A SATURATION BASED INTERPOLATION METHOD FOR FUZZY SYSTEMS

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Abstract: This paper presents an alternative inference-defuzzification algorithm for Takagi-Sugeno fuzzy systems that preserves local-model interpretation and convexity properties. The linear model in the rule consequent is saturated outside the core set of the antecedent membership functions. This allows the interpretation of the consequents of fuzzy rules as a local linearization of the model restricted to the subset where it is valid. The setting has readability advantages over Takagi-Sugeno frameworks, and it is simpler that other interpolation proposals. Some examples illustrate the approach. *Copyright 2005 IFAC*

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1. INTRODUCTION

One of the main successes of fuzzy models is the capacity to express empirical knowledge in the form of semantic rules. Every rule can be seen as a specific and specialised subset of the global knowledge. Then, if a fuzzy system is used for modelling, a rule represents a submodel valid in some situations that are included as the antecedents. This conduces to the local model approach: every rule represents a local model that is valid in a subset defined in the antecedents.

It is particularly important to select suitable criteria to allow us to obtain the referred local model partition, in particular, selecting an accurate number of local models, and to define them in the most important points in order to minimise the total complexity (number of local models) of the overall system description. There are a series of tools able to help in those modelling tasks, such as the ones discussed in (Díez, 2003), including widely-available software (Nelles *et al.*, 2000).

In Takagi-Sugeno (T-S) fuzzy models, the submodel is represented by a parametric linear function of the input variables. The local models are aggregated (interpolated) by means of a weighted sum of the submodel outputs to obtain the global model (Takagi and Sugeno, 1985). Ideally, those submodels should resemble the local Jacobian linearisation of the system to be modelled.

But, even though it has low computing requirements, the usual TS formula is a non convex interpolation method and produces some unsuitable behaviour in the interpolation area (Babuska *et al.*, 1996; Ario and Sala, 2004). The reason for this behaviour is, as widely known, that the TS interpolator gives solutions out of the convex hull of the local models in the overlapping regions: Suitable accuracy in intermediate points in interpolation of Jacobian-linearisation based models requires a different "convex" interpolation technique.

The structure of the paper is as follows: first, the rule interpolation problem in T-S fuzzy models is analysed from the local modelling point of view. Then, different approaches are reviewed and finally, it is presented an alternative interpolating expression that preserves convexity characteristics of the underlying function.

2. INTERPOLATION IN T-S FUZZY MODELS

A T-S fuzzy model of a system may be stated as a set of rules in the form:

$$\text{IF } x \text{ is } A_l, \text{ THEN } y = f_l(x); l = 1 \dots m \quad (1)$$

where m is the number of rules (number of local models), A_l is a fuzzy set defined on the domain of $x \in \Re^n$, with membership function $\mu_l(x)$, and $f_l(x)$ represents the local model in the region defined in the antecedent (fuzzy set A_l). In T-S it is used a local affine model for $f_l(x)$.

$$f_l(x) = a_0^l + \sum_{i=1}^{m} a_i^l x_i \quad (2)$$

The usual way of carrying out the aggregation gives rise to the so-called Takagi-Sugeno (Takagi and Sugeno, 1985; Wang, 1994)inference formula:

$$y(x) = \frac{\sum_{l=1}^{m} \mu_l(x) f_l(x)}{\sum_{l=1}^{m} \mu_l(x)} \quad (3)$$

Usually, it is assumed that $\sum \mu_l = 1$ (add-1 fuzzy partition) and it will be assumed on the sequel:

$$y(x) = \sum_{l=1}^{m} \mu_l f_l \quad (4)$$

Now, let us focus on the analysis of the interpolation. Suppose two adjacent local models, with $l = 1, 2$, and for clarity we will use a SISO system ($n = 1$).

