DIRECT FEEDBACK IN AUTOMATA NETWORKS

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Abstract: This paper investigates the problem posed by direct feedback in automata networks. Such a feedback introduces a direct instantaneous depending of the input of a system upon itself through a signal path within the network. In continuous system theory such a feedback yields an algebraic loop, which may render the overall system ill-posed. In discrete-event system theory this problem has not been investigated in detail. This paper develops criteria to test if direct feedback exists and if the network is well-posed.

Keywords: Feedback loops, automata theory, discrete-time systems, stochastic systems

1. INTRODUCTION

Automata networks have been used for modelling, controlling, and diagnosing discrete-event systems in the last years. A network consists of several interconnected automata and in the networks investigated here all state transitions are synchronised by the clock. This synchronisation may become ill-defined if the state transitions of one or more automata depend upon each other through direct feedback paths within the network. This problem is called the *feedback problem.* The feedback problem occurs whenever a signal depends directly and instantaneously on itself. This may lead to a conflict resulting in blocking or even a not well-defined system.

This paper investigates the problem of direct feedback in automata networks with synchronised signals and develops criteria to test whether a given automata network is well-defined. In literature, so far the problem has been avoided by simply excluding conflicts per definition or by dealing only with Moore automata for which this problem cannot occur (Pearl, 1988; Lee and Varaiya, 2003).

The structure of the paper is as follows. In the following section the automata network is introduced. Section 3 describes the feedback problem for both continuous and discrete systems. In Section 4 criteria for testing whether a feedback leads to a well-defined system are presented.

2. AUTOMATA THEORY

In this section the automaton as well as the automata network are introduced. Automata are finite-state machines and their state transitions are synchronised by the clock. That is, the time line is discretised into a sequence of equally spaced intervals (Schröder, 2003; Lunze, 2003; Cassandras and Lafortune, 1999).

2.1 The Deterministic Automaton

The deterministic automaton is given by the tuple

$$\mathcal{D} = (\mathcal{N}_z, \mathcal{N}_v, \mathcal{N}_w, L_d, z(0)) \qquad (1)$$

with the finite sets $\mathcal{N}_z=\{1,\ldots,N\}$ of automaton states, $\mathcal{N}_v=\{1,\ldots,M\}$ of input symbols, and $\mathcal{N}_w=\{1,\ldots,R\}$ of output symbols. The state of the automaton is denoted by $z\in\mathcal{N}_z$, the input by $v\in\mathcal{N}_v$, and the output by $w\in\mathcal{N}_w$. $z(0)$ is the initial state. The dynamics of the automaton is defined by the behavioural function

$$L_d : \mathcal{N}_z \times \mathcal{N}_w \times \mathcal{N}_z \times \mathcal{N}_v \to \{0,1\}, \tag{2}$$

where $L_d(z', w, z, v) = 1$ if for the input v the state changes from z to z' and the automaton produces the output w. Otherwise $L_d(z', w, z, v) = 0$ holds. The output function H_d can be extracted from L_d:

$$H_d : \mathcal{N}_w \times \mathcal{N}_z \times \mathcal{N}_v \to \{0,1\}, \tag{3}$$

$$H_d(w, z, v) = \sum_{z'=1}^{N} L_d(z', w, z, v). \tag{4}$$

Since the automaton is deterministic

$$\sum_{z'=1}^{N} \sum_{w=1}^{R} L_d(z', w, z, v) = 1,\ \forall z\in\mathcal{N}_z, v\in\mathcal{N}_v \tag{5}$$

holds. Correspondingly, for an autonomous automaton $\mathcal{D} = (\mathcal{N}_z, L_d, z(0))$ the following holds:

$$L_d : \mathcal{N}_z \times \mathcal{N}_z \to \{0,1\},\ \sum_{z'=1}^{N} L_d(z', z) = 1. \tag{6}$$

