REALIZATION OF NONLINEAR DISCRETE-TIME COMPOSITE SYSTEMS: COMPUTATIONAL ASPECTS

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Abstract: The paper is devoted to computational aspects of the problems of realization of discrete-time nonlinear single-input single-output composite systems. The main contributions are made in two directions. First, we study the relations between three different methods that for any non-realizable system construct a "compensating system" which will result in a realizable series or parallel connection. Second, we implement these methods and algorithms in the computer algebra system *Mathematica*. Results are illustrated by examples. *Copyright © 2005 IFAC*

Keywords: Nonlinear control systems, discrete-time systems, compositions, algebraic approaches, state-space realization.

1. INTRODUCTION

Most results on parameter identification are achieved for systems described by input-output(i/o) difference equations. At the same time the majority of techniques for system analysis and control design are based on state-space description and unlike linear systems where a proper i/o difference model is always realizable in the state-space form, nonlinear systems do not always enjoy this property. The realization problem of input-output equations has been extensively studied both in continous-time (Van der Schaft, 1987) (Crouch and Lamnabhi-Lagarrigue, 1988), (Crouch *et al.*, 1995), (Dealaleau and Respondek, 1995) and discrete-time cases (Kotta *et al.*, 2001), (Sadegh, 2001), (Kotta and Sadegh, 2002). The general realizability conditions for discrete-time nonlinear systems together with constructive algorithm (up to integrating the differential one-forms) to find the state coordinates were presented in (Kotta *et al.*, 2001). Similar results for continuous- time systems were obtained in (Moog *et al.*, 2002). The wide and general subclass of realizable i/o difference equations was suggested in (Kotta and Sadegh, 2002).

While majority of contributions consider only single system, real world systems often consist of *compositions* of some simpler systems. For example, in telecommunications, circuits of the devices can be considered as series connections of different (i/o) systems (Nõmm *et al.*, 2004*a*). There is a number of contributions which study different aspects of composite systems (Hammer, 1984), (Sontag and Ingalls, 2002), (Willems, 1997), (Hammer, 1989). In (Nõmm, 2003) preliminary results on realization of series and parallel connections of i/o models were presented, and (Nõmm *et al.*, 2004*b*) proves that for any non-realizable system it is possible to construct a post-compensator which will result in a realizable series connection and a feedback which will result in a realizable

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closed-loop connection. Moreover, the constructive algorithms were given to obtain a compensating system for the cases of series and closed-loop connections. In many cases results obtained for continuous-time nonlinear systems lately were adapted for discrete-time case. At the best of our knowledge there are no similar results for continuous-time case.

In this paper we will prove that, alternatively, a pre-compensator, connected in series to the original system, can also make a compensated system realizable, and prove the equivalence of pre- and post-compensated systems. As another approach, we suggest to use the parallel connection to make the system realizable. Finally, implementation of those algorithms in the computer algebra package *Mathematica* will be described and illustrated by examples.

2. PROBLEM STATEMENT

Consider a single-input single-output system described by a higher order nonlinear i/o difference equation

$$y(t+n) = \phi(y(t), \dots, y(t+n-1), u(t), \dots, u(t+s)) \quad (1)$$

where $u(t)$ is a real-valued scalar input at time instant t, $y(t)$ is a real-valued scalar output at time instant t, ϕ is real-analytic function defined on $\mathbb{R}^{n+s+1}$, n and s are nonnegative integers, $n > s$. We call this original system an object. The realization problem consists in constructing the state equations

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \quad (2)$$

with $x(t) \in \mathbb{R}^n$ such that the i/o sequences generated by (2) are equal to the i/o sequences satisfying equation (1). Then (2) will be called a realization of (1). Input-output system is said to be realizable if there exists a realization of the form (2).

In the present paper the following compositions of input-output difference models of the form (1) are considered:

- **Series connection of two systems**
  Under series connection we understand such composite system that output of the first system is the input for the second system. In other words we connect output of the first system to the input of the second. Two types of series connection will be distinguished throughout the paper.

