INVARIANCE CONTROL DESIGN FOR CONSTRAINED NONLINEAR SYSTEMS

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Abstract: This paper focuses on the control of nonlinear control affine systems that are subject to hard state constraints. The control concept is based on the invariance control approach and is designed as an extension of an ordinary feedback controller. Compliance with the state constraints is achieved by a switching controller that makes a constraint admissible state space region positive invariant. The control method is demonstrated by numerical simulation of an example system.

Keywords: nonlinear control systems, invariance, constraints

1. INTRODUCTION

A controller design method for nonlinear control affine systems that are subject to a single state constraint is presented in (Wolff and Buss, 2004). In this paper those results are generalized towards consideration of multiple constraints of the regarded class. The control concept is based on the invariance control approach (Mareczek *et al.*, 2001; Wollherr *et al.*, 2001) and allows to constrain the system to a state-space region, which is constructed from the constraints.

Various control methods for constrained linear systems can be found including override-control (Glattfelder and Schaufelberger, 2003), the theory of constraint admissible sets (Gilbert and Tan, 1991) and Model Predictive Control (Bemporad *et al.*, 2002). The latter can also be applied to nonlinear systems, but with limitations imposed by the numerical complexity of the on-line optimization. The reference governor approach presented in (Bemporad, 1998; Gilbert and Kolmanovsky, 2001) also considers nonlinear systems but depends on extensive numerical simulation, as the state space is probed for constraint admissibility.

This paper is organized as follows: section 1 gives an introduction to the problem and summarizes previous results for a single constraint. Multiple constraints are considered in section 2 and in section 3 the control design method is applied to an example system.

1.1 *System and Constraint Definition*

In the following, nonlinear control affine systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u, \qquad \boldsymbol{x}(t=0) = \boldsymbol{x}_0 \qquad (1)$$

are regarded with $\boldsymbol{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$ and smooth vector fields $\boldsymbol{f}, \boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$.

Let the state constraints be given by a set of smooth output functions $h_i(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$

$$y_i = h_i(\boldsymbol{x}) \leq 0 \qquad 1 \leq i \leq m \qquad (2)$$

with globally well defined relative degree r_i and for which 0 is a regular value.

A point $\boldsymbol{x} \in \mathbb{R}^n$ in state space is called constraint admissible, if the constraint condition (2) is satisfied for all m constraints.

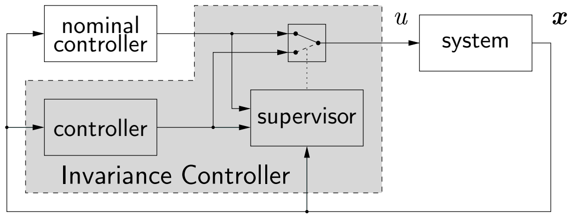

Fig. 1. Control Structure

1.2 Control Concept

The proposed controller is designed as an extension of an ordinary feedback controller, which is further called nominal controller. The nominal controller can be designed by ordinary controller design techniques to achieve the desired control goals while the state constraints are neglected. It is assumed that the nominal controller stabilizes system (1) with respect to an equilibrium point or a trajectory and that a suitable Lyapunov function is known.

To achieve compliance with the constraints, the invariance controller modifies the nominal control signal, such that a pointwise constraint admissible state space region becomes positive invariant. A state space region is called positive (negative) invariant with respect to a dynamic system, if trajectories originating from the region remain therein with increasing (decreasing) time. All following statements concerning invariance properties will implicitly refer to positive invariance. The invariance controller enforces compliance with the constraints by making a state space region invariant that is also pointwise constraint admissible. The structure of the resulting control loop (see Fig. 1) has similarities with the overrides that are common in the anti-windup literature (Glattfelder and Schaufelberger, 2003) and the reference governor approach (Gilbert and Kolmanovsky, 2001), where the reference signal is modified to avoid constraint violation.

1.3 A Single Constraint

In this subsection, the results for a single constraint from Wolff and Buss (2004) are summarized. The index i indicating the specific constraint is omitted for the sake of clarity.

