LOCALLY CONSTRAINED OPTIMAL AND GLOBALLY STABLE BACKSTEPPING DESIGN

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Abstract: Robust nonlinear controller design with constraint on the poles' location of the linear part of closed-loop system is proposed. The design method is based on the integrator backstepping procedure and linear constrained H_∞ for nonlinear strict-feedback systems with disturbance also in strict-feedback form. The resulted closed-loop system will be globally stable, while both local robustness and desired *α-stability* are achieved. An analytic example is used to compare the performance of the proposed methodology with that of the locally optimal backstepping design with no closed-loop poles constraint. *Copyright © 2005 IFAC*

Keywords: Nonlinear control systems, H-infinity control, constrained poles, disturbance attenuation, global stability.

1. INTRODUCTION

Although optimal designs for linear systems have been discussed largely and applicable controllers such as LQR and H_∞ are used for these systems with Quadratic cost functions, for nonlinear systems there are no applicable methods to solve optimal design's problems. The main reason is our inability to solve Hamilton–Jacobi–Bellman (HJB) or the more general Hamilton–Jacobi–Isaacs (HJI) equations. Using linear optimal controller for nonlinear systems with controllable linearized term and quadratic approximation of related cost function leads to optimal closed-loop system in a neighborhood of equilibrium point, see (Isidori, 1995; Khalil 1996), but global stability is not guarantee by such controllers. (Ezal, *et al*., 1997; Ezal, *et al*., 2000) have introduced a nonlinear control design method for systems in strict feedback form which obtains local optimality via linear terms of the controller and global stability via higher order terms of the controller. Also stepwise algorithm is introduced in (Ezal, *et al*., 2000). Linear optimal part of this controller is the solution of the standard time domain linear H_∞ and design of nonlinear parts is based on backstepping procedure (Freeman and Kokotovic, 1996; Krstic, *et al*., 1995).

Linear H_∞ controller has been discussed in (Burl, 1999; Green and Limebeer, 1995). Although this controller leads to stable closed-loop system with an arbitrary level of disturbance attenuation, it may place closed-loop poles near imaginary axis that leads to unrobustness in face of the model uncertainties and unmodeled dynamics. Also transient behavior of closed-loop system deeply depends on the poles places. To avoid of this phenomena, constrained H_∞ controller are suggested in (Adhami, *et al*., 2005; Yedavalli and Liu, 1995). In (Adhami, *et al*., 2005) this controller has been achieved by the solution of an optimization problem with standard H_∞ cost function and α-stability constraint.

In this paper, we design locally constrained optimal controller which guarantees global stability for nonlinear systems in strict feedback form. By using this method the resulted closed-loop system has some specification such as:

- Globally Asymptotically Stability (GAS) in the absence of disturbances.

- Has desired level of disturbance attenuation locally (local optimality).
- Has desired states settling time for its linear part, which is dominant dynamic in neighborhood of the equilibrium point.

This paper combines the methods of (Ezal, *et al.*, 2000) and (Adhami, *et al.*, 2005), to enlarge the validity domain of local optimality, and make the closed-loop response faster.

The organization of the paper is as follows. In Section 2 system specification is presented and the problem is formulated, we introduce in Section 3 the linear recursive design for linear constrained H_∞ controller. Section 4 completes nonlinear design for globally stabilizing controller. Finally, simulation results are employed to show the effectiveness of the proposed method and comparison results are given which clearly indicate the advantages gained by the constrained optimal design. The paper ends with some concluding remarks in Section 6.

2. PROBLEM FORMULATION

Backstepping design method is applicable to systems in strict feedback form. Our procedure is also based on backstepping, thus it needs the same nonlinear system form, (1).

