ON THE INVERTED PENDULUM ON A CART MOVING ALONG AN ARBITRARY PATH

Luca Consolini * **Mario Tosques** **

* *Dipartimento di Ingegneria dell'Informazione, University of Parma, Italy, e-mail:luca.consolini@polirone.mn.it*
** *Dipartimento di Ingegneria Civile, University of Parma, Italy, e-mail: mario.tosques@unipr.it*

Abstract: This article discusses a problem of stabilization for an inverted pendulum on a cart. In this work the cart does not travel along an horizontal line, which is the case commonly considered in the literature, but along an arbitrary path in the horizontal-vertical plane. The main obstacle encountered lies in the fact that the feedback controlled system becomes unstable in some "critical regions" along the cart path curve (close to the points where the path is vertical). The overall closed loop system is characterized by an alternation of periods of stability and instability and its state is proved to be stable under some conditions, which involve the geometry of the path. Simulations results are provided for the case in which the path is a circle. *Copyright 2005 IFAC*

Keywords: Nonlinear control, time-varying systems, feedback stabilization, non-minimum phase systems, mechanical systems.

1. INTRODUCTION

The inverted pendulum on a cart is an important benchmark for nonlinear control techniques and has been widely analized under different points of view, see for instance (Astrom and Furuta, 2000) for an energy based approach, (Bloch *et al.*, 2000) for a controlled Lagrangian one, (Shiriaev *et al.*, 2000) for a passivity based approach, (Holzhuter, 2004) for an Euler-Lagrange backward integration approach, (Angeli, 2001) for an approach based on a continuous feedback, finally a tracking problem is presented in (Mazenc and Bowong, 2003).

In the most common configuration, the cart travels along an horizontal line and, to the best of our knowledge, the general case in which the cart travels on a generic curve on the vertical plane has not been considered in detail in the literature. This article presents an analysis of this more general case, presenting a control strategy that, under some geometrical hypotheses, allows the inverted pendulum to be practically stabilized at the upright position while the cart travels with a given velocity on an arbitrary path. The control function is generated on the basis of the current curve direction and curvature, i.e. the controller uses the same strategy that would stabilize the system when the path is a circle. It is proved that, unless of choosing the reference speed sufficiently low, it is possible to follow any given curve, provided that this curve has a globally bounded curvature and a non null curvature at the points where the tangent is vertical. The main obstacle encountered in this problem lies in the fact that the internal dynamics of the constrained system became unstable in some "critical regions" along the cart path (namely those close to the points where the tangent to the reference curve is vertical). This article is an improvement of (Consolini and Tosques, 2004) that focuses on the VTOL aircraft, in fact as it is shown in Section 3 the path following problem for the VTOL aircraft is strictly connected to the one considered in this article (see also (Fliess *et al.*, 1999) for an explanation of the relationship between the VTOL and the inverted pendulum). Simulations results are provided for the case where the curve is a circle. The main limitation of the work is that the basin of attraction of the equilibrium is rather small.

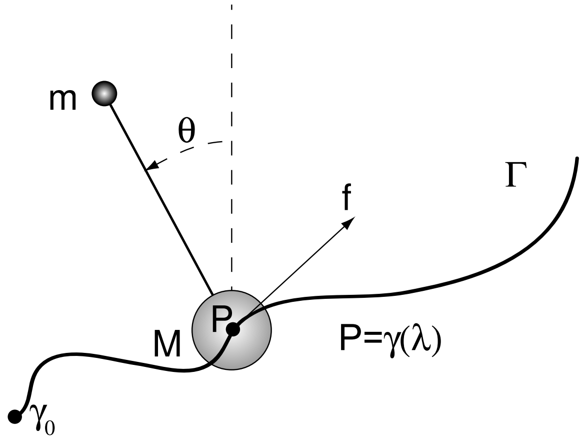

Fig. 1. Inverted pendulum on a cart constrained on a curve.

2. THE MODEL

Let $\gamma = (\gamma_1,\ \gamma_2) \in C^\infty([0,+\infty),\mathbb{R}^2)$, with $\|\gamma'(\lambda)\| = 1$, $\forall\lambda \geq 0$ and $\Gamma = \{\gamma(\lambda)|\lambda \geq 0\}$ be the image of the curve γ. Consider an inverted pendulum of mass m linked to a moving cart of mass M through a massless rod of length l , in Figure 1 the pendulum is represented as the smaller sphere and the cart as the bigger one. It is supposed that, during the motion, the cart center of mass P is constrained to stay on the curve Γ and a force $f(t)$ is applied on it in the direction tangent to Γ. Remark that a pendulum model similar to this one can be found in (Fliess *et al.*, 1999), with the only difference that in that case the pendulum is not constrained and has actually two degrees of freedom.

