NON-SMOOTH DISTURBANCE REJECTION IN NONLINEAR SYSTEMS USING NONLINEAR INTERNAL MODEL

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Abstract: This paper deals with estimation and rejection of unknown non-smooth disturbances in nonlinear systems in the output feedback form. The specific non-smooth disturbances considered in this paper are the square-wave disturbances with known frequencies. The other information such as the amplitudes and phases are unknown. The rejection takes an indirect approach by estimating the disturbance first and then designing the feedback control based on the estimated disturbances. To estimate the disturbance, filters are designed to extract the contribution to the state from the disturbances and a novel nonlinear internal model is constructed using the integrals over half of the period and nonlinear functions. The proposed control design asymptotically rejects the unknown square-wave disturbances and ensures the boundedness of all the variables.

Keywords: Disturbance rejection, Estimation, Nonlinear systems, Uncertainty, Integral control

1. INTRODUCTION

Recently, tremendous progresses have been reported in rejecting sinusoidal disturbances. Even for the case of unknown frequencies, a series of results have been published for rejecting disturbances (Bodson *et al.*, 1994; Bodson and Douglas, 1997; Marino *et al.*, 2003; Ding, 2003). The key to the success is the internal model principle, ie, an internal model is used in the control algorithm to generate the disturbance or its contribution to the critical system state variable in some way. It is easy to model the sinusoidal disturbances as the output of linear exosystems, and the internal model can then be designed accordingly. This is not the case for the non-smooth periodical disturbances such as square-wave disturbances. Until now there is not any report on asymptotically rejecting the square-wave disturbance. If the period is long enough, integral actions in the system may reject the disturbance, but it is not in the sense of asymptotic rejection, ie, after the disturbance changes value, it will take some time to settle down again. Therefore the integral action is not good enough to reject square-wave disturbances.

This paper will address the problem of asymptotically rejecting square-wave disturbances in nonlinear systems which are in the output feedback form. A dynamic model for square-wave disturbance is first introduced. In this model, there are strong nonlinear functions which prevent the direct application of the existing results using linear internal model such as one shown in (Ding, 2001) and even the nonlinear internal model method recently introduced in (Ding, 2004). The idea of the internal model principle is exploited in the sense that an asymptotic estimate of the unknown disturbance is needed for the control design in one way or another. A novel structure of generating an estimate of disturbance is proposed, which exploits the periodical nature of the disturbances. The integrals over half of the period, delay and the sign function are used in the construction of estimator for the disturbance. The estimate is then used in the control design for disturbance rejection and stability. The proposed control guarantees the complete rejection of the square-wave disturbances and the boundedness of the variables in the closed-loop control system. An example is included to illustrate the proposed estimation and control design, together with the simulation results.

2. PROBLEM FORMULATION

Consider a single-input-single-output nonlinear system which can be transformed into the output feedback form

$$\begin{aligned} \dot{x} &= A_c x + \phi(y) + b(u - \mu) \\ y &= Cx \end{aligned} \tag{1}$$

with

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_\rho \\ \vdots \\ b_n \end{bmatrix}$$

where $x \in R^n$ is the state vector, $u \in R$ is the control, ϕ, is a known nonlinear smooth vector field in R^n with $\phi(0) = 0$, $\mu \in R$ is a periodical disturbance.

Assumption 1. the system is minimum phase, ie, the zeros of polynomial $\mathbf{B}(s) = \sum_{i=\rho}^{n} b_i s^{n-i}$ have negative real parts.

Assumption 2. The period and the pattern of the disturbances are known.

To simplify the presentation, only the square-wave disturbance is considered in this paper. The pattern information is the square wave, which means that the disturbance only changes values among the positive and negative amplitude. The disturbance frequency is assumed known, from Assumption 2. However, there is no information of the phase and amplitude of the disturbances.

The disturbance rejection problem considered in this paper is to design a feedback control algorithm so that the unknown disturbance is completely rejected. An indirect approach is adopted, ie, the disturbance is explicitly estimated first and then control algorithm is designed using the estimate to reject the disturbance.

