ON A CHARACTERISTIC VECTOR FIELD FOR SYSTEMS REDUCIBLE TO ORDER TWO

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Abstract: The paper deals with nonlinear control systems with a single input that can be reduced to second order systems. An intrinsic definition of a "characteristic prolongation" is given. Its role in several control theoretic problems, such as controllability, feedback linearization (flatness characterization), group classification, and optimal control is discussed. A batch chemical reactor example is considered.

Keywords: Controllability, differential geometry, feedback linearization, nonlinear systems, optimal control, transformations

1. INTRODUCTION

Preliminary analysis of a given control system requires answering some basic questions on the system structure, among them:

- Is the system controllable?
- By which transformations can the system be linearized?
- What are the group properties of the system — or, which symmetries does it admit?

A typical problem is also an optimal control problem with the question

- Is it possible to construct an optimal control law?

Obviously, the answers to any of the above problems depends on some intrinsic properties of the system and for each problem there are algorithms for getting the answers. However, what is less obvious is that for some classes of control systems the answers to all of the above questions can be given using a single vector field which is defined as the "characteristic prolongation" below.

The theoretical results are illustrated with an example of a chemical reactor operated in the batch mode. Two parallel reactions are assumed to take place independently in the reactor: μ A $\rightarrow$ C and ν B $\rightarrow$ D, with positive integers μ and ν. The temperature of the cooling jacket is used as the control input. The system may, thus, be described by the following single-input model with three states. (An additional state equation may easily be added, if the dynamics of the cooling jacket must be taken into account.)

$$\begin{aligned}\dot{c}_A &= -k_A(T)c_A^\mu \\ \dot{c}_B &= -k_B(T)c_B^\nu \\ \dot{T} &= h_A k_A(T)c_A^\mu + h_B k_B(T)c_B^\nu + \gamma(T - T_c)\end{aligned}$$

where C_A and C_B denote the concentrations of reactants A and B, rescpectively, T the temperature in the reactor, and T_c the temperature in the cooling jacket. The function k_A is of the form $k_A = k_{A0}\exp(-E_A/RT)$. Similarly, $k_B = k_{B0}\exp(-E_B/RT)$, and $k_{A0}, E_A, k_{B0}, E_B, R, h_A, h_B, \gamma$ are real positive constant parameters. It will be shown below how this system can be reduced to a system with two states of the form

$$\dot{x}^1 = u^\alpha =: F(u)$$
$$\dot{x}^2 = u$$

and then the construction of the characteristic prolongation and the transformation to the Brunovský form are illustrated on this model.

2. DEFINITION OF CHARACTERISTIC PROLONGATION FOR SECOND ORDER CONTROL SYSTEMS

Consider the class of second order control systems [1] with a single input u

$$\dot{x}^1 = F(t, x^1, x^2, u) \tag{1}$$
$$\dot{x}^2 = u, \tag{2}$$

where F is an arbitrary function, t is time, and u is the input, locally on an appropriate subset of $\mathbb{R}^4$. From a differential equations theory point of view, system (1) is under-determined. Otherwise stated it is an open loop system.

With system (1) is associated the Pfaffian system $I = \{\omega^1, \omega^2\}$, where

$$\omega^1 = dx^1 - Fdt, \quad \omega^2 = dx^2 - udt \tag{3}$$

on a subset of $\mathbb{R}^4$. With I is associated the so-called derived flag (Bryant *et al.*, 1991). The first derived system $I^{(1)}$ includes the single one-form ω:

$$I^{(1)} = \{\omega = dx^1 + (uF_u - F)dt - F_u dx^2\}. \tag{4}$$

With this the characteristic prolongation can be defined.

Definition 1. The *characteristic prolongation* associated with I, i.e., with system (1), is the vector field X defined by

$$X \rfloor \Omega = \omega \wedge d\omega,$$

with Ω the standard volume form, ω the generator of the first derived flag $I^{(1)}$, $\rfloor$ the inner product, and $\wedge$ the wedge product.

Obviously, this definition is intrinsic. It is connected with the definition of "rigidity" which was introduced in (Bryant and Hsu, 1993) and the notion of "characteristic vector field" introduced in (Jakubczyk and Zhitomirskii, 2003).