The end is to show that starting from the initial point $\gamma_0 = \gamma(0)$ with an initial angle θ_0 it is possible to find a control force $f(t)$, which has to be applied to the center P of mass of the cart, such that the resulting motion $(\lambda(t),\theta(t))$ of that system satisfies that $\lim_{t\to+\infty}\lambda(t) = +\infty$ (in other words all the curve Γ will be covered) and $\theta(t)$ remains close to the upright position. To determine the dynamic equations on motion, take $q = \begin{pmatrix}\lambda\\ \theta\end{pmatrix}$ as coordinates vector, where λ is the arc-length coordinate representing the position of P along the curve Γ and θ is the angle between the rod and the vertical axis. The kinetic energy T of this two-masses system is

$$T = \dot{q}^T H\dot{q}\ ,$$

where the inertia matrix H is given by

$$H(q) = \begin{pmatrix} M+m & -lm(\gamma'(\lambda))^T\nu(\theta)\\ -lm(\gamma'(\lambda))^T\nu(\theta) & ml^2\end{pmatrix}\ ,$$

the potential energy U is given by

$$U(q) = (M+m)g\gamma_2(\lambda) + mgl\cos\theta\ .$$

Setting $L = T - U$ the dynamic equations are derived through the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial\dot{q}} = \tau$$

which implies that

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \qquad (1)$$

where

$$C(q,\dot{q}) = \begin{pmatrix} 0 & -ml(\gamma'(\lambda))^T\dot{\nu}(\theta)\dot{\theta}\\ -ml(\gamma''(\lambda))^T\nu(\theta)\dot{\lambda} & 0\end{pmatrix}$$

is the Coriolis term, $G = \begin{pmatrix}(m+M)g\gamma_2'(\lambda)\\ -mgl\sin(\theta)\end{pmatrix}$ is the gravity term and $\tau = \begin{pmatrix}f\\ 0\end{pmatrix}$ is the vector of forces.

By left-multiplying equation (1) by vectors $\begin{pmatrix}0\\ 1\end{pmatrix}^T$, $\begin{pmatrix}ml^2\\ ml(\gamma'(\lambda))^T\nu(\theta)\end{pmatrix}$, we obtain respectively the following equations:

$$l\ddot{\theta} = \dot{\lambda}^2 < \gamma'' + \begin{pmatrix}0\\ g\end{pmatrix}, \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} > + < \gamma', \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} > \ddot{\lambda}\ , \qquad (2)$$

$$\begin{aligned}\ddot{\lambda} = \frac{ml^2}{\det H}&(ml(\gamma')^T\dot{\nu}\dot{\theta}^2 - (M+m)g\gamma_2' + f +\\ &+m(\gamma'')^T\nu\dot{\lambda}^2 + mg(\gamma')^T\nu\sin\theta).\end{aligned} \qquad (3)$$

where the notation $< a,b >$ with $a,b\in\mathbb{R}^n$ denotes the inner product.

Assuming for simplicity that $l = m = 1$, we get that the following system has to be verified

$$\begin{cases}\ddot{\theta} = < \dot{\lambda}^2\gamma''(\lambda) + \begin{pmatrix}0\\ g\end{pmatrix}, \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} > +\\ \qquad + < \gamma'(\lambda), \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} > u\\ \ddot{\lambda} = u\\ \theta(0) = \theta_0,\ \dot{\theta}(0) = \dot{\theta}_0\\ \lambda(0) = 0,\ \dot{\lambda}(0) = v_0\ ,\end{cases}$$

which represents the pendulum dynamics and the required force is given by

$$\begin{aligned}f = [(M+1)g\gamma_2' - (\gamma'')^T\nu\dot{\lambda}^2 +\\ -g((\gamma')^T\nu\sin\theta - (\gamma')^T\dot{\nu}\dot{\theta}^2] + \det H u\ .\end{aligned}$$

3. EQUIVALENCE WITH THE VTOL AIRCRAFT PATH-FOLLOWING PROBLEM

In this section it is shown that a path-following problem formulation for the VTOL aircraft leads to the same equation as the inverted pendulum, this is related to the fact that these two systems are equivalent by static feedback (as stated in (Fliess *et al.*, 1999)).