Higher order time derivatives are denoted by

$$y^{(j)} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\right]^j y$$

and the Lie derivative of the function $h(\boldsymbol{x})$ along the vector field $\boldsymbol{f}$ by $\mathcal{L}_{\boldsymbol{f}}\, h(\boldsymbol{x})$

$$\begin{aligned}\mathcal{L}_{\boldsymbol{f}}\, h(\boldsymbol{x}) = \mathcal{L}_{\boldsymbol{f}}^1\, h(\boldsymbol{x}) &= \tfrac{\partial h}{\partial x}\, \boldsymbol{f}(\boldsymbol{x}) \\ \mathcal{L}_{\boldsymbol{f}}^j\, h(\boldsymbol{x}) &= \ \mathcal{L}_{\boldsymbol{f}}\, \mathcal{L}_{\boldsymbol{f}}^{j-1}\, h(\boldsymbol{x}).\end{aligned}$$

An analysis of the output dynamics, which is based on the current state $\boldsymbol{x}$ and an assumption on the control u, allows to derive an upper bound of the future output trajectory. The use of τ as time variable expresses the predictive character of the procedure.

The first r time derivatives of y are given by

$$\begin{aligned}y^{(j)} &= \mathcal{L}_{\boldsymbol{f}}^j\, h(\boldsymbol{x}) && 0 < j < r \\ y^{(r)} &= \mathcal{L}_{\boldsymbol{f}}^r\, h(\boldsymbol{x}) + \mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r-1}\, h(\boldsymbol{x})\, u.\end{aligned}$$

Under the assumption, that $y^{(r)}$ is upper bounded by a constant γ

$$\forall_{\tau>0}\ \ y^{(r)}(\tau) \le \gamma,$$

the output trajectory $y(\tau)$ is upper bounded by a polynomial $p(\tau, \boldsymbol{x}, \gamma)$ of order r

$$\begin{aligned}\forall_{\tau\ge 0}\ \ y(\tau) &\le p(\tau, \boldsymbol{x}, \gamma) \\ p(\tau, \boldsymbol{x}, \gamma) &:= \frac{\tau^r}{r!}\gamma + \sum_{i=0}^{r-1} \frac{\tau^i}{i!} y^{(i)}(\boldsymbol{x}).\end{aligned}$$

The design parameter γ is chosen according to

$$\begin{aligned}\gamma &= 0 && \text{for } r = 1 \\ \gamma &< 0 && \text{for } r > 1,\end{aligned}$$

to ensure that $p(\tau, \boldsymbol{x}, \gamma)$ has a maximum for $\tau \ge 0$, which is further denoted by $\Phi(\boldsymbol{x})$

$$\Phi(\boldsymbol{x}) := \max_{\tau\ge 0}\left[p(\tau, \boldsymbol{x}, \gamma)\right].$$

The function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is continuous but not necessarily differentiable and can be analytically determined for low relative degrees:

$$\begin{aligned}r=1:\ \ \Phi(\boldsymbol{x}) &= y \\ r=2:\ \ \Phi(\boldsymbol{x}) &= \begin{cases} y & \dot{y} \le 0 \\ -\dfrac{1}{2\gamma}\dot{y}^2 + y & \dot{y} > 0. \end{cases}\end{aligned} \tag{3}$$

A state space region $\mathcal{G}$ with boundary $\partial\mathcal{G}$ is implicitly defined by

$$\begin{aligned}\mathcal{G} &= \{\boldsymbol{x} \,|\, \Phi(\boldsymbol{x}) \le 0\} \\ \partial\mathcal{G} &= \{\boldsymbol{x} \,|\, \Phi(\boldsymbol{x}) = 0\}\,,\end{aligned} \tag{4}$$

which is pointwise constraint admissible with respect to the regarded single constraint. As $\Phi(\boldsymbol{x})$ is continuous but not necessarily differentiable, the standard invariance condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{x}) \le 0 \qquad\qquad \boldsymbol{x} \in \partial\mathcal{G}$$

is inapplicable. The following generalized invariance condition is used instead.

Definition 1. The function $\Phi(\boldsymbol{x})$ is locally decreasing with respect to an autonomous dynamic system in $\boldsymbol{x}_0$, if

$$\exists_{\epsilon>0}\ \forall_{t\in[0;\epsilon)}\ \ \Phi(\boldsymbol{x}_0) \ge \Phi(\boldsymbol{x}(t)).$$

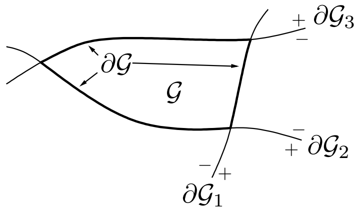

Fig. 2. Diagram of the relationships of $\partial\mathcal{G}_i$ and $\mathcal{G}$

Proposition 1. A region $\mathcal{G}$ as defined by (4) is invariant with respect to a dynamic system, if the function $\Phi(\boldsymbol{x})$ is continuous and locally decreasing with respect to the system on the boundary $\partial\mathcal{G}$.