In Cartesian coordinates one gets the formula

$$X = F_{uu}X_0 + (F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2})U \tag{5}$$

where $X_0 = \partial_t + F\partial_{x^1} + u\partial_{x^2}$ is the vector field associated with the initial system (1) and $U = \partial_u$. Thus, X can be interpreted as a vector field, associated with closed loop system which consist of the open loop system (1) and the additional differential equation

$$\dot{u} = F_{uu}^{-1}\Big(F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2}\Big), \tag{6}$$

with $F_{uu} \neq 0$. It will be shown below, that in several control problems (controllability, flatness characterization, group classification, and optimal control) one encounters the partial differential equation

$$XS = F_{uu}(S_t + FS_{x^1} + uS_{x^2}) + (F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2})S_u = 0. \tag{7}$$

Because of this role the vector field X in (5) will be called the *characteristic prolongation* of the system vector field X_0.

3. CONTROLLABILITY CONDITIONS AS THE SINGULAR SOLUTIONS OF (7).

Controllability (strong accessibility) is known to be directly related to the existence of invariants (first integrals) (Chow, 1940; Rashevsky, 1938; Pavlovsky and Yakovenko, 1982; Schipanov, 1939; Fliess *et al.*, 1997). Here, the attention is put on the connection of these concepts with equation (7), or otherwise stated with the characteristic prolongation. System (1) is called (locally) uncontrollable, if (locally) all its solutions (trajectories) lie in an invariant manifold of the form $\phi\left(t, x^1, x^2\right) = C$.

One may now ask for which $F\left(t, x^1, x^2, u\right)$ in (1) there exist such invariants. To answer this question interpret the invariant $\phi\left(t, x^1, x^2\right) = C$ as a nontrivial solution of the system of partial differential equations

$$X_0\phi = 0, \qquad U\phi = 0. \tag{8}$$

In order to look for the functionally independent solutions of system (8), one must calculate the involutive closure (Nijmeijer and van der Schaft, 1990) for a distribution of vector fields given as (8). One has the Lie brackets

$$X_1 = [U, X_0], \quad X_2 = [U, X_1], \quad X_3 = [X_0, X_1]. \tag{9}$$

System (1) admits a first integral if for the above vector fields the two conditions $U \wedge X_0 \wedge X_1 \wedge$

[1] Notice that this class covers a rather wide set of systems. For instance, all single-input feedback linearizable systems can be reduced to this form (and even further) by feedback and control variable elimination. The difficult problems occur if F is not affine in u, i.e., $F_{uu} \neq 0$.

$X_2 = 0$ and $U \wedge X_0 \wedge X_1 \wedge X_3 = 0$ are fulfilled simultaneously. Therefore, system (1) is uncontrollable if and only if the function F satisfies the following two conditions:

$$F_{uu} = 0 \quad (10)$$

$$F_{ut} + FF_{ux^1} + uF_{ux^2} - F_u F_{x^1} - F_{x^2} = 0. \quad (11)$$

This system coincides with the singularity conditions of the partial differential equation (7). In particular, one direct implication of these results is the following.

Corollary 2. System (1) is uncontrollable if and only if $F = \alpha\left(t, x^1, x^2\right) u + \beta\left(t, x^1, x^2\right)$ and the coefficients α and β satisfy

$$\alpha\beta_{x^1} - \beta\alpha_{x^1} - \alpha_t + \beta_{x^2} = 0.$$

Using differential forms one may get the same as the integrability condition ($d\omega \wedge \omega = 0$) for the form ω defined in (4).