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1) + g_1(x_1)\,w \\ \dot{x}_2 = x_3 + f_2(x_1,x_2) + g_2(x_1,x_2)\,w \\ \quad\vdots \\ \dot{x}_n = f_n(x_1,\dots,x_n) + g_n(x_1,\dots,x_n)w + u \end{cases} \tag{1}$$

(2) shows the compact form of (1)

$$\dot{x} = f(x) + B_u\, u + G(x)\, w \tag{2}$$

where $x = [x_1 \quad x_2 \quad \cdots \quad x_n]'$ is the n-dimensional state vector. $u(t) : [0,\infty) \to R$ and $w(t,x) : [0,\infty)\times R^n \to R^m$ are control and bounded disturbance inputs, respectively. f_i, g_i are assumed smooth functions and $f_i(0) = 0$.

Our procedure has two main goals, global stability and local constrained optimality. To achieve the second goal, the constrained H_∞ controller should be designed for the linearized system, therefore linear and higher order terms in (2) are separated.

$$\dot{x} = Ax + B_u u + B_w w + \breve{f}(x) + \breve{G}(x)\,w \tag{3}$$

where $\breve{f}(x) = f(x) - Ax$ and $\breve{G}(x) = G(x) - B_w = G(x) - G(0)$ contain only nonlinear terms, A is in the form of (4). Obviously irrespect to elements of A, the pair (A, B_u) is controllable.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} a_{11} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 1 \\ a_{n1} & a_{n2} & & \cdots & a_{nn} \end{bmatrix} \tag{4}$$

Here after in this paper the i th submatrix of A and its related state vector are defined as bellow.

$$A_{[i]} = \begin{bmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ii} \end{bmatrix}$$

$$x_{[i]} = [x_1 \quad \cdots \quad x_i]$$

3. LOCALLY OPTIMAL BACKSTEPPING DESIGN

In this section the constrained H_∞ controller for the linear part of (3) is obtained and transformed to the suitable form for the backstepping procedure. Then the linear backstepping procedure is employed to show the stability and optimality achievements for the linear part of (3) closed with the transformed controller.

In (Adhami, *et al.*, 2005), it has shown that, the linear controller that achieves desired disturbance attenuating level γ and places the closed loop poles in the left of the line $L_1 : \mathrm{Re}(s) = -\alpha$ in the $s-plane$ is obtained through the constrained dynamic game $\min_{u_l}\max_{w_l} J_l(u_l, w_l)$ with cost function (5) and lyapunov shaped constraint (6).

$$J_l(u_l, w_l) = \int_0^\infty (x'c'cx + u_l^2 - \gamma^2 w_l' w_l)\, dt \tag{5}$$

$$(A_\alpha x + B_u u)' Lx + x'L(A_\alpha x + B_u u) = 0 \tag{6}$$

where $A_\alpha = A + \alpha I$ and $c'c \neq 0$. (Adhami, *et al.*, 2005) converts this constrained optimization to unconstrained one by Lagrange multipliers, and obtains the optimal control law (7) and the worst case disturbance (8).

$$\mu_l(x) = -B_u'(P + L)x \tag{7}$$

$$v_l(x) = \gamma^{-2} B_w' P x \tag{8}$$

where P and L are symmetric positive definite solutions of two Algebraic Riccati Equations (ARE) (9) and (10).

$$\begin{aligned} A'P + PA - P(B_u B_u' - \gamma^{-2} B_w B_w')P \\ + c'c + LB_u B_u' L = 0 \end{aligned} \tag{9}$$

$$\begin{aligned} (A_\alpha - B_u B_u' P)' L + L(A_\alpha - B_u B_u' P) \\ -2LB_u B_u' L = 0 \end{aligned} \tag{10}$$

The proper transformation that makes (7) and (8) suitable for backstepping is $P = T'\Delta T$, where T and Δ are found by unique cholesky factor of P, and have special forms as (11) and (12).

$$T := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha_{11} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\alpha_{n-1,1} & \cdots & -\alpha_{n-1,n-1} & 1 \end{bmatrix} \tag{11}$$

$$\Delta := diag(\delta_1, \ldots, \delta_n) \tag{12}$$

where δ_i is positive scalar. Now by $z = Tx$ transformation the linear part of (3) will be represented as (13).

$$\dot{z} = \overline{A}z + \overline{B}_w w + B_u u \tag{13}$$

This transformation and transformed system have the following listed properties, which are used in the design procedure. For details see (Ezal, *et al*., 2000).