A main result of (Wolff and Buss, 2004) is that the function $\Phi(\boldsymbol{x})$ is locally decreasing in $\boldsymbol{x}$, if either of the following conditions is satisfied:

$$y^{(r)}(\boldsymbol{x}, u) \leq \gamma \tag{5}$$

$$\forall_{0<j<r} \quad y^{(j)}(\boldsymbol{x}) < 0. \tag{6}$$

Condition (5) can be satisfied by increasing or decreasing u respectively as $\mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r-1}\,h(\boldsymbol{x}) \neq 0$ from the definition of relative degree.

Thus the region $\mathcal{G}$ can be made invariant with respect to the closed control loop by choosing the control u, such that for all $\boldsymbol{x} \in \partial\mathcal{G}$ at least one of the conditions (5) and (6) is satisfied.

2. MULTIPLE CONSTRAINTS

In the following, the index i always refers to one or more of the m state constraints or output functions respectively.

For each constraint (2) a function $\Phi_i(\boldsymbol{x})$ is determined like described in the previous subsection.

Each function $\Phi_i(\boldsymbol{x})$ then defines a region $\mathcal{G}_i$

$$\mathcal{G}_i = \{\boldsymbol{x} \,|\, \Phi_i(\boldsymbol{x}) \leq 0\}$$

that is pointwise constraint admissible with respect to its corresponding constraint.

The region $\mathcal{G}$ defined by the function $\Phi(\boldsymbol{x})$

$$\Phi(\boldsymbol{x}) = \max_i\,[\Phi_i(\boldsymbol{x})]$$
$$\mathcal{G} = \{\boldsymbol{x} \,|\, \Phi(\boldsymbol{x}) \leq 0\}$$

is the intersection of the regions $\mathcal{G}_i$ and is therefore pointwise constraint admissible with respect to all m constraints. It is further called invariance region.

The boundary $\partial\mathcal{G}$ of the region $\mathcal{G}$ consists of subsets of the boundaries $\partial\mathcal{G}_i$ of the regions $\mathcal{G}_i$. The relation of the regions $\mathcal{G}_i$ and $\mathcal{G}$ is illustrated in Fig. 2. The sign of the corresponding function $\Phi_i(\boldsymbol{x})$ is depicted beside the region boundaries.

2.1 Invariance

In the following a sufficient condition for invariance of the region $\mathcal{G}$ is derived from proposition 1 and the conditions (5) and (6).

According to proposition 1 the composite region $\mathcal{G}$ is invariant, if $\Phi(\boldsymbol{x})$ is locally decreasing on the boundary $\partial\mathcal{G}$. Let the set $I(\boldsymbol{x}) \subset \mathbb{N}$

$$I(\boldsymbol{x}) = \{i \,|\, \Phi_i(\boldsymbol{x}) = \Phi(\boldsymbol{x})\}$$

indicate which functions $\Phi_i(\boldsymbol{x})$ are maximal in $\boldsymbol{x}$.

Proposition 2. The function $\Phi(\boldsymbol{x})$

$$\Phi(\boldsymbol{x}) = \max_i\,[\Phi_i(\boldsymbol{x})]$$

is locally decreasing in $\boldsymbol{x}_0$, if the functions $\Phi_i(\boldsymbol{x})$ are locally decreasing in $\boldsymbol{x}_0$ for all $i \in I(\boldsymbol{x}_0)$

$$I(\boldsymbol{x}) = \{i \,|\, \Phi_i(\boldsymbol{x}) = \Phi(\boldsymbol{x})\}\ .$$

Proof of Proposition 2:
In this proof, use of the indices j and k implies

$$\forall_{j \notin I(\boldsymbol{x}_0)} \qquad \forall_{k \in I(\boldsymbol{x}_0)}\ .$$