  Object and post-compensator
  Consider two systems Σ_{obj} and post-compensator Σ_{pst}, described by input-output difference equation (1) and

$$\Sigma_{pst} : \hat{y}(t+m) = = \psi(\hat{y}(t), \dots, \hat{y}(t+m-1), y(t), \dots, y(t+p)) \quad (3)$$

  respectively, where, $y \in \mathbb{R}$ is the output of the system Σ_{obj} and the input of the post-compensator Σ_{pst} and $\hat{y} \in \mathbb{R}$ is the scalar output of the post-compensator system; m and p are nonnegative integers, $m > p$, ψ is real-analytic function defined on $\mathbb{R}^{m+p+1}$. In order to avoid superpositions of the functions ψ and ϕ in the equations of the extended system (5), associated to the i/o equations (1) and (3) (see Section 3), we assume that $n \geq p$. The series connection of systems Σ_{obj} and Σ_{pst} as a single-input single-output system Σ_S with input u and output $\hat{y}$ is shown on Figure 1.

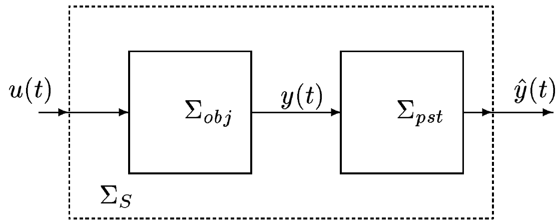

Fig. 1. Series connection of two systems

  Object and pre-compensator
  In Section 4 construction of a pre-compensator will be considered. In this case the series connection of the pre-compensator $\Sigma_{prc} : u(t+q) = v(t)$ and system Σ_{obj} is considered, where $v \in \mathbb{R}$ is the scalar input of the pre-compensator Σ_{prc}, $u \in \mathbb{R}$ is the scalar output of the pre-compensator and the input for system Σ_{obj}.

- **Parallel connection of two systems**
  Consider two systems Σ_{obj} and Σ_{par}, described by input-output difference equation (1) and the equation

$$\Sigma_{par} : \bar{y}(t+m) = = \gamma(\bar{y}(t), \dots, \bar{y}(t+m-1), u(t), \dots, u(t+p)) \quad (4)$$

  respectively, where $u \in \mathbb{R}$ is the scalar input for the both systems and $\bar{y} \in \mathbb{R}$ is the scalar output of the second system; m and p are nonnegative integers, $m > p$, γ is real analytic function defined on $\mathbb{R}^{m+p+1}$. Under parallel connection of these two systems we understand a single-input single-output system Σ_P with input u and output $\tilde{y}(t) = y(t) + \bar{y}(t)$, see Figure 2.

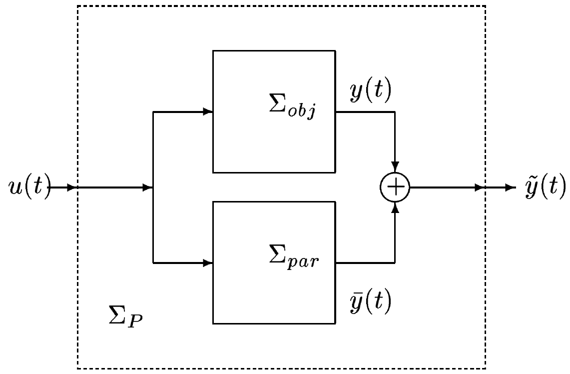

Fig. 2. Parallel connection of two systems

The following problems will be studied in this paper .

(1) For a non-realizable system Σ_{obj} find, if possible, a compensating system Σ_{prc} such that series connection Σ_S of two systems is realizable
(2) For a non-realizable system Σ_{obj} find, if possible, a compensating system Σ_{par} such that the parallel connection Σ_P of two systems is realizable.

3. ALGEBRAIC FORMALISM

In order to make this paper self-contained, we briefly recall the main facts of algebraic formalism, developed in (Aranda-Bricaire *et al.*, 1996), (Kotta *et al.*, 2001) and generalized for the case of composite systems in (Nõmm, 2003).