Consider the simplified VTOL aircraft model introduced in (Sastry *et al.*, 1992):

$$\begin{cases}\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = u_1\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} + \varepsilon u_2\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \begin{pmatrix}0\\ g\end{pmatrix}\\ \ddot{\theta} = \mu u_2\\ (x(0),y(0))^T = \gamma(0)\\ (\dot{x}(0),\dot{y}(0)) = \gamma'(0)v_0\\ \theta(0) = \theta_0,\ \dot{\theta}(0) = \dot{\theta}_0\ ,\end{cases} \qquad (4)$$

where $(x,y)^T$, the output of the system, are the coordinates of the center of mass P of the aircraft on a fixed inertial frame, v_0 the initial scalar velocity, θ is

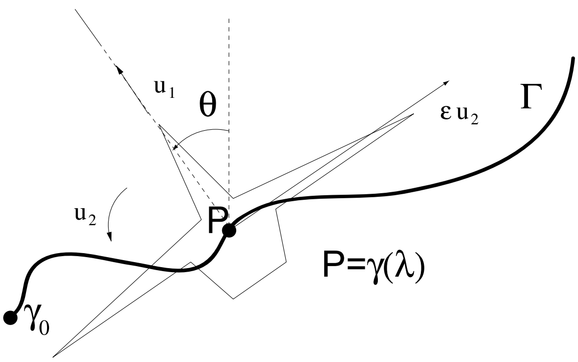

Fig. 2. Vtol aircraft

the angle between the aircraft symmetry axis and the vertical y-axis (see figure 2).

The goal is to show that starting from the initial point $\gamma_0 = \gamma(0)$ with an initial angle θ_0 it is possible to find a control (u_1, u_2) such that the resulting motion $(\lambda(t), \theta(t))$ of the VTOL verifies that $\lim_{t\to+\infty}\lambda(t) = +\infty$ (in other words all the curve Γ is covered) and $\theta(t)$ remains close to 0, that is the VTOL symmetry axis remains close to the vertical axis. To this end, if $\lambda \in \mathscr{C}^\infty([0,+\infty),\mathbb{R})$ is such that $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \gamma(\lambda(t))$, it must be $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \gamma'\dot{\lambda}$, $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \gamma''\dot{\lambda}^2 + \gamma'\ddot{\lambda}$ therefore it follows from (4) that

$$\begin{cases} u_1 \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} + \varepsilon u_2 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \gamma''\dot{\lambda}^2 + \\ \qquad\qquad\qquad\qquad + \gamma'\ddot{\lambda} + \begin{pmatrix} 0 \\ g \end{pmatrix} \\ \ddot{\theta} = \mu u_2 \,, \end{cases}$$

which implies, setting for simplicity $\varepsilon = \mu = 1$, that the following system holds:

$$\begin{cases} \quad\ddot{\theta} = <\dot{\lambda}^2\gamma''(\lambda) + \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > + \\ \qquad\quad + <\gamma'(\lambda), \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > u \\ \quad\ddot{\lambda} = u \\ \theta(0) = \theta_0,\ \dot{\theta}(0) = \dot{\theta}_0 \\ \lambda(0) = 0,\ \dot{\lambda}(0) = v_0 \,, \end{cases} \tag{5}$$

which represents the equation of the internal constrained dynamics for the VTOL and the control $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is given by the following equations:

$$\begin{aligned} u_1 &= <\dot{\lambda}^2\gamma'' + \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} > + \\ &\quad + <\gamma', \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} > u \\ u_2 &= <\dot{\lambda}^2\gamma'' + \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > + \\ &\quad + <\gamma', \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > u \end{aligned}$$

then the solution of system (4) will verify (5), that is the center of mass of the VTOL will stay on γ.

4. PROBLEM FORMULATION

Given the original inverted pendulum system:

$$\begin{cases} \quad\ddot{\theta} = <\dot{\lambda}^2\gamma''(\lambda) + \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > + \\ \qquad\quad + <\gamma'(\lambda), \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} > u \\ \quad\ddot{\lambda} = u \\ \theta(0) = \theta_0,\ \dot{\theta}(0) = \dot{\theta}_0 \\ \lambda(0) = 0,\ \dot{\lambda}(0) = v_0 \,, \end{cases} \tag{6}$$

we want to show that for any $\varepsilon > 0$ there exists a one dimensional smooth manifold $\Sigma \in \mathscr{C}^\infty([0,+\infty),\mathbb{R}^4)$, in the state space and a sufficiently small $r > 0$ such that $\forall(\theta_0, \dot{\theta}_0, v_0) \in B(\Sigma(0), r)$ there exists a feedback $u(\theta,\ \dot{\theta},\ \lambda,\ \dot{\lambda}) \in \mathscr{C}^\infty([0,+\infty),\mathbb{R})$ such that the solution to system (6) has the following properties:

- $\lambda \in \mathscr{C}^\infty([0,+\infty),\mathbb{R})$ is a strictly monotone function such that
$$\lim_{t\to\infty}\lambda(t) = +\infty \tag{7}$$
-
$$(\theta(t), \dot{\theta}(t), \lambda(t), \dot{\lambda}(t)) \in B(\Sigma(\lambda(t)), \varepsilon),\ \forall t \geq 0\,, \tag{8}$$
(where $\forall \bar{x} \in \mathbb{R}^4$, $B(\bar{x}, \varepsilon) = \{x \in \mathbb{R}^4 | \|x - \bar{x}\| < r\}$).

In other words the cart will cover all over the path Γ and the pendulum angle θ remains close to a given reference trajectory Σ, in which, as it will be shown, θ and $\dot{\theta}$ are small and therefore the pendulum remains close to the vertical along the trajectory.

The equation vector field (6), which is the "nominal system" will be denoted shortly as

$$\dot{x} = f(x,u)\,.$$

5. CONTROL PROCEDURE

In this section we give an idea of the main steps of the proposed control procedure. The control action is based on a second order approximation of the path γ. The controller knows at every time the tangent to the curve and its curvature (which define the osculating circle to the curve). A feedback control function is found that stabilizes the internal dynamics if the path is the osculating circle itself. The same causal control law is be used to stabilize the internal dynamics when the output of the systems covers a general path. The osculating circle represents an approximated and incomplete internal model of the path.

Consider the following system

$$\begin{cases} \ddot{\theta} = -v_r^2\kappa\sin(\omega t + \phi) + g\theta + \cos(\omega t + \phi)u \\ \dot{v} = u\,. \end{cases} \tag{9}$$

This system is denoted more shortly as

$$\dot{x} = f^{\omega,\phi}(x)\,,$$

where $x = (\theta,\ \dot{\theta},\ v)^T$ and the subscript a denotes that this is the "approximated system".

A solution to (9) is given by

$$\begin{array}{l} \theta = l\sin(\omega t+\phi) \\ v = v_r \\ u = 0\ , \end{array} \tag{10}$$

where $l = \frac{v_r\omega}{g+\omega^2}$. Denote this solution as as $r^{\omega,\phi} = (\theta,\ \dot{\theta},\ v_r)^T$. Note that this solution has the remarkable property of being obtained with a null control.

Set $x_e = x - r^{\omega,\phi} = (e,\ \dot{e},\ w)^T$, as the difference between the solution to (9) and (10), then

$$\begin{cases} \ddot{e} = ge + \cos(\omega t+\phi)u \\ \dot{w} = u\ , \end{cases} \tag{11}$$

which can be written as

$$\dot{x_e} = f^{\omega,\phi}(x_e,u) = A_e(t)x_e + B_e(t)u\ ,$$

where ω, ϕ indicate the dependence of the vector field f from these two parameters, the subscript e denotes the "error equation" and $A_e(t)$, $B_e(t)$ are matrix functions.

The following Proposition shows that the time varying periodical system (11) can be globally asymptotically stabilized with a static feedback control.

Proposition 1. If $(\omega,\phi) \notin \{(0,\frac{\pi}{2}),(0,-\frac{\pi}{2})\}$, then system (11) is globally asymptotically stabilized by state feedback.

Proof. Consider the coordinate transformation $z = T(t)x_e$, where $T(t)$ is the matrix

$$\begin{pmatrix} -\omega\sin(\omega t+\phi) & \cos(\omega t+\phi) & -cos(\omega t+\phi)^2 \\ (g-\omega^2)cos(\omega t+\phi) & -2\omega\sin(\omega t+\phi) & 2\omega\sin(\omega t+\phi)\cos(\omega t+\phi) \\ (-3\omega g+\omega^3)sin(\omega t+\phi) & (g-3\omega^2)\cos(\omega t+\phi) & 2\omega^2(cos(\omega t+\phi)^2 - sin(\omega t+\phi)^2) \end{pmatrix}$$

in z-coordinates equation (11) takes the form

$$\dddot{z} =< a(t),z > +(b_1+b_2(t))u$$

where $a(t)$ is a zero mean periodic function of t, $b_1 = \frac{g-3\omega^2}{2}$ and $b_2(t) = \frac{g+\omega^2}{2}cos(2\omega t+2\phi))$. Moreover $\det(T(t)) = -(b_1+b_2(t)^2$. Notice that system (11) has not a well defined relative degree, because $b(t)$ zeroes periodically. Choose