3. DISTURBANCE ESTIMATION

An important concept in disturbance rejection is the internal model principle, ie, an internal model is designed to regenerate the unknown disturbance from the output measure. But there is not a ready dynamic model for a square-wave disturbance. A square-wave disturbance can be described by

$$\mu(t) = a\mathrm{sign}(\sin \omega t + \phi) \tag{2}$$

where sign$(\cdot)$ denotes a function of taking the sign of its operant, a and ϕ denote the amplitude and the phase respectively. Since a sinusoidal function can be modeled as the output of a second order critically stable linear system, with reference to (2), a dynamic model can be created for the square-wave disturbance as:

$$\begin{aligned} \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} &= \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} w_1(0) \\ w_2(0) \end{bmatrix} = \begin{bmatrix} w_{1,0} \\ w_{2,0} \end{bmatrix} \\ \mu(t) &= \mathrm{sign}(w_1) \end{aligned} \tag{3}$$

where the initial values $w_{1,0}$ and $w_{2,0}$ decide the amplitude and the phase of the disturbance. Hence the square-wave disturbance can be considered as the output of the above nonlinear exosystem. Since the function sign() is strongly nonlinear, the existing methods for rejection of sinusoidal disturbance with known frequencies in (Ding, 2001) and (Ding, 2004) are not applicable. Even though the nonlinear exosystem does not lead to a direct design of the internal model for disturbance rejection, it provides useful hints to the properties of square-wave disturbances.

Define a half-period integration operator $I_{T/2}$ and a quarter-period delay operator $D_{T/4}$ as

$$\begin{aligned} I_{T/2} \circ f(t) &:= I_{T/2}(f(t)) = \int_{t-T/2}^{t} f(s)ds \\ D_{T/4} \circ f(t) &:= D_{T/4}(f(t)) = f(t - T/4) \end{aligned} \tag{4}$$

It is easy to show the following property for square-wave disturbances

$$\begin{aligned} \mu(t) &= -D_{T/4} \circ \mathrm{sign}(I_{T/2} \circ \mu(t))\frac{T^2}{8} \\ &= D_{T/4}^3 \circ \mathrm{sign}(I_{T/2} \circ \mu(t))\frac{T^2}{8} \end{aligned} \tag{5}$$

where T is the period of the disturbance with $T = \frac{2\pi}{\omega}$. This property provides the foundation for the disturbance estimation algorithm in this paper.

In order to extract the contribution in system state due to the disturbances, the following filter is designed:

$$\dot{p} = (A_c + kC)p + \phi(y) + bu - ky \tag{6}$$

where $p \in R^n$, $k \in R^n$ is chosen so that

$$\begin{aligned} K(s) &:= s^n - \sum_{i=1}^{n} k_i s^{n-i} \\ &= B(s) \prod_{i=1}^{\rho} (s + \lambda_i)/b_\rho \end{aligned} \tag{7}$$

with λ_i being positive real design parameters.

Now it is ready to introduce the estimate of unknown disturbances. The estimation algorithm depends on the relative degree of the system. Consider the relative degree one case, ie, $\rho = 1$,

$$\hat{\nu}(t) = (I - D^2_{T/4} + \lambda_1 I_{T/2}) \circ (p_1 - y) \tag{8}$$

$$\hat{\mu}(t) = \frac{8}{T^2} \mathrm{sign}(D^2_{T/4} \circ \hat{\nu}(t)) I_{T/2} \circ \hat{\nu}(t) \tag{9}$$

where $\hat{\mu}$ is the estimate of disturbance, and I denotes the identity operator, ie, $I \circ f(t) = f(t)$. The disturbance estimates for the cases of high relative degrees can be obtained similarly.

Lemma 3.1 The estimate given in (9) converges to the actual disturbance in L_p, ie $\mu - \hat{\mu} \in L_p$ for $p = 1, 2$ and ∞.