We consider the *series connection* of the systems (1) and (3) as an input-output system Σ_S and associate with it an extended state-space system Σ_{Se} with input $w(t) = u(t+s+1)$ and the state $\theta(t) = [\hat{y}(t), \ldots, \hat{y}(t+m-1), y(t), \ldots, y(t+n-1), u(t), \ldots, u(t+s)]^T$ defined as

$$\theta(t+1) = f_e(\theta(t), w(t)) \tag{5}$$

where

$$f_e(\theta, w) = (\theta_2, \ldots, \theta_m, \psi(\theta_1, \ldots, \theta_{m+p+1}), \theta_{m+2}, \ldots, \theta_{m+n}, \phi(\theta_{m+1}, \ldots, \theta_{m+n+s+1}), \theta_{m+n+2}, \ldots, \theta_{m+n+s+1}, 0)^T + (0, \ldots, 0, 0, 0, \ldots, 0, 0, 0, \ldots, 0, 1)^T w.$$

Below we will use notation of (Kotta *et al.*, 2001). Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of variables $\theta(0), w(t), t \geq 0$. The forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by $\delta\zeta(\theta(t), w(t)) = \zeta(f_e(\theta(t), w(t)), w(t+1))$.

Lemma 1. The following three statements are equivalent

- The operator δ is one-to-one
- The extended system (5) is submersive
- The following condition is satisfied

$$\text{rank}_{\mathcal{K}} \frac{(\partial\phi(\cdot), \partial\psi(\cdot))^T}{\partial(\hat{y}(t), y(t), u(t))} = 2 \tag{6}$$

Under (6) the pair $(\mathcal{K}, \delta)$ is a difference field, and up to an isomorphism, there exists an unique pair $(\mathcal{K}^*, \delta^*)$, called the inversive closure of $(\mathcal{K}, \delta)$, such that $\mathcal{K} \subset \mathcal{K}^*$, $\delta^* : \mathcal{K} \to \mathcal{K}^*$ is an automorphism and the restriction of $\delta^*$to $\mathcal{K}$ equals to δ. We will assume that $(\mathcal{K}^*, \delta^*)$ is given and we will use the same symbol to denote $(\mathcal{K}, \delta)$ and its inversive closure. Over the field $\mathcal{K}$ one can define a difference vector space $\mathcal{E} := \text{span}_{\mathcal{K}}\{\text{d}\varphi \mid \varphi \in \mathcal{K}\}$. The operator δ induces a forward-shift operator $\Delta : \mathcal{E} \to \mathcal{E}$ by $\sum_i a_i \text{d}\phi_i \to \sum_i \delta a_i \text{d}(\delta\phi_i)$, $a_i; \phi_i \in \mathcal{K}$. The relative degree r of a one form $\omega \in \mathcal{E}$ is defined to be the smallest integer such that $\Delta^r\omega \notin \text{span}_{\mathcal{K}}\{\text{d}\theta\}$. If such an integer does not exist, we set $r = \infty$. A sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ is defined by

$$\begin{aligned} \mathcal{H}_1 &= \text{span}_{\mathcal{K}}\{\text{d}\theta(0)\} \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \Delta\omega \in \mathcal{H}_k\} \quad k \geq 1. \end{aligned} \tag{7}$$

Obviously, $\mathcal{H}_k$ is a subspace of one-forms with relative degrees equal to k or higher than k. It is easy to see that sequence (7) is decreasing. Denote by k^* the smallest integer such that

$$\mathcal{H}_1 \supset \ldots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \ldots =: \mathcal{H}_\infty$$

Theorem 2. (Frobenius) Let $\mathcal{V} = \text{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_r\}$ be a subspace of $\mathcal{E}$. $\mathcal{V}$ is closed if and only if $d\omega_k \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0$ for any $k = 1, \ldots, r$. *Here "$\wedge$" denotes the wedge product.*

Under the conditions of the Frobenius theorem there exists locally a system of coordinates$\{\zeta_1, \ldots, \zeta_r\}$ such that $\mathcal{V}$ is generated by $\{\text{d}\zeta_1, \ldots, \text{d}\zeta_r\}$. In this case $\mathcal{V}$ is said to be completely integrable (Choquet-Bruhat *et al.*, 1989).