Using the local model approach, every local model should be valid inside the core ($\mu_l = 1$) of its associated fuzzy set, that is,

$$\|y(x) - f_l(x)\| < \epsilon; \forall x \in \text{core}(A_l) \quad (5)$$

Outside the core set, the submodel is not longer valid, and, usually, the error increases with the distance to the core set. This idea is reflected in the membership function shape, usually decreases monotonically towards zero outside the core set. So, for suitable readability, the two adjacent submodels, the associated membership functions do overlap and no overlap exists in its core set (add-1 hypothesis). The overlapping zone is:

$$B = \{x; 0 < \mu_1(x)\mu_2(x) < 1\} \quad (6)$$

Then, the interpolation is done in B, but if 4 is used, local models are extrapolated outside its validity region, and it may produce some

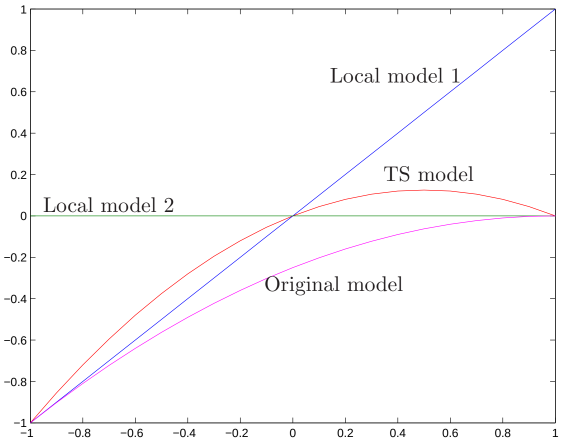

Fig. 1. TS interpolation defects

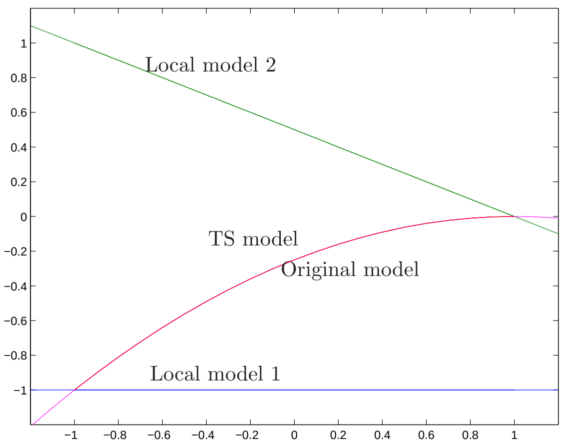

Fig. 2. TS interpretability drawbacks

unsuitable results as, for instance, those in Figure 1.

Conversely, if the model accurately fits a particular function, for instance by parameter identification procedures, then the obtained f_i may bear no relationship to the local Jacobian-linearised models (figure 2 depicts an example where the intermediate curve can be obtained by a suitable choice of membership function values to be applied to the "local" models 1 and 2). In fact, there is a sort of conditioning issue if full freedom for selection of μ is available in the modelling step, as then the variations of μ may be the ones that actually account for the fit of f. This idea is the one behind the bounding-polytopic interpretation of fuzzy systems (Tanaka and Wang, 2001; Taniguchi *et al.*, 2001). This readability *vs.* accuracy trade-off and related issues are discussed in (Abonyi *et al.*, 2000; Díez *et al.*, 2002).

3. OTHER INTERPOLATION METHODS

Due to this non-convex behaviour, different alternatives has been proposed to improve the interpolation in the T-S framework. The properties that should be met are (Babuska, 1998):

- The approximation error in the interpolation set (B) must be better.
- The local gradients in B must be bounded by the gradient in the adjacent local models.
- The surface must be sufficiently smooth, with continuous derivatives until some pre-specified order.

Some of the interpolation proposals are based in methods of non-linear regression (Seber and Wild, 1989):

Transition functions : The idea is to parametrize the transition between two adjacent models by an expression with the sign function. Then, the sign function is substituted by a continuous function, for instance a sigmoid, hyperbolic or tanh function,

Max-min smoothing : Connects two or more adjacent hyperplanes by a smooth convex or concave surface defined by a polynomial (Babuska, 1998),

Splines : polynomials of order n are used to define the transition between the local models. Continuity of order $n-1$ is achieved.

Other methods are based on ad-hoc developments. For instance, in (Ario and Sala, 2004), the first step is to define a piecewise-linear limit function F_{pw}, via the intersection of the local submodels. Then, a softened version is calculated by means of the calculation of a weighted mean of F_{pw} in a ball centered in x. This can be seen as a rounded surface in edges and vertexes.