As the functions $\Phi_k(\boldsymbol{x})$ are locally decreasing in $\boldsymbol{x}_0$, there exists $\epsilon_1 > 0$

$$\forall_{t\in[0;\epsilon_1)}\ \Phi_k(\boldsymbol{x}_0) \geq \Phi_k(\boldsymbol{x}(t)), \tag{7}$$

and with

$$\Phi_k(\boldsymbol{x}_0) > \Phi_j(\boldsymbol{x}_0),$$

there exists ϵ_2 with $0 < \epsilon_2 < \epsilon_1$, such that

$$\forall_{t\in[0;\epsilon_2)}\ \Phi_k(\boldsymbol{x}(t)) > \Phi_j(\boldsymbol{x}(t)). \tag{8}$$

Combination of (7) and (8) yields

$$\begin{aligned}\forall_{t\in[0;\epsilon_2)}\ \Phi(\boldsymbol{x}_0) &= \Phi_k(\boldsymbol{x}_0) && \text{from def. of } I(\boldsymbol{x})\\ &\geq \Phi_k(\boldsymbol{x}(t)) && \text{from (7)}\\ &> \Phi_j(\boldsymbol{x}(t)). && \text{from (8)}\end{aligned}$$

This means that for all $1 \leq i \leq m$

$$\forall_{t\in[0;\epsilon_2)}\ \Phi(\boldsymbol{x}_0) \geq \Phi_i(\boldsymbol{x}(t)).$$

$\Phi(\boldsymbol{x})$ is therefore locally decreasing in $\boldsymbol{x}_0$

$$\forall_{t\in[0;\epsilon_2)}\ \Phi(\boldsymbol{x}_0) \geq \max_i\,[\Phi_i(\boldsymbol{x}(t))] = \Phi(\boldsymbol{x}(t)). \qquad \square$$

Sufficient conditions for $\Phi_i(\boldsymbol{x})$ to be locally decreasing are given by (5) and (6), from which (6) is independent of the control input.

The term $\mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r_i-1}\,h_i(\boldsymbol{x})$ is either positive or negative for each i and all $\boldsymbol{x}$, since $\boldsymbol{f}$, $\boldsymbol{g}$ and h_i are smooth and $\mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r_i-1}\,h(\boldsymbol{x}) \neq 0$ from the definition of relative degree.

Condition (5) can be written as

$$y_i^{(r_i)}(\boldsymbol{x}, u) = \mathcal{L}_{\boldsymbol{f}}^{r_i}\,h_i(\boldsymbol{x}) + \mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r_i-1}\,h_i(\boldsymbol{x})\,u \leq \gamma_i$$

and, depending on the sign of the expression $\mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r_i-1}\,h_i(\boldsymbol{x})$, represents either an upper or lower bound condition for the control input u.

Let the sets $I^0(\boldsymbol{x})$ and $I^{\pm}(\boldsymbol{x})$ be a decomposition of $I(\boldsymbol{x})$ with

$$\begin{aligned} I^0(\boldsymbol{x}) &= \left\{ i \,\middle|\, y_i^{(j)}(\boldsymbol{x}) < 0,\ 0 < j < r_i \right\} \cap I(\boldsymbol{x}) \\ I^{\pm}(\boldsymbol{x}) &= I(\boldsymbol{x}) \setminus I^0(\boldsymbol{x}) \end{aligned}$$

and let $I^+(\boldsymbol{x})$ and $I^-(\boldsymbol{x})$ be a decomposition of $I^{\pm}(\boldsymbol{x})$ with

$$\begin{aligned} I^+(\boldsymbol{x}) &= \left\{ i \,\middle|\, \mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r_i-1}\, h_i(\boldsymbol{x}) > 0 \right\} \cap I^{\pm}(\boldsymbol{x}) \\ I^-(\boldsymbol{x}) &= \left\{ i \,\middle|\, \mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r_i-1}\, h_i(\boldsymbol{x}) < 0 \right\} \cap I^{\pm}(\boldsymbol{x}) \end{aligned}$$

With the introduced notations, a sufficient condition for the function $\Phi(\boldsymbol{x})$ to be is locally decreasing in $\boldsymbol{x}$ is given by

$$\underline{u}^*(\boldsymbol{x}) \le u \le \overline{u}^*(\boldsymbol{x}) \tag{9}$$

$$\begin{aligned} \underline{u}^*(\boldsymbol{x}) &:= \max_{i \in I^-(\boldsymbol{x})} \{u_i^*(\boldsymbol{x}), -\infty\} \\ \overline{u}^*(\boldsymbol{x}) &:= \min_{i \in I^+(\boldsymbol{x})} \{u_i^*(\boldsymbol{x}), +\infty\} \\ u_i^*(\boldsymbol{x}) &:= \frac{\gamma_i - \mathcal{L}_{\boldsymbol{f}}^{r_i}\, h_i(\boldsymbol{x})}{\mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r_i-1}\, h_i(\boldsymbol{x})}. \end{aligned}$$