4. FEEDBACK LINEARIZATION AND ITS CONNECTION WITH EQUATION (7)

In this section the problem of feedback linearization of system (1) is considered, i.e. the existence of transformations

$$\hat{t} = \tau(t, x, u), \quad \hat{x} = \phi(t, x, u), \quad \hat{u} = \psi(t, x, u) \quad (12)$$

which convert (1) into Brunovský normal form

$$\frac{d\hat{x}^1}{d\hat{t}} = \hat{x}^2, \quad \frac{d\hat{x}^2}{d\hat{t}} = \hat{u}. \quad (13)$$

With system (13) one may associate the Pfaffian system in Goursat canonical form (Bryant *et al.*, 1991; Tilbury, 1994)

$$d\hat{x}^1 - \hat{x}^2 d\hat{t} = 0, \quad d\hat{x}^2 - \hat{u} d\hat{t} = 0. \quad (14)$$

Thus, the transformation of system (1) to system (13) can be interpreted as the transformation of the differential forms (3) to the Goursat canonical form (14) using the change of variables $(t, x^1, x^2, u) \to (\hat{t}, \hat{x}^1, \hat{x}^2, \hat{u})$. In order to calculate the new coordinates one can follow, for example, (Gardner and Shadwick, 1992; Tilbury, 1994). In particular, for the "new time" one can use the condition

$$d\omega \wedge \omega \wedge d\hat{t} = 0. \quad (15)$$

Denoting

$$d\hat{t} = \frac{\partial \hat{t}}{\partial t} dt + \frac{\partial \hat{t}}{\partial x^1} dx^1 + \frac{\partial \hat{t}}{\partial x^2} dx^2 + \frac{\partial \hat{t}}{\partial u} du$$

and substituting this differential relation into (15), one gets the partial differential equation

$$F_{uu}(\hat{t}_t + F\hat{t}_{x^1} + u\hat{t}_{x^2}) + (F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2})\hat{t}_u = 0. \quad (16)$$

This is the same as equation (7)! It is clear that for affine systems ($F_{uu} = 0$) there is no need for tranforming time, because $\partial\hat{t}/\partial u = 0$ and $\hat{t} = \tau(t, x^1, x^2)$ is an arbitrary function. Otherwise, for non-affine systems, the "new time" $\hat{t}$ must depend on u, because $\partial\hat{t}/\partial u = 0$ and $X_0\hat{t} = 0$ cannot be fulfilled simultaneously for controlled systems (see Section 3).

5. GROUP CLASSIFICATION FOR SYSTEM (1) AND ITS CONNECTION WITH EQUATION (7).

As a whole class of systems of the form (1) is considered (with arbitrary function F) there naturally occurs a problem of group classification (Ovsiannikov, 1982). Start the analysis deriving the so-called *determining equations*. In order to determine the coefficients $\tau\left(t, x^1, x^2, u\right)$, $\xi^i\left(t, x^1, x^2, u\right)$, and $\varphi\left(t, x^1, x^2, u\right)$ of the infinitesimal symmetry operator

$$Y = \tau\frac{\partial}{\partial t} + \xi^1\frac{\partial}{\partial x^1} + \xi^2\frac{\partial}{\partial x^2} + \varphi\frac{\partial}{\partial u}$$

one needs to solve the system

$$Yf^i - X_0\xi^i + f^i X_0\tau = 0 \quad (17)$$

$$U\xi^i - f^i U\tau = 0, \quad i = 1, 2 \quad (18)$$

where $f^1 = F$, $f^2 = u$ (see also (Lehenkyi, 1996; Lehenkyi and Rudolph, 2001)). Introduce the generating function of symmetries in the form

$$\sigma = Y \rfloor \omega = \xi^1 - F_u\xi^2 + (uF_u - F)\tau, \quad (19)$$

then reduce system (17) to the single equation

$$F_{uu}(\sigma_t + F\sigma_{x^1} + u\sigma_{x^2}) + (F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2})\sigma_u = F_{uu}F_{x^1}\sigma. \quad (20)$$

If one can solve equation (20) for a particular specialization of the function F then all coefficients can be calculated via

$$\begin{aligned}
\tau &= U\left(F_{uu}^{-1}\sigma_u\right) \\
\xi^1 &= \sigma + FU\left(F_{uu}^{-1}\sigma_u\right) - F_u F_{uu}^{-1}\sigma_u \\
\xi^2 &= uU\left(F_{uu}^{-1}\sigma_u\right) - F_{uu}^{-1}\sigma_u \\
\varphi &= -X_0\left(F_{uu}^{-1}\sigma_u\right)
\end{aligned}$$

where $U = \partial_u$. Equation (20) is a quasi-linear inhomogeneous partial differential equation. Its homogenous part just coincides with equation (7)! The conditions (10) for uncontrollability play the role of the classification conditions for equation (20), and if they are met the symmetry algebra is wider (see also (Lehenkyi and Rudolph, 2001) for further details).

