ANALYSIS OF COORDINATION IN MULTI-AGENT SYSTEMS THROUGH PARTIAL DIFFERENCE EQUATIONS. PART I: THE LAPLACIAN CONTROL

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Abstract: In this first part of a two-parts paper we introduce the framework of Partial difference Equations (PdEs) over graphs for analyzing the behavior of multi-agent systems equipped with decentralized control schemes. We generalize the Vicsek's model (Vicsek *et al.*, 1995) by introducing errors in the agent dynamics and analyze agent alignment in leaderless and leader-follower models through the joint use of PdEs and automatic control tools. Moreover, we show that the resulting PdEs enjoy properties that are similar to those of well-known Partial Differential Equations (PDEs) like the heat equation, thus allowing to exploit physical-based reasoning for conjecturing properties of the collective dynamics.

Keywords: Networked autonomous vehicles, Partial difference Equations, Formation control, Cooperative control, Decentralized control.

1. INTRODUCTION

In the last few years, the problem of understanding when the individual actions of interacting agents give rise to a coordinated behavior has received a considerable attention in many fields. For instance, this issue appears in biology, statistical physics and computer graphics. For a thorough review of the literature in various field, we defer the interested reader to (Jadbabaie *et al.*, 2003) and (Tanner *et al.*, 2004).

In the control community, the interest in coordination phenomena has been recently promoted by the need of controlling groups of unmanned autonomous vehicles, like airplanes or robots (Ögren *et al.*, 2002). A fairly simplified setup, adopted for instance in (Jadbabaie *et al.*, 2003) and (Tanner *et al.*, 2003), is to consider a group of N mobile agents, each one described by a dynamical system capturing the evolution of its position and velocity. Different agents share information through a communication network: agents connected by a communication link are *neighbors* and position and velocity of each one is instantaneously available to the others for regulating their own trajectory. When agents communicate with a limited number of neighbors, one faces the problem of designing a *decentralized* control scheme (where each agent uses only the neighbors information) in order to orchestrate the *collective* behavior. Decentralization implies that the control action can be computed in a distributed fashion.

The main purpose of this paper is to propose a new modeling framework for the analysis of

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multi-agent systems. Our approach exploits the formalism of Partial difference Equations (PdEs) over graphs proposed by Bensoussan and Menaldi (2004) and summarized in Section 2. Conceptually, PdEs mimic PDEs (Partial Differential Equations) in spatial domains having a graph structure and, in (Bensoussan and Menaldi, 2004), the basic mathematical framework for studying static problems of elliptic type is provided. In order to account for the temporal dynamics of the agents, we generalize the models of (Bensoussan and Menaldi, 2004) to *continuous-time PdEs*. One major difference between PDEs and PdEs is that the latter can be recast into systems of Ordinary Differential Equations (ODEs). However, we argue that PdE models can be more expressive than their equivalent ODE form, for many reasons. First, many mathematical tools for analyzing PdEs are completely analogous to the ones developed for PDEs. Then, the PdEs formalism establishes a direct link between classic functional analysis and control theory that can be fruitfully exploited for studying systems linked by a communication network. Second, PdEs provide a mathematical description of the collective dynamics where spatial phenomena (due to the graph structure), and temporal evolution of the agent states, are kept separated and described through operators acting either on space or time. In our opinion, this vehicles a precise picture about the interaction between the communication network and the agent dynamics. Third, the PdEs framework leads to equations that may be reminiscent of PDEs arising in physics and this can be of great help for conjecturing sensible properties of the collective dynamics.

In the present paper we introduce the framework of PdEs (Section 2) and discuss their main properties. In Section 3 we exploit PdEs for analyzing alignment in leaderless multi-agent systems equipped with the "Laplacian control law", a control strategy inspired to the one proposed by Vicsek *et al.* (1995). Differently from other results available in literature, we consider also the effect of perturbations on the agent dynamics. Finally, in Section 4, the theory is extended to the case of leader-follower models. In the second part of the paper (Ferrari-Trecate *et al.*, 2005) we show how many of the basic concepts introduced here can be fruitfully exploited for analyzing the behavior of systems equipped with potential field based control laws that guarantee, beside alignment, cohesion of the agents.