$$u = -\frac{\text{Sign }(b_1+b_2(t))}{k_a} < k,z >\ , \tag{12}$$

where k_a is the mean value of function $(b_1+b_2(t))$ Sign $(b_1+b_2(t))$ and $k = (k_1,\ k_2\ k_3)^T$ is a gain vector such that $p(s) = k_1+k_2s+k_3s^2+s^3$ is Hurwitz. The closed loop system takes the form

$$\dddot{z} = A_2(t)z - < k,z >\ ,$$

where $A_2(t)$ is a bounded matrix in which every line is null but the last one and every component has zero mean. Let A_c be the companion form matrix associated to $p(s)$, consider the solution P to the Lyapunov equation

$$A_c^T P + PA_c = -I\ ,$$

and define the potential $V(z) = z^T Pz$. Then

$$\dot{V} = -\|z\|^2 + +z^T(A_2^t(t)P + PA_2^t)z \le -V(\frac{1}{\lambda_M} - m(t))\ ,$$

where λ_M is the maximum eigenvalue of P and $m(t)$ is a periodic and zero mean function. Therefore

$$V(t) \le V(0)e^{-\int_0^t(\frac{1}{\lambda_M}-m(t))dt}\ ,$$

and

$$\lim_{t\to+\infty} V(t) = 0. \tag{13}$$

Remark that being $T(t)$ singular for some t, (13) does not in general imply that $lim_{t\to+\infty}x_e(t) = 0$. To prove this last part remark that because of the periodicity of (9), its solution can be expressed as

$$x_e(t) = P(t)e^{tF}x_e(0)\ ,$$

where $P(t)$ is periodic. Now $z(t) = T(t)x(t) = T(t)P(t)$ $e^{tF}x(0)$ and, being the term $T(t)P(t)$ periodic, F must be negative definite and

$$\lim_{t\to+\infty} x_e(t) = 0\ .$$

which ends the proof.□

Consider system (11) when the reference speed is null, i.e. $v_r = 0$. It reduces to the following time invariant system

$$\dot{x}_e = \begin{pmatrix} 0 & 1 & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos\phi \\ 1 \end{pmatrix} u \tag{14}$$

Proposition 2. For any $\eta >$ there exists gain constants K_1, K_2, K_3 for which

- system (11) is stable
- system (14) is stable for every ϕ such that $|\cos\phi| > \eta$.

Proof omitted for sake of brevity.
The following definition describes the set of paths for which the control procedure presented here works; it consists of those in which the curvature and its derivatives are bounded, the curvature is non-null whenever the curve is vertical and the vertical points are separated by a distance which is always greater then a positive constant.

Definition 1. If $M_1,\ M_2,\ M_3,\ M_4 \in \mathbb{R}^+$ and $M_1 > M_3$, set

$$\begin{array}{c} \mathscr{A}(M_1,M_2,M_3,M_4) = \{\gamma : \|\ddot{\gamma}(\lambda)\| < M_1, \\ \|\dddot{\gamma}(\lambda)\| < M_2,\ \forall\lambda \in [0,+\infty); \\ \|\ddot{\gamma}(\lambda)\| > M_3 \\ \forall\lambda \text{ such that } \phi(\lambda) \triangleq \arg(\dot{\gamma}(\lambda)) \in \{\frac{\pi}{2},-\frac{\pi}{2}\}; \\ |\lambda_1-\lambda_2| > M_4\ , \\ \forall\lambda_1 \neq \lambda_2 : \phi(\lambda_1),\ \phi(\lambda_2) \in \{\frac{\pi}{2},\ -\frac{\pi}{2}\}\}\} \end{array}$$

$\mathscr{A}(M_1,M_2,M_3,M_4)$ will be the set of the "admissible" curves γ which can be followed by the system.

Definition 2. When $(\omega,\ \phi) \notin \{(0,\ \frac{\pi}{2}),(0,\ -\frac{\pi}{2})\}$ the *transition matrix* $\Phi_{\omega,\phi}(\tau,t)$ is the solution to the following differential system

$$\begin{cases} \frac{d\Phi_{\omega,\phi}}{d\tau}(\tau,t) = A_e(\tau)\Phi_{\omega,\phi}(\tau,t) + B_e(\tau)u \\ \Phi(0,t) = I \end{cases}$$

where $u(t)$ is the stabilizing feedback control defined in (12).