All these methods suffer from a higher computational complexity than 4, extra parameters must be determined and, in the case of max-min smoothing, different formulas must be used for convex or concave surfaces. Other setups need a suitable meshing of the input domain that can be cumbersome to carry out with high-dimensional data.

4. SATURATED LOCAL MODELS

An alternative and new method for interpolating local models can be based on the idea that they are only valid in its core set, and no extrapolation must be done, but instead interpolation with some saturated values.

This can be easily done for functions in one variable. In this case, the local model $f_l = a_0^l + a_1^l x$ is saturated outside the core set $\mathrm{core}(A_l) = [x_{\min}^l, x_{\max}^l]$ at the extreme values:

$$f_l(x) = a_0^l + a_1^l \max(\min(x, x_{\max}), x_{\min}) \qquad (7)$$

Then the interpolation is performed among constants. The membership function shapes in B determine the transition between the extreme values. For instance, for two adjacent local models with $x_{\max}^1 \leq x_{\min}^2$, the result is:

$$y(x) = \mu_1(x) f_1(x_{\max}^1) + \mu_2(x) f_2(x_{\min}^2) \qquad (8)$$

For higher input dimension, similar concepts can be used, but it is slight more difficult: the core set extreme values form a $(n-1)$-dimensional boundary surface and there is not a unique solution for saturating the local model.

The more intuitive way is to select the point closest to x in this $(n-1)$-dimensional surface and calculate the local model at this point. There are two cases to be discussed below.

4.1 Cartesian partition

This is the case when the membership functions are separately defined for every input and then they are combined in rule antecedents. For every input, core intervals are obtained based on the input membership function. Then, the Cartesian product of these intervals results in the full n-dimensional core sets, with its sides parallel to the axes (boxes).

An extension for equation (7) can be written as:

$$f_l(x) = a_0^l + \sum_{i=1}^{n} a_i^l \max(\min(x_i, x_{i\,\max}^l), x_{i\,\min}^l) \qquad (9)$$

4.2 Arbitrary partition

This case usually occurs as a result of an identification process from experimental data (Babuska, 1998). N-dimensional membership functions are obtained with different shapes and orientations.

The same idea can be used, *i.e.*, choosing the nearest point to the core set boundary, but in this case, the computation of the referred boundary may be more demanding, especially if the set has not a regular shape (such as the one obtained from clustering methods). In the case of clustering-related results, approximations (using cluster covariance concepts, for instance) or calculation of the convex hull of a finite set of points above a particular value of membership can be used.

4.3 Properties

The proposal under study has some interesting properties:

- Interpolation is carried out with constant values, so the properties of interpolation only depends on the membership functions shape and mutual overlapping: the global model is

differentiable if membership functions are so; the local gradients are a weighted sum of the fuzzy set membership gradients.
- If trapezoidal functions are used, a piecewise linear model is obtained. In the case of high-dimensional input, a ruled surface is obtained.
- For triangular fuzzy sets, the fuzzy model is equivalent to a Mamdani type with singletons in the rule consequents.
- For Cartesian partitions the computational overhead of this method is similar to the classical T-S one.

In order to reduce the computational load in the arbitrary partition case, the core set outlines can be precalculated to improve the algorithm efficiency.

5. EXAMPLES

In this section, some examples on one and two-dimensional input spaces will be described, to show the possible advantages of the proposed approach.

5.1 Example 1: Single input.

Let us have a function:

$$f(x) = \frac{1}{0.1 \cdot x + 0.01} \qquad (10)$$

from which two local linearised models are obtained, centered at $x_1 = 0$ and $x_2 = 1$:

$$m_i = \frac{-0.1}{(0.1 \cdot x_i + 0.01)^2}$$
$$n_i = \frac{1}{0.1 \cdot x_i + 0.01} - \frac{-0.1}{(0.1 \cdot x_i + 0.01)^2} \cdot x_i$$
$$f_i(x) = m_i \cdot x + n_i$$
$$i \,:\, 1, 2$$

The core region for each model is:

$$x \in [a_i, b_i] \; i : 1, 2$$

Where $a_1 = -0.04$ $b_1 = 0.04$ and $a_2 = 0.5$ $b_2 = 4$ The values of the function and its derivative at the centroid points coincide with those from the local models, so these can be properly interpreted as the linearisations of the interpolant function. The two membership functions are defined as shown in figure 3.