2.2 The Nondeterministic Automaton

As opposed to the deterministic automaton the movement of the nondeterministic automaton

$$\mathcal{N} = (\mathcal{N}_z, \mathcal{N}_v, \mathcal{N}_w, L_n, z(0)) \tag{7}$$

is not unambiguous. The behavioural relation

$$L_n : \mathcal{N}_z \times \mathcal{N}_w \times \mathcal{N}_z \times \mathcal{N}_v \to \{0,1\} \tag{8}$$

assumes the value $L_n = 1$ for all *possible* transitions (z', w, z, v). The output relation is given by

$$H_n(w, z, v) = \bigvee_{z'=1}^{N} L_d(z', w, z, v), \tag{9}$$

where $\vee$ denotes the Boolean conjunction. In this paper the automaton is required to be live, meaning transitions are possible in all states:

$$\bigvee_{z'=1}^{N} \bigvee_{w=1}^{R} L_n(z', w, z, v) = 1,\ \forall z\in\mathcal{N}_z, v\in\mathcal{N}_v. \tag{10}$$

Analogously to (6), for the autonomous automaton $\mathcal{N} = (\mathcal{N}_z, L_n, z(0))$ the following holds for liveness:

$$\bigvee_{z'=1}^{N} L_n(z', z) = 1,\ \forall z\in\mathcal{N}_z. \tag{11}$$

2.3 The Stochastic Automaton

For the stochastic automaton

$$\mathcal{S} = (\mathcal{N}_z, \mathcal{N}_v, \mathcal{N}_w, L_s, \boldsymbol{p}(z(0))) \tag{12}$$

the behavioural relation provides information about the probability P for the respective state transitions (Bukharaev, 1995):

$$L_s : \mathcal{N}_z \times \mathcal{N}_w \times \mathcal{N}_z \times \mathcal{N}_v \to [0,1] \tag{13}$$

$$L_s(z', w|z, v) = P(z', w|z, v). \tag{14}$$

The initial state is given as a probability distribution $\boldsymbol{p}(z(0))$. In analogy to the automata described above, the following holds:

$$H_s(w|z, v) = \sum_{z'=1}^{N} L_s(z', w|z, v) \tag{15}$$

$$\sum_{z'=1}^{N} \sum_{w=1}^{R} L_s(z', w|z, v) = 1,\ \forall z\in\mathcal{N}_z, v\in\mathcal{N}_v. \tag{16}$$

2.4 The Automata Network

The definitions of the automata can easily be extended to automata with multiple inputs and outputs by using vectorial signals:

$$v_1\in\mathcal{N}_{v_1}, \ldots, v_\mu\in\mathcal{N}_{v_\mu},\ \boldsymbol{v}\in\mathcal{N}_v=\mathcal{N}_{v_1}\times\cdots\times\mathcal{N}_{v_\mu}$$
$$w_1\in\mathcal{N}_{w_1}, \ldots, w_\rho\in\mathcal{N}_{w_\rho},\ \boldsymbol{w}\in\mathcal{N}_w=\mathcal{N}_{w_1}\times\cdots\times\mathcal{N}_{w_\rho}.$$

An automata network consists of several interconnected automata. As a formal definition of such a network is beyond the scope of this paper, it will be introduced on an intuitive level. The network is completely defined by the set of automata, e.g. $\mathcal{SAN} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ for the stochastic automata network of Fig. 1, presuming that all signals of the network have an unique name. Then the coupling from Automaton 1 to Automaton 2 in Fig. 1 is given by the fact that the signal s^1 is the output signal of Automaton 1 and simultaneously the input signal of Automaton 2. Note that the evaluation of all signals and the state transitions of all automata of the network are synchronised by a clock (cf. (Lunze and Neidig, 2003)). In general, this leads to a very strong coupling among all components. Under the conditions investigated below every component influences many other components at all times k.

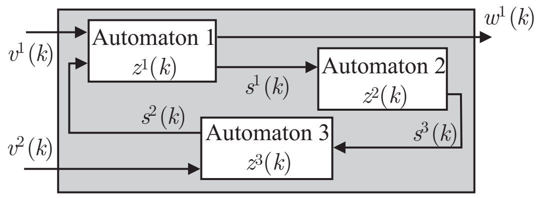

Figure 1. Example of an automaton network

3. FEEDBACK IN INTERCONNECTED SYSTEMS

Direct feedback in interconnected systems means that a signal depends directly on itself: [1]

$$y = f(y). \tag{17}$$

A solution of (17) is called *fixed point.* A direct feedback may have no, one or more than one fixed points. A feedback with a unique fixed point $\bar{y} = f(\bar{y})$ is called *well-formed* (Lee and Varaiya, 2003).