4. MAIN RESULTS

4.1 Three ways to overcome non-realizability

Necessary and sufficient conditions for the series and parallel connections of two systems to be realizable are given in the following theorem.

Theorem 3. (Nõmm, 2003) The series connection Σ_S of systems (1) and (3) admits a state-space realization if and only if for $1 \leq k \leq s+2$, the subspaces $\mathcal{H}_k$ defined by (7), for the extended system Σ_{Se}, are completely integrable.

In (Nõmm *et al.*, 2004*b*) it was proved that for any non-realizable system it is possible to construct a post-compensator of the form $\hat{y}(t+q) = y(t)$ such that the series connection is realizable. Also the algorithm to find the minimal integer q was presented. However, (Nõmm *et al.*, 2004*b*) did not consider the cases of serially added pre-compensator and parallel connection of the system and a compensator. The algorithm below is based on the algorithm proposed in (Nõmm *et al.*, 2004*b*) and allows to construct a "compensating system" either as a series pre- or post-compensator or a compensator added via the parallel connection. The algorithm is given on Figure 3; see also remarks below, describing certain steps in detail.

Remark 1. Implementation of this step is equivalent to calculation the largest integrable subspace of the subspace $\mathcal{H}_r$ and completing its basis such that we get $\mathcal{H}_r$.
Remark 2. The value of N_s is given by the highest order of negative shifts in the basis elements of non-integrable part of $\mathcal{H}_r$. For example, if a basis contains a non-integrable element $\mathrm{d}y(t+k) - a(\xi)\mathrm{d}u(t+j)$ where ξ represents the variable with negative shifts then by adding to this element $u(t+j)\mathrm{d}a(\xi)$ and adding $\mathrm{d}\xi$ (which corresponds to increasing q) to the basis will make the basis element integrable.

Since one has to calculate the finite number of subspaces $\mathcal{H}_k$ to check realizability, the algorithm stops after a finite number of steps with the realizable system.

The following theorem generalizes the results of (Nõmm *et al.*, 2004*b*) including also the cases of pre-compensator and compensator added via the parallel connection.

Theorem 4. For any non-realizable system Σ_{obj} of the form (8) there always exists

(1) a post-compensator Σ_{pst} such that series connection Σ_S, of systems Σ_{obj} and Σ_{pst} shown on Figure 1, is realizable.
(2) a pre-compensator Σ_{prc}such that series connection Σ_S, of systems Σ_{pre} and Σ_{obj}, is realizable.
(3) a compensating system Σ_{par} such that parallel connection Σ_P, of systems Σ_{obj} and Σ_{pst} shown on Figure 2, is realizable.

Sketch of the proof. One can demonstrate that adding forward shifts either in a form of pre-compensator or in a form of a post-compensator will guarantee that formerly nonzero wedge-products $\mathrm{d}\omega_k \wedge \omega_1 \wedge \ldots \wedge \omega_r$ where r is the dimension of the non-integrable subspace,will become zero. Alternatively, this can be achieved by using the parallel compensator. ■