Proof: It is easy to see that both μ and $\hat{\mu}$ are bounded signals, and therefore $\mu - \hat{\mu} \in L_\infty$. To complete the proof, it only needs to show the case for L_1, as $\mu - \hat{\mu} \in L_1 \cap L_\infty$ implies $\mu - \hat{\mu} \in L_2$

Consider a dummy filter

$$\dot{q} = (A_c + kC)q + b\mu \tag{10}$$

where q denotes the steady state only. Let $e = x - (p - q)$, then it is easy to show that

$$\dot{e} = (A_c + kC)e \tag{11}$$

Since $(A_c + kC)$ is Hurwitz, it can be shown that

$$\|e(t)\| \le K_e e^{-\lambda_e t} \tag{12}$$

for some positive real constants K_e and λ_e. From the special structure of k chosen in (7), it can be shown that, for the relative degree one case,

$$\dot{q}_1 = -\lambda_1 q_1 + b_1 \mu \tag{13}$$

Define

$$\nu(t) = (I - D^2_{T/4} + \lambda_1 I_{T/2}) \circ (q_1/b_1) \tag{14}$$

It can be shown that

$$a = \frac{8}{T^2} I_{T/2} \circ \nu(t) \tag{15}$$

$$\mu(t) = \frac{8}{T^2} \mathrm{sign}(D^2_{T/4} \circ \nu(t)) I_{T/2} \circ \nu(t) \tag{16}$$

From (8) and (14), it follows that

$$\begin{aligned} \tilde{\nu}(t) &:= \nu(t) - \hat{\nu}(t) \\ &= (I - D^2_{T/4} + \lambda_1 I_{T/2}) \circ e(t) \end{aligned} \tag{17}$$

From (12), it can be obtained that

$$\|\tilde{\nu}(t)\| \le K_\nu e^{-\lambda_\nu t} \tag{18}$$

for some positive real constants K_ν and λ_ν. Now consider

$$\begin{aligned} \tilde{\mu} &:= \mu - \hat{\mu} \\ &= \frac{8}{T^2} \{ \mathrm{sign}(D^2_{T/4} \circ \nu(t)) I_{T/2} \circ \nu(t) \\ &\qquad - \mathrm{sign}(D^2_{T/4} \circ \hat{\nu}(t)) I_{T/2} \circ \hat{\nu}(t) \} \\ &= a[\mathrm{sign}(D^2_{T/4} \circ \nu(t)) - \mathrm{sign}(D^2_{T/4} \circ \hat{\nu}(t))] \\ &\quad + \frac{8}{T^2} \mathrm{sign}(D^2_{T/4} \circ \hat{\nu}(t)) I_{T/2} \circ \tilde{\nu}(t)] \end{aligned} \tag{19}$$

From (18) it can be shown that the second term in (19) is in L_1, and therefore the proof is completed if one can show the first term is in L_1, which is equivalent to show that

$$I_\infty = \int_0^\infty |\mathrm{sign}(\nu(t)) - \mathrm{sign}(\hat{\nu}(t))| dt < \infty \tag{20}$$

Define

$$\begin{aligned} J_i = \int_{(i-1)T}^{iT} & |\mathrm{sign}(\nu(t)) \\ & - \mathrm{sign}(\nu(t) - \tilde{\nu}(t))| dt \end{aligned} \tag{21}$$

It follows that

$$I_\infty = \sum_{i=0}^{\infty} J_i \tag{22}$$

Notice that $\nu(t)$ is a triangular wave with the peak value $aT/2$. Since $|\tilde{\nu}(t)|$ is bounded by an exponentially decaying function, there exists an $\bar{i}$ such that $|\tilde{\nu}(\bar{i}T)| < aT/2$ for all $i > \bar{i}$. Therefore, for $i > \bar{i}$, it can be shown that

$$J_i < \frac{4}{a} K_\nu e^{-i\lambda_\nu T} \tag{23}$$

Finally, it can be obtained that

$$I_\infty \le 2\bar{i}T + \sum_{i=\bar{i}+1}^{\infty} \frac{4}{a} K_\nu e^{-i\lambda_\nu T} < \infty \quad (24)$$

This completes the proof of Lemma 3.1.

Remark 1: Although the results shown so far are for the systems with relative degree one, a similar result can be obtained for systems with high relative degrees. The control design for disturbance rejection with stabilization in the next section will assume that there exists a disturbance estimate $\hat{\mu}$ which satisfies the conditions specified in Lemma 3.1.