Property 1: For, $1 \le k \le n$; $z_{[k]} = T_{[k]} x_{[k]}$ where $T_{[k]}$ is invertible.

Property 2: For $1 \le k < n$

$$\dot{z}_{[k]} = \overline{A}_{[k]} z_{[k]} + \begin{bmatrix} 0 \\ \vdots \\ z_{k+1} \end{bmatrix} + \overline{B}_{w[k]} w_l \tag{14}$$

Property 3: For $1 \le k < n$

$$\begin{gathered} \overline{A}'_{[k]}\Delta_{[k]} + \Delta_{[k]}\overline{A}_{[k]} + \gamma^{-2}\Delta_{[k]}\overline{B}_{w[k]}\overline{B}'_{w[k]}\Delta_{[k]} \\ + \overline{c}'_{[k]}\overline{c}_{[k]} + (\overline{L}B_u B'_u \overline{L})_{[k]} = 0 \end{gathered} \tag{15}$$

also,

$$\overline{v}_{lk} = \gamma^{-2}\overline{B}'_{w[k]}\Delta_{[k]}z_{[k]} \tag{16}$$

By this transformation (9) and (10) will be changed to (17) and (18), respectively.

$$\begin{gathered} \overline{A}'\Delta + \Delta\overline{A} - \Delta(B_u B'_u - \gamma^{-2}\overline{B}_w \overline{B}'_w)\Delta \\ + \overline{c}'\overline{c} + \overline{L}B_u B'_u \overline{L} = 0 \end{gathered} \tag{17}$$

$$\begin{gathered} (\overline{A}_\alpha - B_u B'_u \Delta)'\overline{L} + \overline{L}(\overline{A}_\alpha - B_u B'_u \Delta) \\ -2\overline{L}B_u B'_u \overline{L} = 0 \end{gathered} \tag{18}$$

where $\overline{L} = {T^{-1}}' L T^{-1}$ is positive definite matrix and $\overline{c} = cT^{-1}$. Now the solution of constrain optimal controller are found through a backstepping procedure. Although this controller could be easily found by transforming (7), using the linear backstepping procedure to find it, is a preparatory for the nonlinear one, which will be discussed in the next section.

In each step of the linear backstepping procedure virtual controller and lyapunov function are in the form of (19) and (20), respectively.

$$\alpha_i(x_{[i]}) = \alpha_{[i]}\, x_{[i]} \tag{19}$$

where $\alpha_{[i]} := [\alpha_{i1} \quad \alpha_{i2} \quad \cdots \quad \alpha_{ii}]$ for $1 \le i < n$.

$$\overline{V}_i(z_{[i]}) = \overline{V}_{i-1}(z_{[i-1]}) + \delta_i z_i^2 = z'_{[i]}\,\Delta_{[i]}\, z_{[i]} \tag{20}$$

here concerning $z = Tx$ transformation, the following required relations and definitions are obtained.

$$\begin{cases} z_1 = x_1 \\ z_i = x_i - \alpha_{[i-1]}x_{[i-1]} = x_i - \overline{\alpha}_{[i-1]}z_{[i-1]};\, i = 2,\ldots,n \end{cases} \tag{21}$$

$$\begin{cases} \dot{x}_i = a_{[i]}x_{[i]} + x_{i+1} + b_{wi}w_l \;;\, i = 1,\ldots,n-1 \\ \dot{x}_n = a_{[n]}x_{[n]} + u_l + b_{wn}w_l \end{cases} \tag{22}$$

where $a_{[i]} := [a_{i1} \quad a_{i2} \quad \cdots \quad a_{ii}]$

Note that in each step, algebraic manipulations are omitted, and we just focus on the new results

Step 1: Define $z_1 = x_1$ and choose $\overline{V}_1 = z'_{[1]}\Delta_{[1]}z_{[1]}$ as lyapunov function. From the property 2 (for $k = 1$), relation (22) and by setting $\overline{a}_{[1]} = a_{[1]} + \overline{\alpha}_{[1]}$, $\overline{b}_{w1} = b_{w1}$ when $z_2 = x_2 - \overline{\alpha}_1$ selected as the virtual control law, the dynamic of z_1 is:

$$\dot{z}_1 = \overline{a}_{[1]}z_{[1]} + z_2 + \overline{b}_{w1}w_l \tag{23}$$

By using property 3 and completing the squares with respect to w_l, Derivation of $\overline{V}_1$ leads to (24).