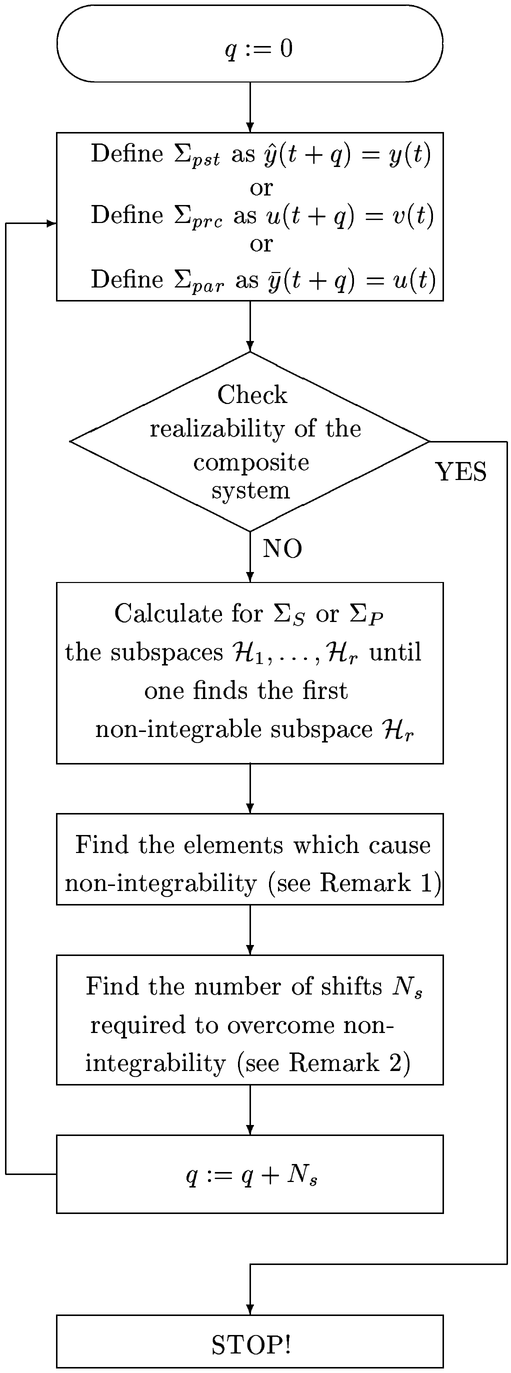

Fig. 3. Algorithm to build the "compensating" system

Theorem 5. For any non-realizable system Σ_{obj} of the form (1) series compositions constructed by the algorithms (3) are equivalent.

4.2 Implementation in computer algebra system Mathematica

In order to illustrate the proposed algorithm and to show its applicability for the computational purposes, algorithm depicted on Figure 3 was implemented in the form of sub-package for computer algebra system *Mathematica*. This sub-package contains the following functions:

- `SeriesConnection[`Σ_1`,`Σ_2`]` returns series composition Σ_S of two given systems. The arguments of this function are Σ_1,Σ_2. To obtain a composition shown on the Figure 1, call the following sequence `SeriesConnection[`Σ_{obj}`,`Σ_{pst}`]`. In order to define systems Σ_{obj} and Σ_{pst} one should use the function `DIO` described in (Kotta and Tõnso, 2003)
- `ParallelConnection[`Σ_1`,`Σ_2`]` returns parallel composition Σ_P of two systems. In order to obtain composition shown on the Figure 2 one should call the following sequence `ParallelConnection[`Σ_{obj}`,`Σ_{par}`]`.
- `ClosedLoopConnection[`Σ_1`,`Σ_2`]` returns closed-loop composition Σ_F of two systems (see Reamark 4). Here system Σ_1 plays a role of an object and Σ_2 a role of a feedback compensator.
- `Realizability[`Σ`]` returns `True` if the composite system Σ is realizable and `False` otherwise. If the system is non-realizable, function also returns the first non-integrable subspace $\mathcal{H}_k$. This function was written for the NLControl-package (Kotta and Tõnso, 2003) and later modified to handle the case of composite systems.
- `PreCompensator[`Σ`,v[t]]` constructs a pre-compensator for the given system Σ , with input variable $v(t)$ such that series connection of it with system Σ results in a realizable composition.
- `PostCompensator[`Σ`, ŷ[`t`]]` constructs a post-compensator for the given system Σ, with output $\hat{y}(t)$, such that series connection of system Σ with it results in a realizable composition.
- `ParallelCompensator[`Σ`, ȳ[`t`]]` constructs a parallel compensator for the given system Σ, with output $\bar{y}(t)$ such that parallel composition is realizable.
- `FeedbackCompensator[`Σ`]` constructs a feedback compensator for the given system Σ, such that closed-loop connection of it with system Σ results in a realizable composition (see Remark 3)
- `SimplifyConnection[`Σ`]` for the given series composition Σ, eliminates variables $u(t),\ldots,$ $u(t+m-1)$ and returns the corresponding SISO system (see Reamark 3)
- `Realization[`Σ`]` returns state coordinates and state equations of the given SISO or composite system Σ. This function was written for the NLControl-package (Kotta and Tõnso, 2003) and later modified to handle the case of composite systems.