2. FUNCTIONAL ANALYSIS AND PARTIAL DIFFERENCE EQUATIONS ON GRAPHS

We introduce basic notions of graph theory (Bollobás, 1998). Let G be an undirected graph defined by a nonempty set $\mathcal{N}$ of N *nodes* and a set $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ of *edges*. In our case, each node represents an agent and without loss of generality we assume that $\mathcal{N} = \{1, 2, \ldots, N\}$. Two nodes x and y are *neighbors* if $(x, y) \in \mathcal{E}$. This means that the agent x and y share the information about their position and velocity. We use the notation $x \sim y$ for neighboring nodes and assume that $x \sim x$ always holds. The neighboring relations are captured by the *adjacency matrix* $\mathcal{A}(G)$, i.e. a square matrix of order N having entries $\mathcal{A}_{x,y} = 1$ iff $x \neq y$ and $x \sim y$ and $\mathcal{A}_{x,y} = 0$ otherwise. The number of neighbors to each node is the node *valency* and the valency matrix $\mathcal{V}(G)$ of the graph is a diagonal matrix of order N with entry $\mathcal{V}_{x,x}$ equal to the valency of the node x. The *Laplacian* matrix of the graph is defined as $D(G) \doteq \mathcal{V}(G) - \mathcal{A}(G)$.

Two nodes x and y are connected by a path if there is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $x_{i-1} \sim x_i$. The graph G is *connected* when each pair of nodes $(x, y) \in G \times G$ is connected by a path.

We summarize the main concepts of functional analysis for vector functions $f : \mathcal{N} \mapsto \mathbb{R}^q$ defined over a graph[2] G by following closely the exposition of (Bensoussan and Menaldi, 2004) where scalar functions are considered. The *partial derivative* of f is defined as

$$\partial_y f(x) \doteq f(y) - f(x) \tag{1}$$

and enjoys the following elementary properties:

$$\partial_y f(x) = -\partial_x f(y) \tag{2a}$$
$$\partial_x f(x) = 0 \tag{2b}$$
$$\partial_y^2 f(x) = \partial_y f(y) - \partial_y f(x) = \partial_x f(y). \tag{2c}$$

The Laplacian of f is given by

$$\Delta f(x) \doteq -\sum_{y \sim x} \partial_y^2 f(x) = +\sum_{y \sim x} \partial_y f(x). \tag{3}$$

where the last identity follows from (2c) and (2a). The integral and the average of f are defined, respectively, as

$$\int_G f \doteq \sum_{x \in \mathcal{N}} f(x), \quad \langle f \rangle \doteq \frac{1}{N} \int_G f. \tag{4}$$

Let $L^2(G|\mathbb{R}^q)$ be the Hilbert space composed by all functions $f : \mathcal{N} \mapsto \mathbb{R}^q$ equipped with the scalar product and the norm

$$(f, g)_{L^2} = \int_G f^T g, \quad \|f\|_{L^2}^2 = \int_G \|f\|^2 \tag{5}$$

where $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^q$. Note that L^2 is isomorphic to $\mathbb{R}^{Nq}$. We introduce now the "Sobolev" space $H^1(G|\mathbb{R}^q)$ composed by all functions in $L^2(G|\mathbb{R}^q)$ with zero average[3]. We will use the shorthand notation L^2 and H^1 when

[2] Rigorously, $\mathrm{dom}\{f\} = \mathcal{N}$. However, as customary in graph theory, we use G instead of $\mathcal{N}$ or $\mathcal{E}$ when no confusion is possible.