For all $i \in I^0(\boldsymbol{x})$ the functions $\Phi_i(\boldsymbol{x})$ are locally decreasing in $\boldsymbol{x}$ because of (6). For all $i \in I^{\pm}(\boldsymbol{x})$ the functions $\Phi_i(\boldsymbol{x})$ are locally decreasing in $\boldsymbol{x}$ because of (5).

2.2 Stability

As the region $\mathcal{G}$ is not necessarily bounded, invariance of $\mathcal{G}$ is not sufficient for stability in the sense of boundedness. In the following, a sufficient condition for existence of a Lyapunov function for the system with switching control is given.

It is assumed that the nominal controller stabilizes system (1) with respect to an equilibrium $\boldsymbol{x}_\mathrm{d}$ or a trajectory $\dot{\boldsymbol{x}}(t)$ and that a suitable Lyapunov function $V(t, \boldsymbol{x}) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is available (Sastry, 1999). Furthermore, the nominal control u_nom is modified to comply with (9) in order to keep $\mathcal{G}$ invariant.

Proposition 3. If the condition

$$\forall_{\boldsymbol{x} \in \partial\mathcal{G},\ i \in I^{\pm}(\boldsymbol{x})} \quad \frac{\mathcal{L}_{\boldsymbol{g}}\, V(t, \boldsymbol{x})}{\mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r_i-1}\, h_i(\boldsymbol{x})} \ge 0 \tag{10}$$

is satisfied, the Lyapunov function $V(t, \boldsymbol{x})$ is also valid for the switching closed loop system.

Proof of Proposition 3:
As $V(t, \boldsymbol{x})$ is a valid Lyapunov function for the nominal control loop, the sufficient conditions for stability of the switching control loop on $V(t, \boldsymbol{x})$ and $\dot{V}(t, \boldsymbol{x}, u)$ with $u = u_\mathrm{nom}$ are satisfied.

If the control u is modified to meet condition (9), there exists $i \in I^{\pm}$ with

$$\begin{aligned} y_i^{(r_i)}(\boldsymbol{x}, u) &\le \gamma_i \\ \gamma_i &< y_i^{(r_i)}(\boldsymbol{x}, u_\mathrm{nom}), \end{aligned}$$

which leads to

$$\begin{aligned} \mathcal{L}_{\boldsymbol{f}}^{r_i}\, h_i(\boldsymbol{x}) + \mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r_i-1}\, h_i(\boldsymbol{x})\, u &\le \gamma_i \\ \gamma_i &< \mathcal{L}_{\boldsymbol{f}}^{r_i}\, h_i(\boldsymbol{x}) + \mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r_i-1}\, h_i(\boldsymbol{x})\, u_\mathrm{nom} \end{aligned}$$

$$\mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r-1}\, h_i(\boldsymbol{x})\, u < \mathcal{L}_{\boldsymbol{g}}\, \mathcal{L}_{\boldsymbol{f}}^{r-1}\, h_i(\boldsymbol{x})\, u_\mathrm{nom}$$

and, with (10),

$$\mathcal{L}_{\boldsymbol{g}}\, V(t, \boldsymbol{x})\, u \le \mathcal{L}_{\boldsymbol{g}}\, V(t, \boldsymbol{x})\, u_\mathrm{nom}. \tag{11}$$

With

$$\begin{aligned} \dot{V}(t, \boldsymbol{x}, u) &= \tfrac{\partial V}{\partial t} + \tfrac{\partial V}{\partial \boldsymbol{x}} \left[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u \right] \\ &= \tfrac{\partial V}{\partial t} + \mathcal{L}_{\boldsymbol{f}}\, V(t, \boldsymbol{x}) + \mathcal{L}_{\boldsymbol{g}}\, V(t, \boldsymbol{x}) u \\ \text{with (11)} \quad &\le \tfrac{\partial V}{\partial t} + \mathcal{L}_{\boldsymbol{f}}\, V(t, \boldsymbol{x}) + \mathcal{L}_{\boldsymbol{g}}\, V(t, \boldsymbol{x}) u_\mathrm{nom} \\ &= \dot{V}(t, \boldsymbol{x}, u_\mathrm{nom}), \end{aligned}$$

the function $V(t, \boldsymbol{x})$ is also valid for the switching control loop. □

