} \subset \mathcal{V} + \operatorname{Im} B.$$

Definition 2. A subspace $\mathcal{S}$ of $\mathcal{X}$ is called (C, A)–invariant if

$$A(\mathcal{S} \cap \operatorname{Ker} C) \subset \mathcal{S}.$$

All properties on these subspaces and on the computation of algorithms to characterize these invariants can be found in references given above.

As seen in Section 4, the problem of attenuating the effect of the disturbance by a distributed predictive control law is reduced to the analysis of an equivalent linear system without delay, and this theory can be applied. The subspaces $\mathcal{V}^*_{A,B,\operatorname{Ker} G}$, $\mathcal{R}^*_{A,B,\operatorname{Ker} G}$, and $\mathcal{S}^*_{C,A,\operatorname{Im} B}$, which are respectively the maximal (A, B)–invariant subspace contained in $\operatorname{Ker} G$, the maximal controllability subspace contained in $\operatorname{Ker} G$, and the smallest (C, A)–invariant subspace containing $\operatorname{Im} B$, play a fundamental role in geometric approach.
Any invariant subspace is associated to a spectrum, and the subspace is called stabilizing if the associated spectrum is stable. In the sequel, $\mathcal{V}^*_{g,A,B,\operatorname{Ker} G}$ and $\mathcal{S}^*_{g,A,B,\operatorname{Ker} G}$ respectively denote the maximal stabilizing (A, B)-invariant subspace contained in $\operatorname{Ker} G$, and the smallest stabilizing (C, A)-invariant subspace containing $\operatorname{Im} B$.

Consider the disturbance attenuation problem by state feedback distributed control law. Then, by Lemmas 1 and 2, the transfer matrix $T_1(s)$ is independent from any control action, and $T_2(s)$ is a linear system without delay. The problem comes down to finding a static state feedback F such that $T_2(s)$ is zero. Then applying the classical results of the geometric approach leads to the following.

Theorem 1.
(i) There exists a state feedback F such that the closed loop system (1)-(2)-(3) is so that $T_2(s) = 0$ in (6) if and only if

$$\operatorname{Im}(\mathrm{e}^{Ah}E) \subset \mathcal{V}^*_{A,B,\operatorname{Ker} G}. \qquad (14)$$

(ii) There exists F such that the closed loop system (1)-(2)-(3) is internally stable and so that $T_2(s) = 0$ if and only if

$$\operatorname{Im}(\mathrm{e}^{Ah}E) \subset \mathcal{V}^*_{g,A,B,\operatorname{Ker} G}. \qquad (15)$$

Proof. This theorem is a direct consequence of Section 4. In fact, in closed–loop, the following equivalent linear system is a state-space representation of $T_2(s)$,

$$\begin{cases} \dot{\psi}(t) = A\psi(t) + Bu(t) + \mathrm{e}^{Ah}Ew(t) \\ u(t) = F\psi(t) \\ z_2(t) = G\psi(t) \end{cases}, \quad (16)$$

where the disturbance attenuation problem by distributed control law is equivalent to solve $\frac{\hat{z}_2(s)}{\hat{w}(s)} = 0$ by static state feedback. Conditions (i) and (ii) are then a direct consequence of the classical works (Willems & Commault, 1981), (Basile & Marro, 1992), (Wonham, 1985).

Theorem 1 gives necessary and sufficient conditions for an exact disturbance decoupling problem on $T_2(s)$ with static state feedback distributed control law, with eventually internal stability. Under these conditions, equality (13) is satisfied.
We can get further conditions using the concept of almost invariance and the associated subspaces.
Consider the problem of almost disturbance decoupling by static state feedback distributed control law, that is to obtain in closed–loop for (16), an impulse response $h_2(t)$ of the transfer function from w to z_2 such that

$$\forall \varepsilon > 0, \exists F \text{ s.t. } \|h_2\|_{\mathcal{A}} \leq \varepsilon,$$

i.e. $\forall \varepsilon > 0$, $\|z_2\|_{\mathcal{A}} \leq \epsilon \|w\|_{\mathcal{A}}$ in closed–loop, where z_2 is the corresponding output of $T_2(s)$. Then, applying the results of (Willems, 1981) to the system (16), we obtain the following.

Theorem 2.
(i) There exists a feedback F such that the closed-loop system (1)-(2)-(3) is so that

$$\inf_F \|T_{zw}\|_1 = \|h_1\|_{\mathcal{A}},$$

if and only if

$$\mathrm{Im}(\mathrm{e}^{Ah}E) \subset \mathcal{V}^*_{A,B,\mathrm{Ker}\, G} + \mathcal{S}^*_{G,A,\mathrm{Im}\, B}. \quad (17)$$

In that case, for every $\epsilon > 0$, there exists a feedback F such that $\|T_2\|_1 < \epsilon$.
(ii) There exists a feedback F such that the closed-loop system (1)-(2)-(3) is stable and so that

$$\inf_F \|T_{zw}\|_1 = \|h_1\|_{\mathcal{A}}\,,$$

if and only if

$$\mathrm{Im}(\mathrm{e}^{Ah}E) \subset \mathcal{V}^*_{g,A,B,\mathrm{Ker}\, G} + \mathcal{S}^*_{G,A,\mathrm{Im}\, B}. \quad (18)$$

Proof. The result immediately comes from (16) and (Willems, 1981).