4. DISTURBANCE REJECTION WITH STABILIZATION

A control algorithm is to be presented for disturbance rejection for a disturbance estimate $\hat{\mu}$ which satisfes $\tilde{\mu} \in L_p$ with $p = 1, 2, \infty$. A state observer is designed as

$$\dot{\hat{x}} = (A_c + kC)\hat{x} + \phi(y) + b(u - \hat{\mu}) - ky \quad (25)$$

Let

$$\tilde{x} := x - \hat{x} \quad (26)$$

It follows that

$$\dot{\tilde{x}} := (A_c + kC)\tilde{x} - b\tilde{\mu} \quad (27)$$

Control design can then be carried out using backstepping based on (25). For the backstepping design, the following notations are used:

$$z_1 = y = x_1 \quad (28)$$

$$z_i = \hat{x}_i - \alpha_{i-1}, i = 2, \dots \rho \quad (29)$$

where α_i are the stabilizing functions designed in the backstepping procedures. The design starts from the dynamics of z_1 given by

$$\begin{aligned} \dot{z}_1 &= x_2 + \phi_1(y) \\ &= z_2 + \alpha_1 + \phi_1(y) + \tilde{x}_1 \end{aligned} \quad (30)$$

The first stabilizing function α_1 is designed as

$$\alpha_1 = -c_1 z_1 - d_1 z_1 - \phi_1(y) \quad (31)$$

where c_i and d_i are the positive real design parameters, for $i = 1, \dots, \rho$. The subsequent stabilizing functions are designed as

$$\begin{aligned} \alpha_i &= z_{i-1} - c_i z_i - d_i\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 z_i \\ &\quad + k_i \hat{x}_1 - k_i y - \phi_i(y) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{x}_j} \dot{\hat{x}}_j \\ &\quad + \frac{\partial \alpha_{i-1}}{\partial y}(\hat{x}_2 + \phi_1(y)) \end{aligned} \quad (32)$$

for $\rho = 2, \dots, \rho$. Finally the control input is given by

$$u = \hat{\mu} + \frac{\alpha_\rho - x_{\rho+1}}{b_\rho} \quad (33)$$

The proposed control ensures the asymptotic rejection of the disturbance and the boundedness of all the variables in the closed-loop system. The stability result is summarized in the following theorem.

Theorem 4.1 The control input u given in (33) with the estimated disturbance $\hat{\mu}$ ensures the asymptotic rejection of the unknown disturbance in the system (1), ie, $\lim_{t\to\infty} y(t) = 0$, and the boundedness of the other variables in the system.

Proof: Define

$$V_x = \tilde{x}^T P \tilde{x} \quad (34)$$

where P is a positive definite matrix satisfying

$$P(A_o + kC) + (A_o + kC)^T P = -3I \quad (35)$$

From (27), it can be obtained

$$\begin{aligned} \dot{V}_x &= -3\tilde{x}^T\tilde{x} - 2\tilde{x}^T P b \tilde{\mu} \\ &\le -2\tilde{x}^T\tilde{x} + \|Pb\|^2 \tilde{\mu}^2 \end{aligned} \quad (36)$$

Define

$$V_z = \frac{1}{2}\sum_{i=1}^{\rho} z_i^2 \quad (37)$$

It can be shown that

$$\begin{aligned} \dot{V}_z &= \sum_{i=1}^{\rho}(-c_i z_i^2 - d_1\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 z_i^2 \\ &\quad - \frac{\partial \alpha_{i-1}}{\partial y} z_i \tilde{x}_2) \\ &\le \sum_{i=1}^{\rho}(-c_i z_i^2) + \beta \tilde{x}_i^2 \end{aligned} \quad (38)$$

where $\beta = \sum_{i=1}^{\rho} 1/d_i$ and $\frac{\partial \alpha_{-1}}{\partial y} = -1$. Let

$$V = V_z + \beta V_x \quad (39)$$

From (36) and (38), it can be obtained that

$$\begin{aligned} \dot{V} &\le -\sum_{i=1}^{\rho} c_i z_i^2 - \beta\tilde{x}^T\tilde{x} + \beta\|Pb\|^2\tilde{\mu}^2 \\ &\le -\lambda V + \beta\|Pb\|^2\tilde{\mu}^2 \end{aligned} \quad (40)$$

where

$$\lambda = \min\{2 \min_{i=1,\dots,\rho} c_i, \frac{\beta}{\lambda_{max}(P)}\} \quad (41)$$

with $\lambda_{max}(P)$ being the maximum eigenvalue of P. it can be concluded, using the comparison lemma (Khalil, 2002), that

$$V(t) \leq \bar{V}(t) \quad (42)$$

where $\bar{V}(t)$ is generated by

$$\dot{\bar{V}} = -\lambda\bar{V} + \beta\|Pb\|^2\tilde{\mu}^2, \bar{V}(0) = V(0) \quad (43)$$