2.3 Invariance Control

Compliance with the constraints defined by (2) is achieved by making the constraint admissible region $\mathcal{G}$ invariant with respect to system (1). This is realized by modifying the control signal to comply with (9) on the region boundary $\partial\mathcal{G}$, which is always possible if the stability condition (10) is met, because in this case either $I^+(\boldsymbol{x})$ or $I^-(\boldsymbol{x})$ is empty.

The resulting invariance controller can be interpreted as a nonlinear, state dependent saturator

$$\forall_{\Phi(\boldsymbol{x}) \ge 0} \quad \underline{u}^*(\boldsymbol{x}) \le u \le \overline{u}^*(\boldsymbol{x})$$

or as a switching controller

$$u = \begin{cases} \overline{u}^*(\boldsymbol{x}) & u_\mathrm{nom} > \overline{u}^*(\boldsymbol{x}) \wedge \Phi(\boldsymbol{x}) \ge 0 \\ \underline{u}^*(\boldsymbol{x}) & u_\mathrm{nom} < \underline{u}^*(\boldsymbol{x}) \wedge \Phi(\boldsymbol{x}) \ge 0 \\ u_\mathrm{nom} & \text{otherwise} \end{cases}.$$

In the design process of the invariance controller, the output functions h_i and design parameters γ_i must be chosen, such that the stability condition (10) is satisfied. This is usually the case for constraints like min-max-limits on states of the system with relative degree $r \le 2$. In some cases, condition (10) can be met by tuning of the parameters γ_i or by adding artificial constraints to modify the shape of $\partial\mathcal{G}$.

3. SIMULATION EXAMPLE

The presented invariance control method is applied to the model of the electromagnetically actuated mass spring damper system regarded in (Gilbert and Kolmanovsky, 2001), which is depicted in figure 3.

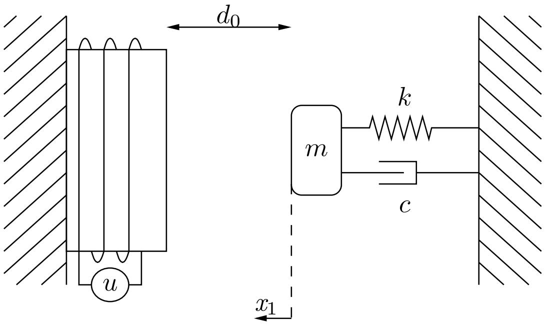

Fig. 3. Diagram of the Example System

3.1 System Model and Nominal Control

The system model is given by

$$\dot{\boldsymbol{x}} = \underbrace{\begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{x})} + \underbrace{\begin{bmatrix} 0 \\ \frac{\alpha}{m(d_0 - x_1)^\delta} \end{bmatrix}}_{\boldsymbol{g}(\boldsymbol{x})} u$$

with the parameters $\alpha = 4.5 \cdot 10^{-5}$, $\delta = 1.99$, $c = 0.659$, $k = 38.94$, $d_0 = 0.0102$, $m = 1.54$ given in SI-units.

The state x_1 corresponds to the position, x_2 to the velocity of the piece of mass.

The constraints are given by operational limits of the position and a maximum velocity:

$$0.001 \leq x_1 \leq 0.008 \qquad |x_2| \leq 0.01.$$

With the nominal control law $u_{\mathrm{nom}}(\boldsymbol{x})$

$$u_{\mathrm{nom}}(\boldsymbol{x}) = \frac{(d_0 - x_1)^\delta}{\alpha}(k\,v - c_d\,x_2) \qquad c_d = 4,$$

the system has a stable equilibrium $\boldsymbol{x}_{\mathrm{d}} = [v,\,0]^{\mathrm{T}}$.