$$\begin{gathered} \dot{\overline{V}}_1 = -z'_{[1]}(\overline{c}'\overline{c} + \overline{L}B_u B'_u \overline{L})_{[1]}\, z_{[1]} + \\ \gamma^2 w'_l w_l - \gamma^2 \left|w_l - \overline{v}_{l1}\right|^2 + 2z_1\delta_1 z_2 \end{gathered} \tag{24}$$

where $\overline{v}_{l1}$ is defined by (16). It is obvious that $(\overline{c}'\overline{c} + \overline{L}B_u B'_u \overline{L})_{[1]}$ is (symmetric) positive definite scalar (matrix in the next steps). So that if $z_2 \equiv 0$, $z_1(t) \to 0$ as $t \to \infty$ for all $w_l(t) \in L_2$, and $z_1(t)$ is bounded for all $w_l(t) \in L_\infty$, (Teel, 1999). Also in the absence of a disturbance, $z_1 = 0$ is GAS.

Step i: is similar to (Ezal, *et al*., 2000), but with the similar changes as step1.

Step n: In this step $\overline{V} = z'\Delta z$ is chosen as candidate lyapunov function for transformed system (13). Using relation (17) and completing the squares with respect to w_l and u_l to the derivative of $\overline{V}$, we obtain:

$$\begin{gathered} \dot{\overline{V}} = -z'\,\overline{c}'\overline{c}\,z - {u_l}^2 + (u_l - \overline{\mu}_l)^2 + \gamma^2 w'_l w_l \\ -2z'\overline{L}B_u(u_l - \overline{\mu}_l) - \gamma^2\left|w_l - v_l\right|^2 \end{gathered} \tag{25}$$

where

$$\overline{\mu}_l(z) = -B'_u(\Delta + \overline{L})z \tag{26}$$

By setting $u_l = \overline{\mu}_l(z)$, $\dot{\overline{V}}$ satisfies the following inequality.

$$\dot{\overline{V}} \le -z'\,\overline{c}'\overline{c}\,z - {u_l}^2 + \gamma^2 w_l' w_l \tag{27}$$

This final step complete our linear backstepping design. According to the result (Teel, 1999), (27) shows that the transformed system (13) with controller (26) and its states:

- is GAS when $w_l = 0$
- remain bounded for bounded disturbances
- converge to zero for L_2 disturbances.

4. GLOBALLY STABILIZING BACKSTEPPING DESIGN

The nonlinear backstepping design is based on linear one which discussed in section 3. In this procedure at step i, the linear transformation (19) are completed by nonlinear terms in the form of (28).

$$\overline{\alpha}_i(z_{[i]}) = \overline{\alpha}_{[i]} z_{[i]} + \hat{\alpha}_i(z_{[i]}) \tag{28}$$

where $\hat{\alpha}_i$ contains only higher order terms which is employed to cancel undesirable nonlinear terms in $\dot{\overline{V}}_i$ and attenuate the disturbances. At the end of the step i, $z_{i+1} = \phi_{i+1}(x_{[i+1]}) := x_{i+1} - \overline{\alpha}_i(x_{[i]})$ is chosen as virtual control low for the next step. At the final step, this constructive procedure leads to the lower triangular Diffeomorphism, $z = \Phi(x)$. Unlike to (Ezal, *et al.*, 2000) and the other common backstepping methods, at this final step the Control Lyapunov Function (CLF) is not constructively completed. As these steps are similar to the same steps in (Ezal, *et al.*, 2000), we just focus on the main and new results.