[3] As shown in (Bensoussan and Menaldi, 2004), the space H^1 can be taught as the quotient space $L^2 \backslash \sim$ with respect the equivalence relation $f \sim g$ iff $f - g$ is constant. This construction is analogous to the one adopted in functional analysis for defining Sobolev spaces (see (Dautray and Lions, 1992)).

there is no ambiguity on the underlying domain and range of the functions. In (Bensoussan and Menaldi, 2004) it is shown that if G is connected, H^1 is an Hilbert space endowed with norm

$$\|f\|^2_{H^1} = \sum_{x\in\mathcal{N}}\sum_{y\sim x}\|\partial_y f(x)\|^2. \tag{6}$$

Note that $\|\cdot\|_{H^1}$ is only a semi-norm on L^2. In fact, if $f \in L^2$ is constant, then $\|f\|_{H^1} = 0$. It is also easy to prove (Bensoussan and Menaldi, 2004) that if G is connected and $\|f\|_{H^1} = 0$, then f is constant. Let $H^1_\perp(G|\mathbb{R}^q)$ denote the space orthogonal to $H^1(G|\mathbb{R}^q)$. Apparently, $H^1_\perp$ is the space of *constant* functions on G and $\dim(H^1_\perp) = q$. Moreover, the decomposition $L^2 = H^1 \oplus H^1_\perp$ is L^2- orthogonal, i.e., $\int_G f^T c = 0\,,\ \forall f \in H^1\,,\ c \in H^1_\perp$.

We conclude the section by summarizing the fundamental properties of the Laplacian operator introduced in (3). First, note that Δ is strongly related to the Laplacian matrix $D(G)$. In fact, we have that $[(\Delta f(1))^T,\dots,(\Delta f(N))^T]^T = -(D(G)\otimes I_q)[f(1)^T,\dots,f(N)^T]^T$, where $\otimes$ is the Kronecker product and I_q is the identity matrix of order q. Thus, $-(D(G)\otimes I_q)$ is the matrix representation of the linear operator Δ. The next theorem clarifies the eigenstructure of the Laplacian operator (Bollobás, 1998), (Bensoussan and Menaldi, 2004).

Theorem 1. Let G be a connected graph. Then,

(1) the operator $\Delta : H^1 \mapsto H^1$ has $(N-1)q$ strictly negative eigenvalues and the corresponding eigenfunctions form a basis for H^1;
(2) for $f \in L^2$, $\Delta f = 0$ if and only if $f \in H^1_\perp$.

Remark 1. Note that when Δ is defined on L^2, it has Nq eigenvalues. In particular, in view of the decomposition $L^2 = H^1 \oplus H^1_\perp$, $(N-1)q$ eigenvalues are those considered in point (1) of Theorem 1 and the remaining q eigenvaues are zeros (this follows directly from point (2) of Theorem 1). We stress that the eigenvalues of Δ (defined on L^2) are easy to compute since they coincide with those of the matrix $-(D(G)\otimes I_q)$.

Remark 2. It is worth to highlight the analogy between Theorem 1 and the corresponding results for the Laplace operator on Sobolev spaces. Indeed, let G be an open, bounded and regular subset of $\mathbb{R}^N$ and let $\underline{\mathrm{n}}$ be the outer unit normal at the boundary ∂G. Let $L^2(G)$ and $H^1(G)$ denote the standard Hilbert spaces as defined in (Dautray and Lions, 1992). Moreover, consider the subspace $V = \{f \in H^1(G) :\ \int_G f = 0,\ \Delta f \in L^2(G),\ \partial_{\underline{\mathrm{n}}} f = 0\}$. Then, the Laplace operator $\Delta : V \mapsto L^2(G)$, $\Delta f \doteq \sum_{i=1}^N \frac{\partial^2 f}{\partial x_i^2}$ has countably many, strictly negative eigenvalues and the corresponding eigenfunctions form a basis for the Hilbert space $\{f \in H^1(G), \int_G f = 0\}$.