The Lyapunov function

$$V(\boldsymbol{x}) = \frac{k}{2}(x_1 - v)^2 + \frac{m}{2}x_2^2$$

is positive definite. Although its time derivative

$$\begin{aligned} \dot{V}(\boldsymbol{x}, u_{\mathrm{nom}}) &= k\,x_1\dot{x}_1 + m\,x_2\dot{x}_2 \\ &= -(c + c_d)\,x_2^2 \leq 0 \end{aligned}$$

is only negative semidefinite, we can conclude asymptotic stability with respect to $\boldsymbol{x}_{\mathrm{d}}$ from LaSalle's Principle (Sastry, 1999), as the set

$$\{\boldsymbol{x} |\, \dot{V}(\boldsymbol{x}, u_{\mathrm{nom}}) = 0\} = \{\boldsymbol{x} |\, x_2 = 0\}$$

contains no other invariant set than $\boldsymbol{x}_{\mathrm{d}}$.

3.2 Invariance Control Design

The output functions

$$\begin{aligned} y_1 &= h_1(\boldsymbol{x}) = x_1 - \overline{x}_1 & \overline{x}_1 &:= 0.008 \\ y_2 &= h_2(\boldsymbol{x}) = x_2 - \overline{x}_2 & \overline{x}_2 &:= 0.01 \\ y_3 &= h_3(\boldsymbol{x}) = -(x_1 - \underline{x}_1) & \underline{x}_1 &:= 0.001 \\ y_4 &= h_4(\boldsymbol{x}) = -(x_2 - \underline{x}_2) & \underline{x}_2 &:= -0.01. \end{aligned}$$

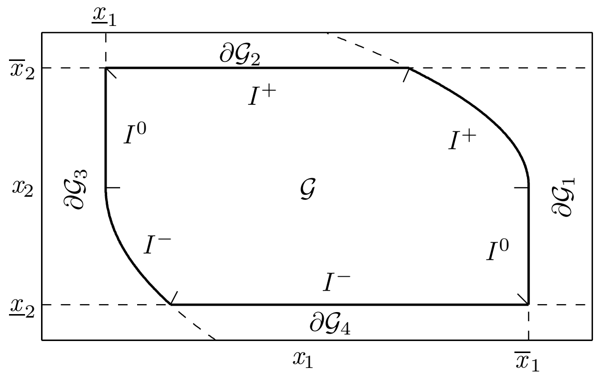

Fig. 4. Shape of the region $\mathcal{G}$ in the x_1–x_2-plane

are selected to represent the regarded constraints, which have global relative degree $r_1 = r_3 = 2$, $r_2 = r_4 = 1$.

The functions $\Phi_i(\boldsymbol{x})$ can be directly obtained by substitution of $y(\boldsymbol{x})$ and $\dot{y}(\boldsymbol{x})$ in (3):

$$\begin{aligned} \Phi_1(\boldsymbol{x}) &= \begin{cases} x_1 - \overline{x}_1 & x_2 \leq 0 \\ -\frac{1}{2\gamma_1}\,x_2^2 + x_1 - \overline{x}_1 & x_2 > 0 \end{cases} \\ \Phi_2(\boldsymbol{x}) &= x_2 - \overline{x}_2 \\ \Phi_3(\boldsymbol{x}) &= \begin{cases} -x_1 + \underline{x}_1 & x_2 \geq 0 \\ -\frac{1}{2\gamma_3}\,x_2^2 - x_1 + \underline{x}_1 & x_2 < 0 \end{cases} \\ \Phi_4(\boldsymbol{x}) &= -x_2 + \underline{x}_2. \end{aligned}$$

The resulting invariance region is depicted in Fig. 4. Marks on the inner side of the boundary $\partial\mathcal{G}$ indicate, which of the sets I^0, I^+, I^- is nonempty on that part of $\partial\mathcal{G}$.

The stability condition (10) is satisfied
for $x_2 > 0$, $I^\pm \subseteq \{1, 2\}$

$$\frac{\mathcal{L}_{\boldsymbol{g}}\,V(\boldsymbol{x})}{\mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r_1-1}\,h_1(\boldsymbol{x})} = \frac{\mathcal{L}_{\boldsymbol{g}}\,V(\boldsymbol{x})}{\mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r_2-1}\,h_2(\boldsymbol{x})} = x_2 > 0;$$

for $x_2 = 0$, $I^\pm = \emptyset$; and
for $x_2 < 0$, $I^\pm \subseteq \{3, 4\}$

$$\frac{\mathcal{L}_{\boldsymbol{g}}\,V(\boldsymbol{x})}{\mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r_3-1}\,h_3(\boldsymbol{x})} = \frac{\mathcal{L}_{\boldsymbol{g}}\,V(\boldsymbol{x})}{\mathcal{L}_{\boldsymbol{g}}\,\mathcal{L}_{\boldsymbol{f}}^{r_4-1}\,h_4(\boldsymbol{x})} = -x_2 > 0.$$