Step 1: Define $z_1 := x_1$ and choose $\overline{V}_1 = z_{[1]}' \Delta_{[1]} z_{[1]}$ as storage function. By adding and subtracting $\overline{\alpha}_1(z_1)$ which defined in (28), dynamics of z_1 is rewritten as (29).

$$\begin{aligned} \dot{z}_1 = \overline{a}_{[1]} z_{[1]} + \breve{f}_1(z_{[1]}) + \breve{\alpha}_1(z_{[1]}) + (x_2 - \overline{\alpha}_1(z_{[1]})) \\ + \overline{g}_1(x_2 - \overline{\alpha}_1(z_{[1]}))w \end{aligned} \tag{29}$$

Concerning (29), derivation of $\overline{V}_1$ leads to (30).

$$\begin{aligned} \dot{\overline{V}}_1 = -z_{[1]}'(\overline{c}'\overline{c} + \overline{L} B_u B_u' \overline{L})_{[1]} z_{[1]} + \gamma^2 w'w \\ -\gamma^2 |w - \overline{v}_1|^2 + 2 z_1 \delta_1 (x_2 - \overline{\alpha}_1) \\ + 2 z_{[1]}' \Delta_{[1]} \left[\breve{\alpha}_1 + \breve{f}_1 + .5\gamma^{-2} (\overline{g}_1 \overline{g}_1' - \overline{b}_{w1} \overline{b}_{w1}') \Delta_{[1]} z_{[1]} \right] \end{aligned} \tag{30}$$

where

$$\overline{v}_1(z_{[1]}) := \gamma^{-2}\, \overline{G}_{[1]}'(z_{[1]}) \Delta_{[1]}\, z_{[1]} \tag{31}$$

Now, by choosing $\breve{\alpha}_1$ as (32) and assuming $x_2 \equiv \overline{\alpha}_1(z_{[1]})$, the inequality (33) will be satisfied.

$$\breve{\alpha}_1(z_{[1]}) = -\breve{f}_1 - .5\gamma^{-2} (\overline{g}_1 \overline{g}_1' - \overline{b}_{w1} \overline{b}_{w1}') \Delta_{[1]} z_{[1]} \tag{32}$$

$$\dot{\overline{V}}_1 \le -z_{[1]}'(\overline{c}'\overline{c} + \overline{L} B_u B_u' \overline{L})_{[1]} z_{[1]} + \gamma^2 w'w \tag{33}$$

According to (Teel, 1999), (33) shows that z_1 subsystem is GAS in the absence of any disturbance. Also L_2 and L_∞ disturbances leads to converged and bounded z_1, respectively.

Step i: In this step, upon the obtained relations and results in the previous step with $z_i := x_i - \overline{\alpha}_{i-1}(z_{[i-1]})$ and $\overline{V}_i = z_{[i]}' \Delta_{[i]} z_{[i]}$ definitions, the $z_{[i]}$ dynamics and $\dot{\overline{V}}_i$ are calculated. By selecting $\hat{\alpha}_i$ as (34), the dissipative form of $\dot{\overline{V}}_i$ is achieved, (35).

$$\begin{aligned} \hat{\alpha}_i(z_{[i]}) = -a_{[i]} \left(\Phi_{[i]}^{-1}(z_{[i]}) - L_{[i]}^{-1} z_{[i]} \right) \\ - \breve{f}_i(\Phi_{[i]}^{-1}(z_{[i]})) + \frac{\partial \hat{\alpha}_{i-1}}{\partial z_{[i-1]}} \left[\overline{A}_{[i-1]} z_{[i-1]} + \begin{bmatrix} 0_{i-2} \\ z_i \end{bmatrix} \right] \\ + \frac{\partial \overline{\alpha}_{i-1}}{\partial z_{[i-1]}} \hat{f}_{[i-1]}(z_{[i-1]}) - (\overline{g}_i \overline{v}_{i-1} - \overline{b}_{wi} \overline{v}_{l,i-1}) \\ - 0.5\gamma^{-2} (\overline{g}_i \overline{g}_i' - \overline{b}_{wi} \overline{b}_{wi}') \delta_i z_i \end{aligned} \tag{34}$$

Assuming $x_{i+1} \equiv \overline{\alpha}_i(z_{[i]})$, so that:

$$\dot{\overline{V}}_i \le -z_{[i]}'(\overline{c}'\overline{c} + \overline{L} B_u B_u' \overline{L})_{[i]} z_{[i]} + \gamma^2 w'w \tag{35}$$

(35) shows that $z_{[i]}$ subsystem satisfies the conditions of (Teel, 1999), thus this subsystem has the required specifications similar to $z_{[1]}$ (see step 1).