3.1 Direct Feedback in Continuous Systems

In continuous systems theory, direct feedback is well known. The relation (17) is the result of an "algebraic loop" within interconnected systems. If (17) has no or more than one solution, the overall system is ill-defined. Methods to detect algebraic loops and to avoid them has been studied in literature about simulation techniques and large-scale systems (Schmidt, 1980; Lunze, 1992). However, in discrete systems the signal values are often symbols and therefore algebraic operations are not defined. This prohibits to transfer the methods of continuous system theory to discrete systems.

3.2 Direct Feedback in Discrete Systems

An analogy to continuous systems the direct feedback results when the output w depends directly on itself as shown in Fig. 2. Closing the loop forces the input to be equal to the output: $v(k) = w(k)$. [2] If the output $w(k)$ depends directly on the input $v(k)$ by means of the output relation H, a relation of the form (17) occurs. Whenever the output relation forces the output to assume a value different from the input, a conflict occurs. The main aim of this paper is to investigate this situation.

Note, that this problem is restricted to synchronous systems. As in asynchronous systems the inputs and outputs are not evaluated at the same time, it is impossible for conflicts as described above to occur (Lamperti and Zanella, 2003).

Example 1: Given is a deterministic automaton $\mathcal{D}$ with $\mathcal{N}_v = \mathcal{N}_w = \{1, 2\}$ and a feedback connection as depicted in Figure 2. If the automata's output relation H_d generates the output $w = 1$ whenever the input is $v = 2$ and $w = 2$ whenever $v = 1$, the feedback condition $w = v$ cannot be fulfilled. The loop is not well-formed. □

[1] This is not to be confused with closed loops as used in control theory which contain integrators where the signal depends on its temporal derivative and not directly on itself.

[2] Therefore $\mathcal{N}_v \supseteq \mathcal{N}_w$ must hold. This will be implied for the remainder of the paper.

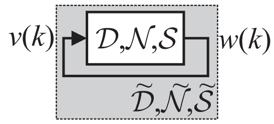

Figure 2. Feedback in discrete systems

3.3 The Origin of Loops

Algebraic loops do not exist in nature. Their origin is a direct consequence of the modelling assumption which creates a simplified description of the physically existing system. Neglecting details, which seem not to contribute to the main properties of the system, leads to the mathematical deficiency. In continuous systems such loops usually occur when neglecting parasitical resistors or when approximating quick dynamical effects by static elements.

In discrete systems such loops may occur, because the natural system is simplified by sampling and quantising the signals. Figure 3 shows two examples for loops in discrete systems. In the left example the exit of the XOR-gate is not stable and the loop is not well-formed. The circuit can be designed, but what happens depends on the neglected propagation and delay times. In the right example a discrete system is controlled by a discrete controller. If the system has a direct feed-through and the controller is a simple static controller ($v(k) = h(w(k))$), the loop may not be well-formed. The cause of this lies in the assumption that the signal values can change infinitely fast. In large coupled systems it may not be intuitively clear if a loop is well-formed or if the modeling process has to account for additional details.

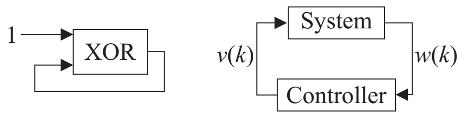

Figure 3. Examples of loops

3.4 Direct Feedback Condition

The direct feedback problem occurs only in components which are *strongly connected* and have *direct feed-through.* Two components i and j are strongly connected iff there exists a signal path p from component i to j as well as from component j to i. Both paths together form a *loop.* A component has a direct feed-through iff its output $w(k)$ depends on its input $v(k)$.