With $\tilde{\mu}^2 \in L_1 \cap L_\infty$, as from Lemma 3.1, it can be concluded that $\bar{V} \in L_1 \cap L_\infty$, and hence $V \in L_1 \cap L_\infty$. The boundedness of V implies the boundedness of $\tilde{x}$ and z_i for $i = 1, \dots \rho$. Since $\dot{V} \in L_\infty$, it can be concluded from Babalat's lemma $\lim_{t\to\infty} V(t) = 0$, which further implies $\lim_{t\to\infty} \tilde{x}(t) = 0$, and $\lim_{t\to\infty} z_i(t) = 0$. The boundedness of y and $\lim_{t\to\infty} y(t) = 0$ follow the results of z_i with $i = 1$. The boundedness of other state variables can be established using the boundedness of y and the minimum phase assumption in Assumption 1. This concludes the proof of the theorem.

If the system (1) is linear, a simpler control design without invoking backstepping can be proposed. For the linear system,

$$\phi(y) = fy \quad (44)$$

with $f \in R^n$. In this case, the following control design is proposed

$$u_l = k_l^T\hat{x} + \hat{\mu} \quad (45)$$

where k_l is chosen so that $A_l = (A_c + fC + bk_l^T)$ is Hurwitz. The following theorem addresses the stability of the proposed control for linear systems.

Theorem 4.2: The control input shown in (45) stabilizes the system (1) and completely rejects the unknown disturbance if $\phi(y) = fy$.

Proof: Under the control (45), the system dynamics are given by

$$\dot{x} = A_l x + bk_l^T\tilde{x} - b\tilde{\mu} \quad (46)$$

Consider

$$V_l = x^T P_l x \quad (47)$$

where P_l satisfies

$$A_l^T P_l + P_l A_l^T = -2I \quad (48)$$

its derivative along (46) is given by

$$\begin{aligned}\dot{V}_l &= -2x^T x + 2x^T Pb(k_l^T\tilde{x} - \tilde{\mu} \\ &\leq -x^T x + \|Pb[k_l^T\tilde{x} - \tilde{\mu}]\|^2 \\ &\leq -\frac{1}{\lambda_{max}(P_l)}V + \|Pb[k_l^T\tilde{x} - \tilde{\mu}]\|^2 \quad (49)\end{aligned}$$

where $\lambda_{max}(P_l)$ denotes the maximum eigenvalue of P_l. Notice that $\|P_l b[k_l^T\tilde{x} - \tilde{\mu}]\|^2$ can be shown in L_p for $p = 1, \infty$. The remaining proof follows in a similar way as in the later part of the proof of Theorem 4.1.

5. AN EXAMPLE

Consider a nonlinear system in output feedback form

$$\begin{aligned}\dot{x}_1 &= x_2 - y^3 + (u - \mu) \\ \dot{x}_2 &= (u - \mu) \\ y &= x_1 \quad (50)\end{aligned}$$

where μ is a square-wave disturbance. It is easy to see that the system (50) are in the format of (1) with $\phi(y) = [y^3\ 0]^T$ and $b = [1\ 1]^T$. The system is minimum phase.

The filters for disturbance estimation are designed as

$$\dot{p} = \begin{bmatrix} k_1 & 1 \\ k_2 & 0 \end{bmatrix} p + \begin{bmatrix} y^3 \\ 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} y \quad (51)$$

$$+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \quad (52)$$

$$\dot{\tilde{x}} = \begin{bmatrix} k_1 & 1 \\ k_2 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} y^3 \\ 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} y \quad (53)$$

$$+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} (u - \hat{\mu}) \quad (54)$$

with $\hat{\mu}$ being generated in exactly the same way as shown in Section 3. Following the control design introduced in Section 4, the control input is designed as

$$u = \hat{\mu} - c_1 y - d_1 y - y^3 - \hat{x}_2 \quad (55)$$

The simulation study has been carried out for the estimation and control design shown in this example. The simulation results shown below are for the settings $k_1 = -2$, $k_2 = -1$, $c_1 = d_1 = \lambda_1 = 1$. The settings for the disturbance are the period $T = 2$, and the amplitude $a = 1$. The control input and the system output are shown in Figure 1, in which the output converges to zero with the input to asymptotically cancel the disturbance. The disturbance μ and its estimate $\hat{\mu}$ are shown in Figure 2.