We are now in a position to introduce *Partial difference Equations (PdEs)* on graphs. Let $z(x,t) : G\times\mathbb{R}^+ \mapsto \mathbb{R}^q$ be a function of two variables and consider the initial value problem

$$\dot z(x,t) = F(z(\cdot,t)) \tag{7a}$$
$$z(x,0) = \tilde z(x) \tag{7b}$$

where $F : L^2(G|\mathbb{R}^q) \mapsto L^2(G|\mathbb{R}^q)$ is a continuous operator. We call the equality (7a) a continuous-time PdE with initial conditions (7b) and refer to $z(x,t)$ as the *state* of the PdE. Note that, for example, one can have $F = \Delta$ thus motivating the term “PdE” used for (7).

As shown in (Ferrari-Trecate *et al.*, 2004), PdEs can be always recast in a system of ODE and existence and uniqueness of the solutions to (7) follow from the corresponding results for ODEs. In the sequel we assume that there exists a unique function z verifying (7) for $t \in [0,+\infty]$. We are interested in the effect of perturbations on the *projection* of $z(x,t)$ on suitable subspaces. Assume that $F(0) = 0$ and consider a subspace $\mathcal{V} \subset L^2(G|\mathbb{R}^q)$. We denote by

$$f_{\mathcal{V}} = P_{\mathcal{V}} f$$

the projection of $f \in L^2(G|\mathbb{R}^q)$ on $\mathcal{V}$.

Definition 1. The origin of (7) is *stable on* $\mathcal{V}$ if for all $t \ge 0$

$$\forall\epsilon > 0,\ \exists\delta > 0 :\ \|\tilde z_{\mathcal{V}}\|_{L^2} \le \delta\ \Rightarrow\ \|z_{\mathcal{V}}(\cdot,t)\|_{L^2} \le \epsilon. \tag{8}$$

If, in addition, there exists $k > 0$, $\eta > 0$ such that $\forall \tilde z \in L^2$, $\forall t \ge 0$ it holds

$$\|z_{\mathcal{V}}(\cdot,t)\|_{L^2} \le k e^{-\eta t}\|\tilde z_{\mathcal{V}}\|_{L^2} \tag{9}$$

then, the origin is globally exponentially stable on $\mathcal{V}$.

Note that if $\mathcal{V} = L^2$, stability on $\mathcal{V}$ coincides with the standard notion of stability of the origin (Khalil, 1996). The next Theorem, that is a straightforward generalization of the second method of Lyapunov (see (Khalil, 1996)), can be used for checking exponential stability of the origin on $\mathcal{V}$.

Theorem 2. Assume that there exists a unique solution $z(x,t)$ to (7), $\forall\tilde z \in L^2$, $\forall t \ge 0$. If there exist a continuously differentiable functional $W : \mathcal{V} \mapsto \mathbb{R}$ and constants $k_1, k_2, k_3, a > 0$ such that

$$k_1\|\xi\|^a_{L^2} \le W(\xi) \le k_2\|\xi\|^a_{L^2}, \quad \forall\xi\in\mathcal{V} \tag{10a}$$
$$\dot W(z_{\mathcal{V}}) \le -k_3\|z_{\mathcal{V}}\|^a_{L^2} \tag{10b}$$

then, the origin of (7) is globally exponentially stable on $\mathcal{V}$.

Proof: The proof is reported in (Ferrari-Trecate *et al.*, 2004). ∎

3. ALIGNMENT IN LEADERLESS MODELS

The communication network between agents is modeled in form of a graph G, which is supposed to be connected. A simple agent model is the approximated double integrator dynamics

$$\dot{r}_x = v_x \tag{11a}$$
$$\dot{v}_x = u_x + \beta e_x \tag{11b}$$
$$\dot{e}_x = -\alpha e_x \tag{11c}$$