6. OPTIMAL SYNTHESIS PROBLEM AND ITS CONNECTION WITH EQUATION (7)

Now consider an optimal synthesis problem for system (1).

Problem 3. Find an optimal control law $u(t, x^1, x^2)$ which minimizes the criterion

$$I = \min_{u \in U^*} t_f$$

and drives system (1) from $x^1(0) = x^1_0, x^2(0) = x^2_0$ to $x^1(t_f) = x^1_f, x^2(t_f) = x^2_f$.

For Bellman's function $S(t, x^1, x^2)$ one gets the necessary condition (Bellman, 1957)

$$\frac{\partial S}{\partial t} + \max_{u \in U^*} \left(F(t, x^1, x^2, u) \frac{\partial S}{\partial x^1} + u \frac{\partial S}{\partial x^2} \right) = 0. \tag{21}$$

Let the optimal control u_{opt} lie inside the set U^* of admissible controls, i.e., $u_{\text{opt}} \in \text{Int } U^*$ is fulfilled. Then, from (21)

$$S_t + F S_{x^1} + u S_{x^2} = 0 \tag{22}$$

$$F_u S_{x^1} + S_{x^2} = 0. \tag{23}$$

Being interested in the synthesis problem (without preliminary determination of the function S), one may eliminate S from (22), (23). The main idea of such an approach (Lehenkyi, 1990*b*) is to interprete S as the unique solution of the equations [2]

$$X_0 S = 0, \qquad X_1 S = 0, \tag{24}$$

where $X_0 = \partial_t + F\partial_{x^1} + u\partial_{x^2}, X_1 = F_u \partial_{x^1} + \partial_{x^2}$ are vector fields parameterized by the function $u = u(t, x^1, x^2)$.

In that case one can also apply the procedure of determining the involutive closure of the distribution spanned by X_1 and X_2 described in Section 3 and can get the equation

$$F_{uu}(u_t + F u_{x^1} + u u_{x^2}) + (F_u F_{x^1} + F_{x^2} - F_{tu} - F F_{ux^1} - u F_{ux^2}) = 0, \tag{25}$$

which again coincides with equation (7)! Therefore, the optimal synthesis problem is reduced to the Cauchy problem for the equation (25) with boundary conditions $x^1(t_f) = x^1_f, x^2(t_f) = x^2_f$. The generalization of this approach to n-th order control systems with vectorial inputs can be found in (Lehenkyi, 1996; Lehenkyi, 1990*a*; Lehenkyi, 1992; Lehenkyi, 1991).

[2] Particular methods for determining such an equation for u were developed by R. Bellman himself (see (Bellman *et al.*, 1958) e.g.) and some other authors (see, for example, (Bryson and Ho, 1975)). The general case, probably, at first was considered in (Lehenkyi, 1990*b*). Similar ideas can be found in mathematical physics (A.F. Sidorov and Yanenko, 1984), and for the non-holonomic mechanical problems this approach has been described in (Vershik and Gershkovich, 1986).

If the condition $F_{uu} = 0$ holds true, then one cannot use equation (25) for the optimal synthesis. In this case the equation $F_u F_{x^1} + F_{x^2} - F_{tu} - F F_{ux^1} - u F_{ux^2} = 0$ defines a singular manifold (see, for instance, (Bryson and Ho, 1975)). If the control u of system (1) is unbounded the system trajectories lie in this surface, otherwise it plays the role of a switching surface.

Remark 4. It is worth underlining the difference between equation (25) and the classical Euler-Lagrange equation for the extremals of the integral

$$I = \int L(t, x, \dot{x}) dt.$$

Redefining the variables in (25) as $L = F, x = x^2, \dot{x} = u$ and using $d/dt = \partial_t + \dot{x}\partial_x + \ddot{x}\partial_{\dot{x}} + L\partial_I$, one can rewrite (25) as

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial I}\frac{\partial L}{\partial \dot{x}} = 0. \tag{26}$$

Thus, the classical Euler-Lagrange equation is a particular case of (25), under the condition $\partial L/\partial I = 0$.