Next step is saturating the output of the local models.

$$x_{sati}(x) = \max(\min(x, b_i), a_i) \; i : 1, 2$$
$$f_{sati}(x) = f_i(x_{sati}(x)) \; i : 1, 2$$

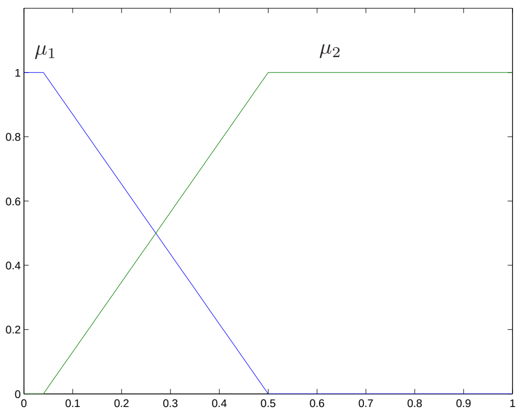

Fig. 3. The membership functions

The inference methodology is, then, the Takagi-Sugeno framework

$$F(x) = \mu_1(x) \cdot f_{sat1}(x) + \mu_2(x) \cdot f_{sat2}(x) \qquad (11)$$

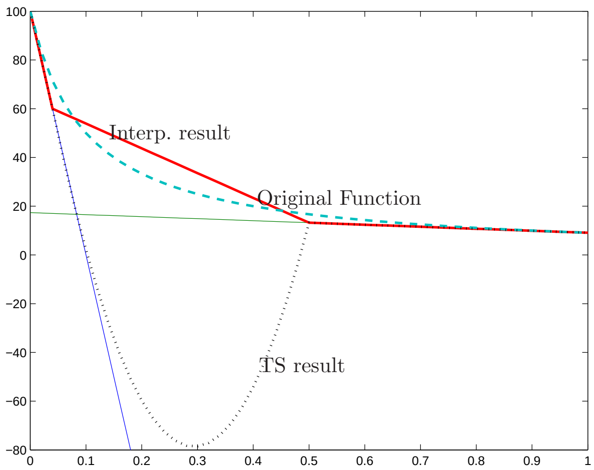

Fig. 4. Interpolation result

The result is depicted in Figure 4 jointly with the local models (f_1 and f_2) and the Takagi-Sugeno inference. In the points with a low reliability for all local models, the inference using the saturation methodology improves significantly over the standard T-S approach. If the membership functions were smoother, the resulting curve would also have been so.

5.2 Example 2: 2 inputs, 1 output.

The system to be modelled is described by:

$$f(x, y) = \frac{1}{0.1 \cdot x + 0.02 \cdot y + 0.01} \qquad (12)$$

Let us have four local models (linearisations), at points $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (0, 1)$ and $p_4 = (1, 1)$

$$m_i = \frac{-0.1}{(0.1 \cdot x_i + 0.02 \cdot y_i + 0.01)^2}$$
$$p_i = \frac{-0.02}{(0.1 \cdot x_i + 0.02 \cdot y_i + 0.01)^2}$$
$$n_i = \frac{1}{0.1 \cdot x_i + 0.02 \cdot y_i + 0.01} - m_i \cdot x_i - p_i \cdot y_i$$
$$f_i(x, y) = m_i \cdot x + p_i \cdot y + n_i$$
$$i : 1..4$$

The core intervals for each model and variable are:

$$x \in [a_i, b_i] \; i : 1, 2$$
$$y \in [c_j, d_j] \; j : 1, 2$$

Where $a_1 = -0.04$, $b_1 = 0.04$; $c_1 = -0.1$, $d_1 = 0.1$; $a_2 = 0.5$, $b_2 = 4$ and $c_2 = 0.4$, $d_2 = 2$.

The 2D fuzzy membership functions μ_i are plotted in figure 5.