where $x \in \mathcal{N}$ is the agent index, $r_x \in \mathbb{R}^q$, $v_x \in \mathbb{R}^q$, $u_x \in \mathbb{R}^q$ are the agent position, velocity and inputs, respectively, $q \in \mathbb{N}$ is the space dimension (usually $q = 2$ or $q = 3$) and $\beta \neq 0$. The state e_x represent an error on the velocity dynamics, exponentially decreasing with the same rate $\alpha > 0$ for all agents. The *errorless* model is obtained by setting the initial errors as $e_x(0) = 0$, $\forall x \in \mathcal{N}$. We consider the control law:

$$u_x = u_x^L \doteq -\sum_{x\sim y}(v_x - v_y) \tag{12}$$

that will be referred to as *Laplacian* control and that is similar to the one proposed by Vicsek (Vicsek *et al.*, 1995) for discrete-time agent models.

We also say that *alignment* is achieved if there exists a constant velocity v^* such that $v_x \to v^*$ as $t \to +\infty$ for all agents $x \in G$. In the errorless case, a number of results on alignment in multi-agent systems described by (11) and (12) are available (Jadbabaie *et al.*, 2003), (Saber and Murray, 2003).

Remark 3. The error model (11c) can be justified as follows. Consider the *uncertain* agent model

$$\dot{r}_x = v_x \tag{13a}$$
$$\dot{v}_x = \epsilon v_x + u_x + \tilde{u}_x \tag{13b}$$

where $\epsilon \in \mathbb{R}\backslash\{0\}$ represents an unknown perturbation coefficient and $\tilde{u}_x$ an internal feedback action. In Remark 4, we show that if $u_x = u_x^L$ and no correcting action is taken (i.e. $\tilde{u}_x = 0$), the control u_x can not guarantee alignment to a non zero velocity.

This shows the necessity of designing $\tilde{u}_x$ in order to compensate for the effect of the perturbation. In (Ferrari-Trecate *et al.*, 2004), we exploit variable structure control for designing an internal feedback which guarantees that, after a *finite* time, each agent behaves according to model (11) with *prescribed* error decrease rate α and with $\beta = \alpha^2$.

The agent velocity, input and errors can be seen as vector-valued functions $v(x,t) \in \mathbb{R}^q$, $u(x,t) \in \mathbb{R}^q$, $e(x,t) \in \mathbb{R}^q$ for $x \in \mathcal{N}$, $t \geq 0$. By using the modeling framework presented in Section 2 and the agent model (11), the collective dynamics is captured by the PdEs:

$$\dot{v} = \Delta v + \beta e \quad v(\cdot,0) = \tilde{v} \in L^2 \tag{14a}$$
$$\dot{e} = -\alpha e \qquad e(\cdot,0) = \tilde{e} \in L^2 \tag{14b}$$

where the agent positions have been neglected since they do not influence the velocities and then do not affect alignment. Note that, in absence of errors and for a scalar function v, the PdE (14a) formally coincide with the *heat equation* where x is a point in an open, bounded and regular set $G \subset \mathbb{R}^N$, v is the temperature and $\tilde{v}$ denotes an initial temperature distribution [4]. Due to *diffusion* effect of the Laplacian, it is not surprising that the temperature becomes asymptotically constant on G. In view of the similarities between classic Laplacian and the Laplacian on graphs, highlighted in Remark 2, asymptotic convergence of the velocity $v(x,t)$ to a function in $H^1_\perp$ is expected. This means that, asymptotically, all agents will move with the same velocity, or, in other words, that alignment will be achieved.

Remark 4. When the velocity dynamics is affected by a persistent perturbation as in (13b) for $\tilde{u}_x = 0$, there are two possibilities: if $\epsilon < 0$ the only equilibrium of (13b) is $v_x = 0$; if $\epsilon > 0$ no equilibrium is compatible with (13b). This can be easily seen by recasting (13b) into the PdE

$$\dot{v} = (\epsilon + \Delta)v. \tag{15}$$

In fact, by using Theorem 1 one immediately verifies that if $\epsilon < 0$, all the eigenvalues of the operator $(\epsilon + \Delta)$ (defined on L^2) are negative. Moreover, if $\epsilon > 0$ at least one eigenvalue of $(\epsilon+\Delta)$ is positive. These results admit an intuitive interpretation motivated by the analogy between PdEs and PDEs. Indeed, when $\epsilon < 0$ [resp. $\epsilon > 0$], equation (15) formally coincides with the heat equation with dissipation [resp. heat generation], and convergence of v to zero [resp. to infinity] can be easily proved.