7. A CHEMICAL REACTOR EXAMPLE

The transformation of the system

$$\begin{aligned} \dot{c}_A &= -k_A(T) c_A^\mu \\ \dot{c}_B &= -k_B(T) c_B^\nu \\ \dot{T} &= h_A k_A(T) c_A^\mu + h_B k_B(T) c_B^\nu + \gamma(T - T_c) \end{aligned}$$

into the second order system

$$\begin{aligned} \dot{x}^1 &= u^\alpha =: F(u) \\ \dot{x}^2 &= u \end{aligned}$$

is done as follows. First, one eliminates the variable T_c by considering T as the (virtual) control. The calculations are further simplified by using the transformation $u = k_B(T)$. With $k_A = k_{A0}\exp(-E_A/RT)$ and $k_B = k_{B0}\exp(-E_B/RT)$, this leads to $k_A(T) = \beta u^\alpha$, with $\beta = k_{A0}/k_{B0}^\alpha$ and $\alpha = E_A/E_B > 0$. Furthermore, the new state variables are introduced as follows. If $\mu, \nu \neq 1$

$$\begin{aligned} x^1 &= c_A^{1-\mu}/(\beta(\mu - 1)) \\ x^2 &= c_B^{1-\nu}/(\nu - 1) \end{aligned}$$

and $x^1 = -(\ln c_A)/\beta$ if $\mu = 1$, resp. $x^2 = -\ln c_B$ if $\nu = 1$.

First, one immediately observes that the system is uncontrollable in case $\alpha = 1$, which means $E_A = E_B$.

The additional differential equation (6) reads

$$\dot{u} = 0,$$

and the characteristic prolongation is

$$X = \frac{\partial}{\partial t} + F(u)\frac{\partial}{\partial x^1} + u\frac{\partial}{\partial x^2}.$$

The following, thus, are first integrals C_1, C_2, C_3 of the p.d.e. $X(q) = 0$:

$$C_1 = u,$$
$$C_2 = x^2 - ut,$$
$$C_3 = x^1 - F(u)t.$$

From these one deduces the new coordinates

$$\hat{t} = x^2 - ut,$$
$$y := \hat{x}^1 = x^1 - F(u)t = x^1 - u^\alpha t$$

and gets

$$d\hat{t} = dx^2 - udt - tdu = -tdu$$

and, using the latter,

$$d\hat{x}^1 = dy = dx^1 - F(u)dt - F_u(u)tdu = F_u(u)d\hat{t}$$

Thus,

$$\hat{x}^2 := y' = dy/d\hat{t} = F_u(u) = \alpha u^{\alpha-1}$$
$$\hat{u} := y'' = dy'/d\hat{t} = F_{uu}(u)du/d\hat{t}$$
$$= -F_{uu}(u)/t = \alpha(1-\alpha)u^{\alpha-2}/t$$

if $\alpha \neq 1$. If $\alpha = 1$, one observes that $\hat{x}^2 = \alpha$ is an invariant of the system.

With the definition of $y = \hat{x}^1, \hat{x}^2$, and $\hat{u}$ this introduces the Brunovský form with the new independent variable (“new time”) $\hat{t}$.

In case $\alpha \neq 1$ one may invert the above formulae in order to express x^1, x^2, t, and u in terms of $\hat{t}, y, y'$, and y'', which yields:

$$u = \left(\frac{y'}{\alpha}\right)^{\frac{1}{\alpha-1}}$$
$$t = \frac{-\alpha(\alpha-1)}{y''}\left(\frac{y'}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}}$$
$$x^1 = y - \frac{\alpha(\alpha-1)}{y''}\left(\frac{y'}{\alpha}\right)^{\frac{\alpha(\alpha-2)}{(\alpha-1)^2}}$$
$$x^2 = \hat{t} - \frac{\alpha(\alpha-1)}{y''}\left(\frac{y'}{\alpha}\right)^{\frac{\alpha-2}{(\alpha-1)^2}}.$$

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