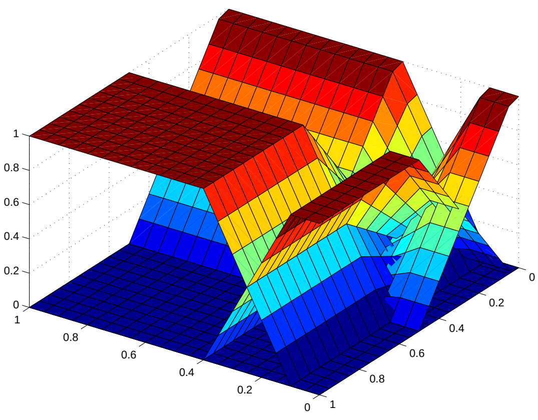

Fig. 5. membership functions

Now, the saturated functions are:

$$x_{sati}(x) = \max(\min(x, b_i), a_i) \; i : 1, 2$$
$$y_{sati}(y) = \max(\min(y, d_i), c_i) \; i : 1, 2$$

The proposed algorithm results in an interpolated surface plotted in figure 6, clearly approaching the intuitively expected shape of the original function, in figure 8, in a better way than that from the TS interpolation in figure 7.

6. CONCLUSIONS

In this paper, the local model approach has been used to obtain a new interpolation method for Takagi-Sugeno fuzzy systems. The proposal agrees with the natural interpretation of local models: they are only valid in the domain where they are defined and model extrapolations must be avoided because model imprecision.

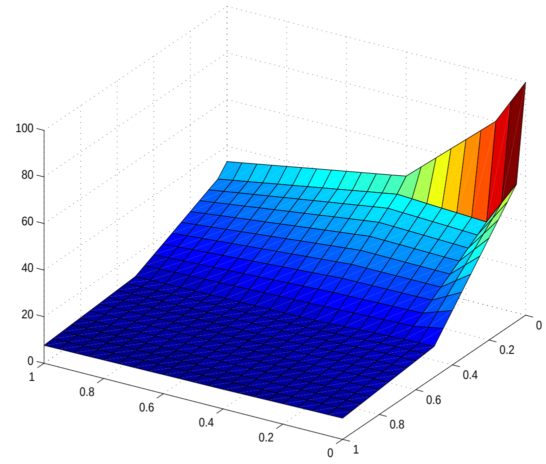

Fig. 6. Interpolation function

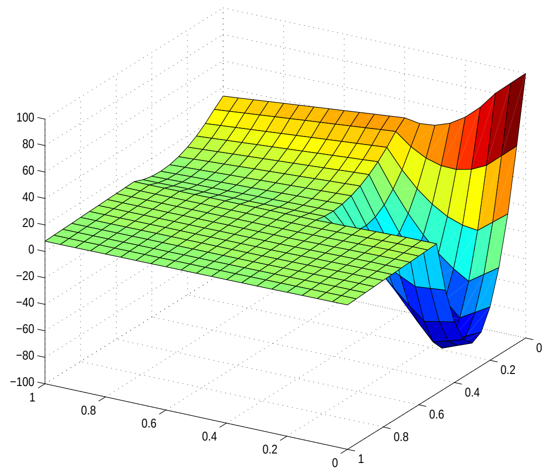

Fig. 7. TS interpolation

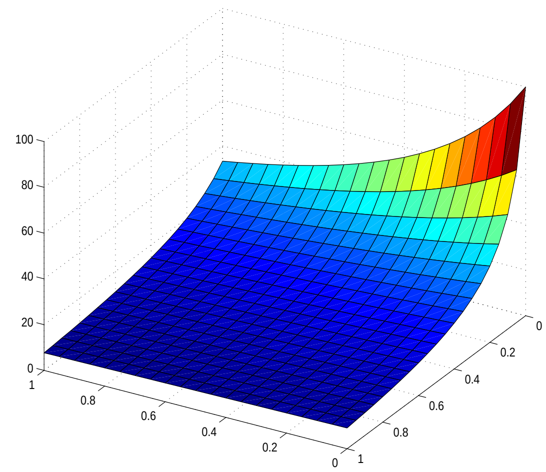

Fig. 8. Original function

This method can be used for models of any number of inputs and it is especially useful when Cartesian partition is employed. Method properties has been analyzed and some examples are drawn to show how it works.

Further work must be done to improve the efficiency for arbitrary fuzzy sets partitions.

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