Definition 3. If $(\omega,\,\phi) \notin \{(0,\,\frac{\pi}{2}),(0,\,-\frac{\pi}{2})\}$, the quadratic form $P_{v_r}(\omega,\phi)$ is defined as follows

$$P_{v_r}(\omega,\phi) = \int_0^{+\infty} \Phi_{\omega,\phi}(\tau,t)^T \Phi_{\omega,\phi}(\tau,t)dt$$

Remark that $P(\omega,\phi)$ is the solution at time $t=0$ to the Lyapunov differential equation associated to (11). From Proposition 1, $P_{v_r}(\omega,\phi)$ is well defined whenever $(\omega,\,\phi) \notin \{(0,\,\frac{\pi}{2}),(0,\,-\frac{\pi}{2})\}$.

Definition 4. Given $\gamma \in \mathscr{A}(M_1,M_2,M_3,M_4)$ the $\forall \lambda \geq 0$ define the following potential

$$V_{v_r}(\lambda,x) = x^T P_{v_r}(v_r\kappa(\lambda),\phi(\lambda))x$$

where

where $\kappa(\lambda)$ is the scalar curvature of Γ at λ and $\phi(\lambda) = \arg(\dot{\gamma}(\lambda))$.

Remark that if $v_r \neq 0$, $P_{v_r}(\omega,\phi)$ is well defined for every curve $\gamma \in \mathscr{A}(M_1,M_2,M_3,M_4)$ by Proposition 1 because $(v_r\kappa(\lambda),\ \phi(\lambda)) \notin \{(0,\frac{\pi}{2}),(0,-\frac{\pi}{2})\}$, $\forall \lambda \in [0,+\infty)$.

Definition 5. For any $\eta > 0$, the *critical set* $\mathscr{C}_\eta$ is

$$\mathscr{C}_\eta = \{\lambda : \cos\phi(\lambda) \leq \eta\ \}.$$

Given a function $V(x)$, the derivative of V, with respect to a vector field $f(x)$ is denoted as

$$L_f V(x) = V'(x) f(x)\ ,$$

where $V'(x)$ is the Jacobian matrix of V at x.

Lemma 1. If the path γ is a circle of curvature κ and such that $\arg(\dot{\gamma}(\lambda)) = \phi(\lambda)$, then the derivative of the potential $V_{v_r}(\lambda,x)$ with respect to the vector field $f^{0,\phi(tv_r)}(x,u(x))$, satisfies

$$L_{f^{\kappa v_r,\phi}} V_{v_r}(\lambda,x) = -\|x\|^2$$

Proof. This comes directly from the definition of $V_{\omega,\phi}$, being the solution to the Lyapunov differential equation associated to the time varying system (11). $\square$.

Proposition 3. If $r^{v_r\kappa(\lambda),\phi(\lambda)}$ is the reference solution and x is the solution of the nominal system $f(x,u)$, then

$$\frac{d}{dt}(x - r^{v_r\kappa(\lambda),\phi(\lambda)}) = f^{v_r\kappa(\lambda),\phi(\lambda)}(x,u) + \psi_1(x) + \psi_2(x)u \tag{15}$$

where $\psi_1(x)$ and $\psi_2(x)$ are suitable functions such that there exists $c_1, c_2 > 0$:

$$\psi_1(x) \leq v_r^2 c_1 \|x\|\ , \psi_2(x) \leq v_r^2 c_2 \|x\|\ ,$$

for $\|x\|$ and v_r sufficiently small.

Proof omitted for sake of brevity.

Lemma 2. The derivative of the potential $V_{v_r}(\lambda,x)$ with respect to system (15), computed along a generic curve $\gamma(\lambda)$, if $\lambda \in \mathscr{C}_\eta$ can be written as by

$$L_{f_n} V_{v_r}(\lambda,x) \leq -\|x\| + \sigma_1(x) + \sigma_2 \frac{d\kappa}{d\lambda}$$

where $\sigma_1(x)$ and $\sigma_2(x)$ are suitable functions such that $\lim_{V_r\to 0}(\sigma_1(x),\sigma_2(x)) = 0$, uniformly on $\mathbb{R}^3$.

Proof omitted for sake of brevity.

Lemma 3. Given a curve γ, if $\lambda_1,\ \lambda_2 \notin \mathscr{C}_\eta$ are such that $\cos\phi(\lambda_1) = \cos\phi(\lambda_2)$ then

$$\lim_{v_r\to 0} P_{v_r}(v_r\kappa(\lambda_1),\phi(\lambda_1)) = \lim_{v_r\to 0} P_{v_r}(v_r\kappa(\lambda_2),\phi(\lambda_2))$$

Proof. It comes from the fact that, as v_r approaches 0, $P_{v_r}(v_r\kappa(\lambda_1),\phi(\lambda_1))$ and $P_{v_r}(v_r\kappa(\lambda_1),\phi(\lambda_2))$ are the solution to the same differential equation. $\square$.