Consider now the case of a static output injection distributed control law. Following (Wonham, 1985), the first problem to be treated is the dual notion of (A, B)–invariance, that is to make the error of the observer–predictor $e_p(t)$ insensible to the disturbance w. According to the results of Section 3, and more precisely using the decomposition (7), it appears that minimizing the observed predicted output estimation error on the time delay system (1) comes down to a disturbance estimation decoupling problem on the following system without delay

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + Bu(t) + Ew(t) \\ u(t) = LC\xi(t) \\ z_2(t) = G\mathrm{e}^{Ah}\xi(t) \end{cases}, \quad (19)$$

where the problem is reduced to estimate $z_2(t)$ from the measure $y_2(t)$ and to impose a zero transfer matrix between w and z_2. We then obtain the dual notions of Theorem 1, for the system (19), using the results of (Wonham, 1985).

Theorem 3.
(i) There exists an output injection L such that the closed–loop (1)-(4)-(5) verifies $T_3(s) = 0$ if and only if

$$\mathcal{S}^*_{C,A,\mathrm{Im}\, E} \subset \mathrm{Ker}(G\mathrm{e}^{Ah}). \quad (20)$$

(ii) There exists an output injection L such that the closed–loop (1)-(4)-(5) is internally stable and such that $T_3(s) = 0$ if and only if

$$\mathcal{S}^*_{g,C,A,\mathrm{Im}\, E} \subset \mathrm{Ker}(G\mathrm{e}^{Ah}). \quad (21)$$

As in Theorem 2, we can solve an almost decoupling disturbance decoupling estimation problem for the observer–predictor, by a direct adaptation of the results of (Willems, 1981).

Theorem 4.
(i) There exists an output injection L such that the closed–loop system (1)-(4)-(5) is so that

$$\inf_L \|T_{ze_p}\|_1 = \|h_1\|_{\mathcal{A}},$$

if and only if

$$\mathcal{V}^*_{A,E,\mathrm{Ker}\, C} \cap \mathcal{S}^*_{C,A,\mathrm{Im}\, E} \subset \mathrm{Ker}(G\mathrm{e}^{Ah}). \quad (22)$$

(ii) There exists an output injection L such that the closed–loop system (1)-(4)-(5) is stable and so that

$$\inf_L \|T_{ze_p}\|_1 = \|T_1(s)\|_{\mathcal{A}},$$

if and only if

$$\mathcal{V}^*_{A,E,\mathrm{Ker}\, C} \cap \mathcal{S}^*_{g,C,A,\mathrm{Im}\, E} \subset \mathrm{Ker}(G\mathrm{e}^{Ah}). \quad (23)$$

All these conditions are numerically computable, and easy to verify.

6. ILLUSTRATIVE EXAMPLE

Let Σ be the system defined by

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t-1) + Ew(t)$$

$z(t) = [\ 0\ 1\ 0\ ]x(t)$, and $y(t) = x(t)$, which is of the form (1). Then, we obtain

$$\mathcal{V}^*_{A,B,\mathrm{Ker}\,G} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},\ \mathcal{R}^*_{A,B,\mathrm{Ker}\,G} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and $\mathcal{S}^*_{G,A,\mathrm{Im}\,B} = \mathcal{V}^*_{A,B,\mathrm{Ker}\,G}$. Consider the disturbance attenuation by state feedback distributed control law with condition (14). Whereas e^A is upper-triangular, if $E = [\ 0\ \ 0\ \ 1\ ]^T$, then

$$\mathrm{Im}(\mathrm{e}^A E) = [\ \mathrm{e}^{-1}\ \ 1\ \ 1\ ]^\top \not\subset \mathcal{V}^*_{A,B,\mathrm{Ker}\,G},$$

and then the problem is not solvable. However, if $E = [\ 1\ \ 0\ \ 0\ ]^T$, then this problem is solvable, and moreover is solvable with internal stability.

Considering now the almost problem in Theorem 2, it is easy to see that it is solvable for all matrices E, with internal stability. Indeed, for $E = [\ 0\ 0\ 1\ ]^T$, take a feedback F of the form

$$F = \begin{bmatrix} 0 & 0 & f_1 \\ 0 & f_2 & -1 \end{bmatrix},$$

with $f_1, f_2 \in \mathbb{R}_-$ to ensure internal stability. Then, it is easy to determine $\|h_2\|_{L_1} = -\frac{1}{f_2}$. To solve the almost disturbance decoupling problem by state feedback with internal stability, take $\varepsilon > 0$, and impose $\|h_2\|_{L_1} \leq \varepsilon$. We obtain $f_2 \leq -\frac{1}{\varepsilon}$, according with high gain theory if $\varepsilon \to 0$ (Willems, 1981). The corresponding feedback is also given by

$$u(t) = \begin{bmatrix} 0 & 0 & f_1 \\ 0 & f_2 & -1 \end{bmatrix} \left[\mathrm{e}^A x(t) + \int_{t-1}^{t} \mathrm{e}^{A(t-\tau)} B u(\tau)\, \mathrm{d}\tau \right].$$

7. CONCLUSION

This paper addresses the disturbance decoupling problem in linear multivariable systems with a delayed input.

To solve this problem, a static predictive control law is used, that allow to work in closed–loop on an equivalent linear system without delay.

Then, geometric conditions are provided to solve various formulations of the disturbance decoupling.

Furthermore, it is shown that any retroaction will act on the system only after a determined time, which corresponds evidently to the initial delay.

The case of systems with multiple delays is not more simple, and is under investigation. Theoretical links with geometric conditions for systems with coefficients over a ring are also investigated.

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