A *Moore automaton* has no direct feed-through. Instead, its output depends only on the automaton state $z(k)$. For the deterministic Moore automaton there exists a function

$$\bar{H}_d : \mathcal{N}_w \times \mathcal{N}_z \to \{0, 1\} \text{ with}$$
$$\bar{H}_d(w, z) = H_d(w, z, v) \quad \forall v$$

that does not depend on v. Analogously for the nondeterministic and stochastic automaton functions $\bar{H}_n$ and $\bar{H}_s$ exist such that

$$\bar{H}_n(w,z) = H_n(w,z,v), \quad \bar{H}_s(w|z) = H_s(w|z,v)$$

hold.

Theorem 1: *A loop in an automata network that contains one or more Moore automata is always well-formed (cf. (Lee and Varaiya, 2003)).*

The question to be answered in the next section is under what conditions ill-formed loops occur in coupled automata for which the output function depends explicitely on the input v.

4. SOLUTION TO THE FEEDBACK PROBLEM

The basic problem can be studied for a single automaton with feedback (Fig. 2). The question is whether the closed feedback represents a well-defined autonomous automaton. The monolithic model of this autonomous system can be obtained by applying known composition rules. However, only if the feedback loop is well-formed the composition results in an automaton according to the definitions given in Section 2. In the following section it is investigated under what conditions the composition of a deterministic, nondeterministic or stochastic automaton with feedback is well-formed. The results are then extended to automata networks and illustrated by extensive examples (Pache, 2004).

4.1 Feedback in Deterministic Automata

The formal composition of a deterministic automaton (1) with a feedback connection (Fig. 2) results in an autonomous system $\tilde{\mathcal{D}} = (\mathcal{N}_z, \tilde{L}_d, z(0))$. The relation $\tilde{L}_d$ is given by

$$\tilde{L}_d(z', z) = \sum_{w=1}^{R} L_d(z', w, z, w), \qquad (18)$$

assuring $v = w$ for all w.

Definition: *The composition of a deterministic automaton with a feedback connection is said to be* well-formed *if for its output function the relation*

$$\sum_{w=1}^{R} H_d(w, z, w) = 1, \ \forall z \in \mathcal{N}_z \qquad (19)$$

holds.

Theorem 2: *A system consisting of a deterministic automaton with a feedback connection is a deterministic automaton iff the feedback composition is well-formed.*

Proof:

$$\sum_{w=1}^{R} H_d(w,z,w) = 1 \Leftrightarrow \sum_{w=1}^{R}\sum_{z'=1}^{N} L_d(z',w,z,w) = 1$$
$$\Leftrightarrow \sum_{z'=1}^{N} \tilde{L}_d(z',z) = 1,$$

satisfying equation (6). ■

Corollary: If the relation $\sum_{w=1}^{R} H_d(w,z,w) > 1$ is satisfied, the feedback connection leads to a nondeterministic automaton. On the other hand, $\sum_{w=1}^{R} H_d(w,z,w) = 0$ results in an automaton which is not live. □

Example 2: Consider the deterministic automaton $\mathcal{D}_2$ with $\mathcal{N}_z = \{1,2\}$ and $\mathcal{N}_v = \mathcal{N}_w = \{1,2\}$. Its dynamics is given in Fig. 4(a) as an automaton graph. The states are depicted as nodes and the transitions as arcs. The arcs are labeled with the respective input/output pair v/w. A feedback loop as shown in Fig. 2 is introduced, resulting in the autonomous system shown in Fig. 4(b). As Theorem 2 is satisfied the resulting system is a deterministic automaton. Obviously, the system is obtained by eliminating all transitions with $v \neq w$. □

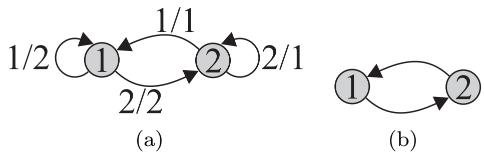

Figure 4. Automata graphs for Example 2

Example 3: Consider the deterministic automaton $\mathcal{D}_3$ with $\mathcal{N}_z, \mathcal{N}_v, \mathcal{N}_w$ defined as above, whose dynamics is given in Fig. 5(a). The composition of $\mathcal{D}_3$ with a feedback loop results in the system shown in Fig. 5(b). Theorem 2 is not fulfilled, therefore the resulting system is not a deterministic automaton. □