Proposition 4. Set M_1, M_2, M_3, $M_4 \in \mathbb{R}^+$, with $M_1 > M_3$, for any $\gamma \in \mathscr{A}(M_1,M_2,M_3,M_4)$ and for any $\varepsilon > 0$ there exists a one dimensional smooth manifold $\Sigma \in \mathscr{C}^\infty([0,+\infty),\mathbb{R}^3)$, in the state space and a sufficiently small $r > 0$ such that $\forall(\theta_0,\dot{\theta}_0,v_0) \in B(\Sigma(0),r)$ there exists a feedback $u(\theta,\ \dot{\theta},\ \lambda,\ \dot{\lambda}) \in \mathscr{C}^\infty([0,+\infty),\mathbb{R})$ such that the solution to system (6) has the following properties:

- $\lambda \in \mathscr{C}^\infty([0,+\infty),\mathbb{R})$ is a strictly monotone function such that
$$\lim_{t\to\infty} \lambda(t) = +\infty \tag{16}$$
-
$$(\theta(t),\dot{\theta}(t),\lambda(t),\dot{\lambda}(t)) \in B(\Sigma(\lambda(t)),\varepsilon),\ \forall t \geq 0\ , \tag{17}$$

Sketch of the proof Given $\varepsilon > 0$, set

$$V_\varepsilon = \frac{\|\varepsilon\|^2}{\max_\lambda \mu_{max}(P_{v_r}(v_r\kappa(\lambda),\phi(\lambda)))}$$

where $\mu_{max}(P)$ is the maximum eigenvalue of P, then if $V_{v_r}(e(t)) < V_\varepsilon$ for every $t > 0$, then $\|e(t)\| < \varepsilon$ and the thesis is proved.

Given any $\delta > 0$, choose η such that, for any $\lambda \in \mathscr{C}_0$, $L_u(\lambda) < \delta$.

Given $\gamma \in \mathscr{A}(M_1,M_2,M_3,M_3)$, for any $\lambda \notin \mathscr{C}_\eta$ it is from Lemma 2

$$L_f V_{v_r}(\lambda,x_e) \leq -\|x_e\| + \sigma(x_e)\ ,$$

with $\lim_{v_r\to 0}\sigma(x_e) = 0$. When $\lambda \in \mathscr{C}_\eta$, unless of reducing η there exists t_1, t_2 such that $\lambda(t_1) = \lambda_1$, $\lambda(t_2) = \lambda_2$, with $\lambda_2 = \inf_{\lambda\notin\mathscr{C}_\eta:\lambda>\lambda_2}$, $\lambda_1 = \sup_{\lambda\notin\mathscr{C}_\eta:\lambda<\lambda_1}$.

Consider the difference

$$\begin{aligned} V_{v_r}(\lambda_2,x_e(t_2)) - V_{v_r}(\lambda_1,x_e(t_1)) = \\ = x_e(t_2)^T P(v_r\kappa(\lambda_2),\phi(\lambda_2))x_e(t_2) + \\ -x_e(t_1)^T P(v_r\kappa(\lambda_1),\phi(\lambda_1))x_e(t_1)\ , \end{aligned}$$

from Lemma 3 it is

$$\lim_{v_r \to 0} P_{v_r}(v_r \kappa(\lambda_2), \phi(\lambda_2)) = \lim_{v_r \to 0} P_{v_r}(v_r \kappa(\lambda_1), \phi(\lambda_1)) .$$

By combining these equations is possible to prove that by reducing η and v_r and choosing a sufficiently small radius r, then for any initial condition $e(0)$ with $\|e(0)\| < R$ we have that $V_{v_r}(\lambda(t), x_e(t)) < V_\varepsilon$, for every $t > 0$. Now it is sufficient to define $\Sigma(\lambda) = (r^{(0,\arg(\dot{\gamma}(\lambda)))}, \lambda)$, $\forall \lambda \geq 0.\Box$

6. SIMULATIONS

This section presents a simulations for the case where the cart lies on a circle. The circle has a radius $r = 5$m, the reference speed is $v = 0.4$m/s, $(x(0), y(0)) = (0,0)$, the initial speed is $(\dot{x}(0), \dot{y}(0)) = (0.4, 0)$, and $\theta_0 = 0$, $\dot{\theta}_0 = 0$. The assigned eigenvalues are $\{-10, -12, -14\}$ and the threshold value for $\det R$ is -0.1.