In the sequel, we prove that a *globally exponentially stable alignment* can be guaranteed even in presence of errors.

First, we introduce the decompositions

$$v = v_1 + \bar{v},\ v_1(\cdot,t) \in H^1,\ \bar{v} \doteq \langle v\rangle \in H^1_\perp \tag{16a}$$
$$e = e_1 + \bar{e},\ e_1(\cdot,t) \in H^1,\ \bar{e} \doteq \langle e\rangle \in H^1_\perp. \tag{16b}$$

For finding the dynamics of $\bar{v}$ we test each side of (14a) against all $c \in H^1_\perp$, i.e. we form the integrals

$$\int_G \dot{v}^T c = \int_G (\Delta v)^T c + \beta \int_G e^T c \tag{17}$$

Since $\Delta\bar{v} = 0$, and

$$e_1, \Delta v_1 \in H^1 \ \Rightarrow \int_G (\Delta v_1)^T c = \int_G e_1^T c = 0 \tag{18}$$

one has that $\dot{v}$ verifies

$$\int_G \dot{v}^T c = \int_G \dot{\bar{v}}^T c = \beta \int_G \bar{e}^T c \tag{19}$$

[4] More rigorously, it coincides with the heat equation on G with homogeneous Neumann boundary conditions (Dautray and Lions, 1992), i.e. with null heat flow through the boundary of G.

thus showing that the average velocity obeys to the equation

$$\dot{\bar{v}} = \beta\bar{e}. \tag{20}$$

From (14a) we also have

$$\dot{v}_1 + \dot{\bar{v}} = \Delta v_1 + \beta e_1 + \beta\bar{e} \tag{21}$$

and by using (20) we obtain

$$\dot{v}_1 = \Delta v_1 + \beta e_1. \tag{22}$$

In summary, the evolution of errors and velocities is captured by the PdEs

$$\Sigma_1 : \begin{cases} \dot{v}_1 = \Delta v_1 + \beta e_1 \\ \dot{e}_1 = -\alpha e_1 \end{cases} \quad \bar{\Sigma} : \begin{cases} \dot{\bar{v}} = \beta\bar{e} \\ \dot{\bar{e}} = -\alpha\bar{e} \end{cases} \tag{23}$$

where we assume that the initial conditions verify the decomposition (16), i.e. $v_1(\cdot,0) = P_{H^1}\tilde{v}$, $e_1(\cdot,0) = P_{H^1}\tilde{e}$, $\bar{v}(0) = P_{H^1_\perp}\tilde{v}$ and $\bar{e}(0) = P_{H^1_\perp}\tilde{e}$.

In view of (23), alignment is achieved if $v_1 \to 0$ as $t \to \infty$. In fact, for the PdE $\bar{\Sigma}$, it is easy to check that $\bar{v}$ converges to a function $\bar{v}^* \in H^1$. Then, if v_1 converges to zero, one would obtain $v \to \bar{v}^*$ as $t \to \infty$.

Remark 5. For the errorless agent model, $\bar{v}$ is constant in time and equal to the average of the initial velocities. On the other hand, if $\bar{e}(x,0) \neq 0$ the asymptotic velocity of the formation will be affected also by the error dynamics.