6. CONCLUSIONS

An novel estimation algorithm is proposed for regenerating the unknown square-wave disturbances, and the estimated disturbance is then used in the proposed control design for complete disturbance rejection with stabilization of the closed-loop system. The proposed estimation algorithm makes good use of nonlinearities in the systems, and the proposed control design still follows the internal principle, with a novel interpretation of the internal model. The enclosed simulation results demonstrate the efficiency of the proposed design in rejecting non-smooth disturbances. Although the proposed estimation and control algorithms only consider the case of square-wave disturbances, the results can be easily extended to other non-smooth periodical disturbances such as triangular disturbances.

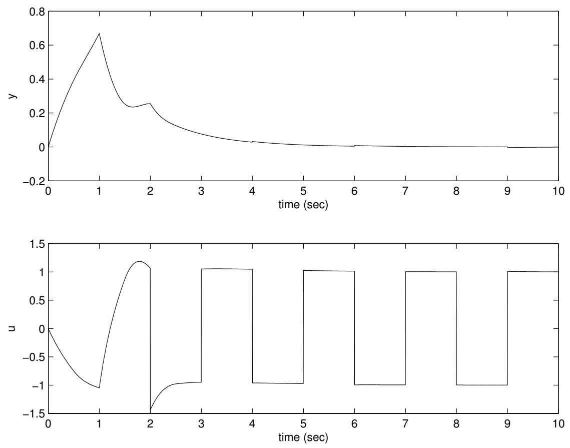

Fig. 1. Control input and system output

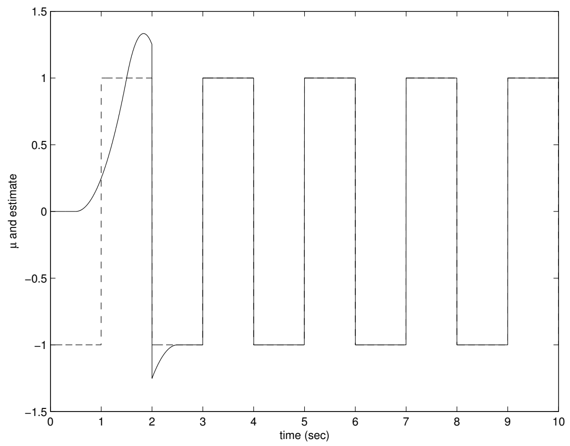

Fig. 2. Disturbance and its estimate

REFERENCES

Bodson, M., A. Sacks and P. Khosla (1994). Harmonic generation in adaptive feedforward cancellation schemes. *IEEE Trans. Automat. Contr.* **39**(9), 1939–1944.

Bodson, M. and S. C. Douglas (1997). Adaptive algorithms for the rejection of sinusoidal disturbances with unknown frequencies. *Automatica* **33**(10), 2213–2221.

Ding, Z. (2001). Global output regulation of uncertain nonlinear systems with exogenous signals. *Automatica* **37**, 113–119.

Ding, Z. (2003). Global stabilization and disturbance suppression of a class of nonlinear systems with uncertain internal model. *Automatica* **39**(3), 471–479.

Ding, Z. (2004). Output regulation of uncertain nonlinear systems with nonlinear exosystems. In: *Proceeding of the 2004 American Control Conference*. Boston, USA. pp. 3677–3682.

Khalil, H. K. (2002). *Nonlinear Systems*. 3rd ed.. Prentice Hall. New Jersey.

Marino, R., G. L. Santosuosso and P. Tomei (2003). Robust adaptive compensation of biased sinusoidal disturbances with unknown frequencies. *Automatica* **39**, 1755–1761.