The simulation results are reported in figure 3, where it can be noticed that the small perturbations of the speed along the trajectory happen when the determinant approaches zero and the tangent to the trajectory becomes parallel to the y-axis.

7. CONCLUSIONS

This article is about the control of an inverted pendulum on a cart that moves along an arbitrary curve. A control strategy which proves to be effective also in simulation, has been introduced and a theoretical explanation has been given. This particular problem seems very interesting and the approach may be generalized to more general non minimum phase systems.

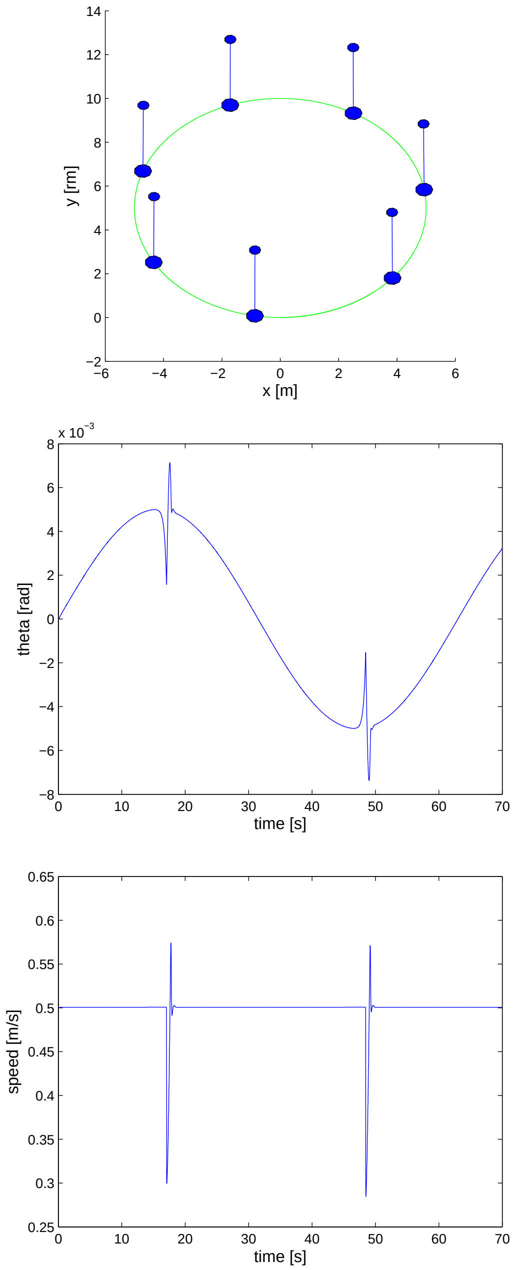

Fig. 3. Inverted pendulum trajectory, orientation, speed for the circle

REFERENCES

Angeli, David (2001). Almost global stabilization of the inverted pendulum via continuous state feedback. *Automatica* **37**(7), 1103–1108.

Astrom, K. J. and K. Furuta (2000). Swinging up a pendulum by energy control. *Automatica* **36**(2), 287–295.

Bloch, AM, NE Leonard and JE Marsden (2000). Controlled lagrangians and the stabilization of mechanical systems i: The first matching theorem. *IEEE Transactions on Automatic Control* **45**(12), 2253–2270.

Consolini, L. and M. Tosques (2004). A controlled invariance problem for the vtol aircraft with bounded internal dynamics. In: *Proceedings of the 43th IEEE Conference on Decision and Control*. Paradise Island, Bahamas.

Fliess, M, J Levine, P Martin and P Rouchon (1999). A lie-backlund approach to equivalence and flatness of nonlinear systems. *IEEE Transactions on Automatic Control* **44**(5), 922–937.

Holzhuter, Thomas (2004). Optimal regulator for the inverted pendulum via euler-lagrange backward integration. *Automatica* **40**(9), 1613–1620.

Mazenc, F and S Bowong (2003). Tracking trajectories of the cart-pendulum system. *Automatica* **39**(4), 677–684.

Sastry, H., J.Hauser and G. Meyer (1992). Nonlinear control design for slightly non-minimum phase systems: application to v/stol aircraft. *Automatica* **28**, 665–679.

Shiriaev, A, A Pogromsky, H Ludvigsen and O Egeland (2000). On global properties of passivity-based control of an inverted pendulum. *International Journal of Robust and Nonlinear Control* **10**(4), 283–300.