The design parameters γ_i are chosen such that the switched control signal does not violate the control constraints given by

$$0 \leq u \leq 0.3$$

to make the simulation results comparable to (Gilbert and Kolmanovsky, 2001):

$$\begin{aligned} \gamma_1 &= -0.0253 & \gamma_2 &= 0 \\ \gamma_3 &= -0.0461 & \gamma_4 &= 0. \end{aligned}$$

This is possible by precalculating the control on the boundary $\partial\mathcal{G}$ and choosing the parameters γ_i, such that u meets the control constraints.

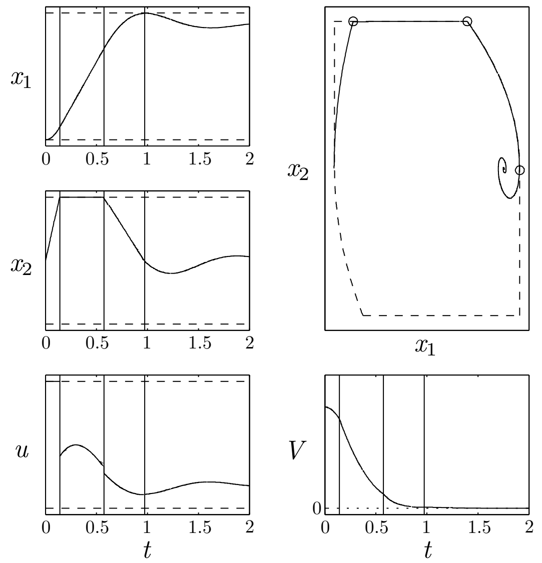

Fig. 5. Simulation results for $\boldsymbol{x}_0 = [0.001, 0]^{\mathrm{T}}$ and $\boldsymbol{x}_{\mathrm{d}} = [0.0074, 0]^{\mathrm{T}}$

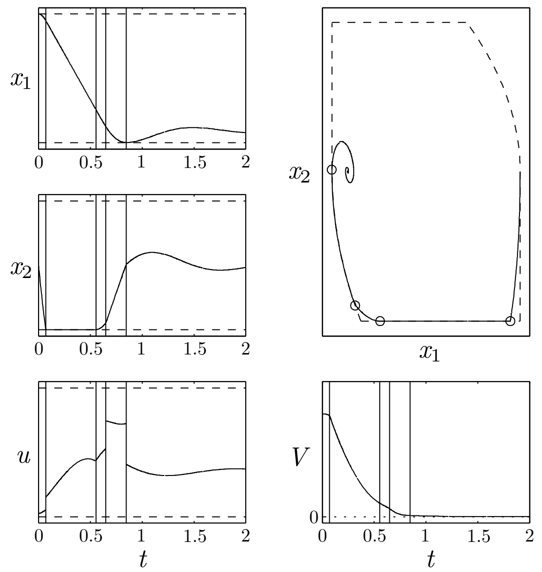

Fig. 6. Simulation results for $\boldsymbol{x}_0 = [0.0074, 0]^{\mathrm{T}}$ and $\boldsymbol{x}_{\mathrm{d}} = [0.001, 0]^{\mathrm{T}}$

3.3 Simulations

Fig. 5 and Fig. 6 show results obtained by numerical simulation with Matlab. The event-function option of the ode-suite was used to detect the switching times and to restart the ODE45-solver, which is based on a Runge-Kutta method. In the simulation plots, dashed lines indicate the region boundary and state or control constraints respectively. Switching times are indicated by vertical lines and circles.

In both scenarios the goal to enforce compliance with all state constraints was achieved. In the first scenario, which is comparable to the one in (Gilbert and Kolmanovsky, 2001), the performance of the invariance controller and the reference governor is equal.

4. CONCLUSION

An invariance control design method for nonlinear control affine systems that are subject to multiple state constraints is presented. The control concept is based on switching of the control and is designed as an extension of an ordinary feedback controller. A sufficient condition for stability of the switching closed loop systems is given. Future research focuses on the digital implementation of invariance controllers and robustness issues.

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