The PdEs (23) open also the way to the study of the *stability* of the alignment. In fact, one is tempted to say that the alignment is stable if the origin of Σ_1 enjoys the same property. However, since v_1 and e_1 belongs to the subspace H^1, one must resort to the notion of stability provided by Definition 1. To this purpose, note that the state of the PdE (14) is $z = \left[\, v^T \;\; e^T \,\right]^T$ and one is interested in proving stability of the origin of (14) on the subspace

$$\mathcal{V} = \left\{ \left[\, v_1^T \;\; e_1^T \,\right] \in H^1(G|\mathbb{R}^q) \times H^1(G|\mathbb{R}^q) \right\} \tag{24}$$

As a candidate Lyapunov functional we take the energy $W : \mathcal{V} \mapsto \mathbb{R}$ given by

$$W(v_1, e_1) = \frac{1}{2}\|v_1\|^2_{L^2} + \frac{\gamma}{2}\|e_1\|^2_{L^2} \tag{25}$$

where $\gamma > 0$ is a parameter. The main stability result is stated in the next theorem.

Theorem 3. The origin of (14) is globally exponentially stable on $\mathcal{V}$.

Proof: The functional W verifies the bounds (10a) for $k_1 = \min\{\frac{1}{2}, \frac{\gamma}{2}\}$, $k_2 = \max\{\frac{1}{2}, \frac{\gamma}{2}\}$ and $a = 2$. By computing $\dot{W}$, one obtains

$$\dot{W} = \int_G v_1^T(\Delta v_1 + \beta e_1) - \alpha\gamma\int_G \|e_1\|^2 \leq \tag{26a}$$

$$\leq \lambda\int_G \|v_1\|^2 + \beta\int_G v_1^T e_1 - \alpha\gamma\int_G \|e_1\|^2 \tag{26b}$$

where λ is the maximum eigenvalue of the Laplacian operator defined on H^1. In view of Theorem 1, we have $\lambda < 0$. It can be shown that the bound (10b) is verified with

$$k_3 = \min\left\{ -\frac{\lambda}{2}, \alpha\gamma + \frac{|\beta|^2}{2\lambda} \right\}.$$

By choosing γ big enough, one obtains $k_3 > 0$ and global exponential stability of the origin on $\mathcal{V}$ follows from Theorem 2. ■

We highlight that Theorem 3 guarantees exponentially stable alignment irrespectively of the magnitude of the error rate α.

4. ALIGNMENT PROPERTIES IN LEADER-FOLLOWER MODELS

In this Section we use PdEs for analyzing the collective motion of the agents in presence of a leader. By leader, we mean a vehicle that moves according to a prescribed constant velocity, independently of the motion of all other vehicles. However, followers connected to the leader use information on the leader state in order to compute their control inputs.

Let S be a subgraph of the connected graph G and let the boundary of S be defined by: $\partial S \doteq \{y \in G\backslash S \;:\; \exists x \in S \;:\; x \sim y\}$. The leader and the follower are indexed by the nodes of ∂S and S respectively. Since we assume that the leader is unique, we have $\partial S = \{x_L\}$. The closure of S is given by $\bar{S} \doteq S \cup \partial S = G$.

We will show that the collective dynamics can be modeled through *Dirichlet boundary value problems*, i.e. PdEs endowed with suitable boundary conditions. To this purpose we first introduce the relevant functional spaces. As in (Bensoussan and Menaldi, 2004), we consider the Hilbert space $H^1_0(S) = \{u \in L^2(\bar{S}) \;:\; u_{|\partial S} = 0\}$, equipped with the norm

$$\|f\|^2_{H^1_0} = \sum_{x\in\mathcal{N}}\sum_{y\sim x} \|\partial_y f(x)\|^2. \tag{27}$$

Note that a function $f \in H^1_0(S)$ is defined on $\bar{S}$ and possibly non null only on S. The next theorem is the counterpart of Theorem 1.

Theorem 4. Let G be a connected graph. Then, the operator $\Delta : H^1_0(S|\mathbb{R}^q) \mapsto L^2(\bar{S}|\mathbb{R}^q)$ has $|S|q$ strictly negative eigenvalues where $|S|$ denotes the number of nodes of S. Moreover, the corresponding eigenfunctions form a basis for $H^1_0(S|\mathbb{R}^q)$.