Remark 3. Theoretical background for those functions is explained in (Nõmm *et al.*, 2004*b*).
Remark 4.
Functions `SeriesConnection` and `ParallelConnection` are used separately when one has to assemble the composite system or they are called by the functions `PreCompensator, PostCompensator, ParallelCompensator`.

As an example let us consider the following non-realizable i/o equation

$$y(t+3) = y(t+2)u(t+1) + y(t+1)u(t) + u(t+2)y(t) \quad (8)$$

and construct a post-compensator such that series connection of two systems is realizable. According to the algorithm on Figure 3 we have to connect to system (1) another system $\hat{y}(t+q) = y(t)$ serially and set the value of q to zero. In *Mathematica* this can be done in the following manner
`ioeq=DIO[y[t+3]-->y[t+2]u[t+1]+y[t+1]u[t] +u[t+2]y[t],u[t],y[t],t]`
calling function `PostCompensator[ioeq]` will initiate the following sequence.
`q=0` the order of post-compensator q is set to zero
`pst=DIO[ŷ[t+q]=y[t]]` defines post-compensator of order q
Σ_s`= SeriesConnection[ioeq,pst]` returns series connection of the initial system and post-compensator
`Realizability[`Σ_s`]`checks if composition Σ_s is realizable
`False`
`H3=Span{dŷ[t],dŷ[t+1],dŷ[t+2] -ŷ[t-1]du[t+1],du[t]}` Composition is not realizable since the subspace $\mathcal{H}_3$ is not integrable. As the next step, function searches for the element of the basis which causes non-integrability. Non-integrability is caused by the element $\mathrm{d}\hat{y}(t+2) - \hat{y}(t-1)\mathrm{d}u(t+1)$. Because of the backward shift term $y(t-1)$ the order of post-compensator q is increased by one. The function `PostCompensator` increases now q by 1
`q=q+1` now the order of the post-compensator $\hat{y}(t+1) = y(t)$ is equal to 1. The function `PostCompensator` calls again the sequence of functions described above.
`pst=DIO[ŷ[t+q]=y[t]]`
Σ_s`= SeriesConnection[ioeq,pst]`
`Realizability[`Σ_s`]`
`False`
`H4=span {dŷ[t+1],dŷ[t+2]-ŷ[t-1] du(t), dŷ[t+3]-ŷ[t]du[t+1]-ŷ[t+2],du[t],dŷ(t)}`
The system is not realizable since the subspace $\mathcal{H}_4$ of the composite system is not integrable. Non-integrability is caused by the element $\hat{y}(t+2) - \hat{y}(t-1)du(t)$. The function `PostCompensator` again calls the following sequence.
`q=q+1`
`pst=DIO[ŷ[t+q]=y[t]]`

Σ_s= `SeriesConnection[ioeq,pst]`
`Realizability[`Σ_s`]`
`True` the composition is realizable.
`PostCompensator` returns the post-compensator Σ_{pst}=$\hat{y}$`[t+2]=y[t]`. Series connection of the system (8) with the post-compensator $\hat{y}(t+2) = y(t)$ is realizable. By applying function `Realization[`Σ_S`]` one can get the state coordinates and state-space equations for the series connection of the system (8) and the post-compensator $\hat{y}(t+2) = y(t)$.

$x_1(t+1)=x_2(t)$ $x_2(t+1)=x_3(t)$
$x_3(t+1)=x_4(t)+x_1(t)u(t)$
$x_4(t+1)=x_5(t)+u(t)(x_4(t)+x_1(t)u(t))$
$x_5(t+1)=(x_4+x_1(t)u(t))u(t)$ $\hat{y}(t)=x_1(t)$

5. CONCLUSIONS

This paper studies the realization of the composite systems of the discrete-time nonlinear input-output systems. It has been proved that for any non- realizable system there exists a "compensating system" such that the series connection or/and parallel connections are realizable. A constructive algorithm to obtain a compensating system is presented. Relationship between the pre and post-compensated systems are characterized. Implementation of the algorithm in the computer algebra system is discussed and illustrated by the example. Note that realization of continuous-time nonlinear composite systems is still an open problem.

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