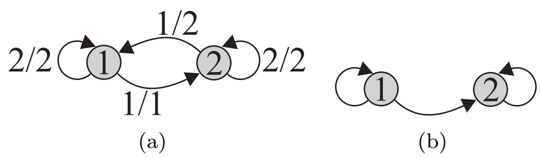

Figure 5. Automata graphs for Example 3

4.2 Feedback in Nondeterministic Automata

Analogously to the deterministic case the nondeterministic automaton (7) with feedback is an autonomous system $\tilde{\mathcal{N}} = (\mathcal{N}_z, \tilde{L}_n, z(0))$ with the relation

$$\tilde{L}_n(z', z) = \bigvee_{w=1}^{R} L_n(z', w, z, w), \qquad (20)$$

assuring $v = w$ for all w.

Definition: *The composition of a nondeterministic automaton with a feedback connection is said to be* well-formed *if for its output function the relation*

$$\bigvee_{w=1}^{R} H_n(w, z, w) = 1,\ \forall z \in \mathcal{N}_z \tag{21}$$

holds.

Theorem 3: *A system consisting of a nondeterministic automaton with a feedback connection is a nondeterministic automaton iff the feedback composition is well-formed.*

Proof:

$$\bigvee_{w=1}^{R} H_n(w, z, w) = 1 \Leftrightarrow \bigvee_{w=1}^{R} \bigvee_{z'=1}^{N} L_n(z', w, z, w) = 1$$

$$\Leftrightarrow \bigvee_{z'=1}^{N} \tilde{L}_n(z', z) = 1,$$

satisfying equation (11). ■

Corollary: If $\sum_{w=1}^{R} H_n(w, z, w) = 0$, the feedback connection leads to a nondeterministic automaton which is not live. □

4.3 Feedback in Stochastic Automata

The feedback of a stochastic automaton (12) results in an autonomous system $\tilde{\mathcal{S}} = (\mathcal{N}_z, \tilde{L}_s, z(0))$ (Schröder, 2003; Plateau and Atif, 1991) with the relation

$$\tilde{L}_s(z'|z) = \sum_{w=1}^{R} L_s(z', w|z, w). \tag{22}$$

Definition: *The composition of a stochastic automaton with a feedback connection is said to be* well-formed *if for its output function*

$$\sum_{w=1}^{R} H_s(w|z, w) = 1,\ \forall z \in \mathcal{N}_z \tag{23}$$

holds.

Theorem 4: *A system consisting of a stochastic automaton with a feedback connection is a stochastic automaton iff the feedback composition is well-formed.*

Proof:

$$\sum_{w=1}^{R} H_s(w|z, w) = 1 \Leftrightarrow \sum_{w=1}^{R} \sum_{z'=1}^{N} L_s(z', w|z, w) = 1$$

$$\Leftrightarrow \sum_{z'=1}^{N} \tilde{L}_n(z'|z) = 1,$$

satisfying equation (16). ■

Interestingly, if equation (16) is not satisfied the resulting process (22) is not a stochastic process any more. As opposed to the deterministic or nondeterministic case, the class of automata which fulfills the condition of the above theorem is expected to be very small compared to the class of stochastic automata in general. Also, a well-formed loop with a nondeterministic automaton embedded in a given stochastic automaton is a necessary but not a sufficient condition for the stochastic automaton to be well-formed.

Example 4: Consider a stochastic automaton $\mathcal{S}_4$ with $\mathcal{N}_z = \{1, 2\}$ and $\mathcal{N}_v = \mathcal{N}_w = \{1, 2\}$. Its dynamics is given in Fig. 6(a) as an automaton graph. The arcs are labelled with the respective input/output pair and the probability for the transition $v/w/P$. The introduction of a feedback connection results in the system shown in Fig. 6(b). As

$$\sum_{w=1}^{2} H(w|z, w) = 0.85 \neq 1 \quad \forall z \in \{1, 2\}$$

holds, the system is not well-formed. Therefore the resulting system (Fig. 6(b)) is not a stochastic process. Clearly, the sum of the probabilities of the transitions leaving a state should be 1. □

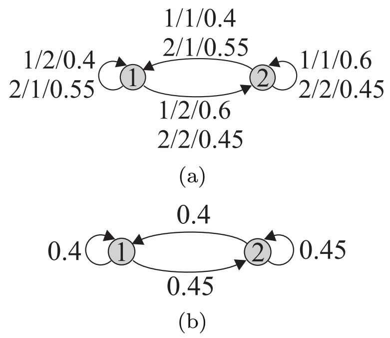

Figure 6. Automata graphs for Example 4

4.4 Feedback in Automata Networks

The above definitions can easily be extended to cover automata with multiple inputs and outputs and automata networks. This will be done in this section for stochastic automata only, bearing in mind that the definitions for the deterministic and nondeterministic case can be derived easily therefrom.