Suppose, for simplicity, that the leader x_L has a constant velocity v_L. Let, by abuse of notation $v_L(x) \doteq v_L$ for all $x \in \bar{S}$. Note that $\Delta v_L = 0$, because $v_L \in H^1_\perp(\bar{S})$. It turns out that the velocity $v \in L^2(\bar{S})$ can be split as

$$v = v_0 + v_L, \quad v_0 \in H^1_0(S) \tag{28}$$

and alignment to the leader velocity corresponds to the condition $v_0 \to 0$ as $t \to \infty$. If the followers make use of Laplacian control, the collective dynamics results in the following *PdE with boundary conditions*

$$\dot{r} = v_0 + v_L \qquad x \in S \tag{29a}$$
$$\dot{v}_0 = \Delta v_0 + \beta e \qquad x \in S \tag{29b}$$
$$\dot{e} = -\alpha e \qquad x \in S \tag{29c}$$
$$v_0 = 0 \qquad x \in \partial S \tag{29d}$$

endowed with the initial conditions $r(\cdot, 0) = \tilde{r} \in L^2$, $v_0(\cdot, 0) = \tilde{v}_0 \in H_0^1$, $e(\cdot, 0) = \tilde{e} \in L^2$. In fact, (29c) corresponds to (14b). The velocity equation (29b) is obtained from (14a), by using the decomposition (28).

Remark 6. Once more, the analogy between PdEs and PDEs provides useful hints on the achievement of alignment. Note that, in absence of errors and for $v(x, t) \in \mathbb{R}$, the PdE (29b) with (29d) formally coincide with the *heat equation* on an open, bounded and regular set $S \subset \mathbb{R}^N$ with inhomogeneous Dirichlet boundary conditions. By interpreting v_0 as a temperature, physical reasoning leads to the conclusion that v_0 converges to zero, as $t \to \infty$. By reading the result in terms of the collective dynamics, this means that alignment to the leader velocity will be achieved.

For proving alignment, we adopt a Lyapunov argument completely analogous to the one used in Section 3. To this purpose, let $z = \left[\, v_0^T \; e^T \, \right]^T$ be the state of (29) and consider the subspace

$$\mathcal{V}_L \doteq \left\{ \left[\, v_0^T \; e^T \, \right]^T \in H_0^1(S) \times L^2(G|\mathbb{R}^q) \right\}. \tag{30}$$

along with the candidate Lyapunov functional $W_L : \mathcal{V}_L \mapsto R$ given by

$$W_L = \frac{1}{2}\|v_0\|_{L^2}^2 + \frac{\gamma}{2}\|e\|_{L^2}^2, \quad \text{for some } \gamma > 0. \tag{31}$$

Theorem 5. The origin of (29) is globally exponentially stable on $\mathcal{V}$.

Proof: The proof, provided in (Ferrari-Trecate *et al.*, 2004), exploits the same arguments used in the proof of Theorem 3. ■

5. DISCUSSION AND CONCLUSIONS

In this paper we proposed the framework of continuous-time PdEs for analyzing coordination phenomena in multi-agent systems. Although we considered a fairly simplified setup (i.e. agents move according to a point-mass dynamics perturbed by errors and the structure of the communication network is time-invariant) we believe that PdEs provide a useful mathematical framework also for dealing with more complex models. First, many tools developed for continuous-time PdEs can be straightforwardly applied to discrete-time PdEs. For instance, all the results presented in Sections 2 and 4 characterizing the Laplace operator can be directly used in the discrete-time case. Also the generalization of such properties to weighted Laplacian operators is pretty easy and already provided in (Bensoussan and Menaldi, 2004). In Part II of the present paper, we show that PdEs can be fruitfully applied for investigating the effect of nonlinear control schemes. Future research will focus on the use of PdEs for studying the impact of communication delays and multiple leaders with non-constant velocities on the collective dynamics.

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