Definition: *The composition of a feedback loop containing γ stochastic automata is said to be* well-formed *if*

$$\sum_{w_1=1}^{R_1} \cdots \sum_{w_\rho=1}^{R_\rho} \sum_{s_1=1}^{K_1} \cdots \sum_{s_\kappa=1}^{K_\kappa} \prod_{i=1}^{\gamma} H_{si} = 1 \tag{24}$$

holds for all state and input values of the network.

This means that the output relations H_{si} of all automata i in the feedback loop are evaluated for all outputs w and coupling signals s that occur in the loop. The relations are then multiplied (element by element) and the result is summed up for all possible signal values.

Theorem 5: *A system consisting of a feedback loop containing stochastic automata is a stochastic process iff the loop is well-formed. (without proof due to place limitations)*

As in general the stochastic automata of a network are not stochastically independent the above definition cannot be reformulated as a criterion of the form

$$\sum H_1 \cdot \sum H_2 \cdot \dots \cdot \sum H_\gamma,$$

which would make it possible to test the automata separately. Hence, for the network to be well-formed it is neither necessary nor sufficient that the single automata of the network are well-formed.

Example 5: Consider a stochastic automata network with two automata $\mathcal{S}_1$ and $\mathcal{S}_2$ as depicted in Figure 7 (cf. (Schröder, 2003)). To test the conditions of Theorem 5 the sum (24) has to be determined. For the state $z_1 = 1$ of automaton S_1 and state $z_2 = 1$ of automaton $\mathcal{S}_2$ this results to

$$\begin{aligned}
&H_1(s=1|z_1=1,w=1)H_2(w=1|z_2=1,s=1)+\\
&H_1(s=1|z_1=1,w=2)H_2(w=2|z_2=1,s=1)+\\
&H_1(s=2|z_1=1,w=1)H_2(w=1|z_2=1,s=2)+\\
&H_1(s=2|z_1=1,w=2)H_2(w=2|z_2=1,s=2)=\\
&\quad 1\cdot 0.5+0.5\cdot 0.5+0\cdot 0.5+0.5\cdot 0.5=1.
\end{aligned}$$

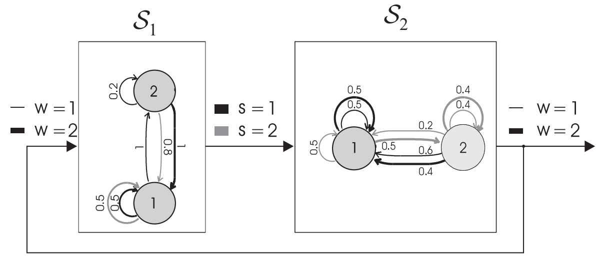

Figure 7. Stochastic automata network for Example 5

As the sum for the remaining state combinations $(z_1 = 1, z_2 = 2)$, $(z_1 = 2, z_2 = 1)$, and $(z_1 = 2, z_2 = 2)$ is also 1, the theorem is fulfilled and the composition is well-formed. Hence the automaton network describes a stochastic process. The same result could have been obtained by applying Theorem 1, because automaton $\mathcal{S}_2$ has no direct feed-through meaning its output $w(k)$ does not depend on its input $s(k)$. $\square$

5. CONCLUSION

The feedback problem has been posed for discrete-event systems. Necessary and sufficient conditions have been found for determining if the composition of a direct feedback loop is well-formed. As these conditions can be implemented easily they are an enhancement to the "brute-force" testing methods that have been use so far. However, because the criteria are rather restrictive, it is to be expected that in general a network of synchronised stochastic automata is not well-formed.

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