Step n: Now the $z = \Phi(x)$ Diffeomorphism is completed by calculating $\hat{\alpha}_n$ from (34) for $i = n$. Clearly the linear part of this Diffeomorphism is Tx, therefore the nonlinear transformed system (36) includes (13) as its linear part.

$$\dot{z} = \overline{A} z + \overline{B}_w w + B_u u + \hat{f}(z) + \hat{G}(z) w \tag{36}$$

In the next section the design procedure is completed by founding suitable control law.

5. NONLINEAR CONTROL LAW

Desired nonlinear controller is found through a theorem.

Theorem 1: A positive definite function $\overline{r}(z)$ with property $\overline{r}(0) = 1$ exists, such that the closed-loop system (36) with following feedback control law

$$u = \overline{\mu}(z) = -\overline{r}^{-1}(z) B_u' (\Delta + \overline{L}) z \tag{37}$$

Achieves

- Local optimality with respect to the cost functional (5).
- Local $\alpha - stability$, (satisfies constraint (6)).
- $z(t) \to 0$ as $t \to \infty$ for all $w(t) \in L_2$.
- $z(t) \in L_\infty$ for all $w(t) \in L_\infty$.
- GAS equilibrium point in the absence of disturbance.

Proof: We will find $\bar{r}(z)$ such that the derivation of $\bar{V} = z'(\Delta + \bar{L})z > 0$ (CLF) with respect to dynamical system (36) goes negative. This derivation could be written in the form of (38).

$$\dot{\bar{V}} \le -\bar{q}(z) - \bar{r}\,u^2 + \gamma^2 w'w - (\bar{r}^{-1} - 1)\; z'\bar{L}B_u B_u'\bar{L}z \\ -2\bar{r}^{-1}\, z'\,\bar{L}B_u B_u'\Delta\, z - 2\alpha\, z'\bar{L}z \qquad (38)$$

where

$$\bar{q}(z) := z'\,\bar{c}'\bar{c}\,z + (\bar{r}^{-1} - 1)\delta_n^2 z_n^2 - 2z_n\delta_n\hat{\eta} \qquad (39)$$

and

$$\hat{\eta}(z) := \hat{f}_n + (\bar{g}_n\bar{v}_{n-1} - \bar{b}_{wn}\bar{v}_{wn-1}) \\ +.5\gamma^{-2}(\bar{g}_n\bar{g}_n' - \bar{b}_{wn}\bar{b}_{wn}')\delta_n z_n \qquad (40)$$

If $\bar{r}^{-1} \ge 1$, the 4th term and the last one in the right hand side of (38) are nonpositive and negative functions. Since δ_n and $\bar{l}_{nn}$ are positive scalars, (41) shows that, if $\bar{r}^{-1} \ge 0$, the 5th term of (38) is also nonpositive.

$$\bar{L}B_u B_u'\Delta = \begin{bmatrix} 0_{n-1,n-1} & d_{n-1,1} \\ d_{1,n-1}' & \delta_n\,\bar{l}_{nn} \end{bmatrix} \ge 0 \qquad (41)$$

Now the proof will be completed if the $\bar{r}(z)$ which makes $\bar{q}(z)$ positive, is found. In (Ezal, *et al.*, 2000) by factoring out of $\hat{\eta}$ as (42) and the same factorization of $\bar{Q} := \bar{c}'\bar{c}$ (43), and definition (44),

$$\hat{\eta}(z) = \bar{\eta}_1(z_{[n-1]})z_{[n-1]} + \bar{\eta}_2(z)z_n \qquad (42)$$

$$\bar{Q} = \begin{bmatrix} \bar{Q}_{[n-1]} & \bar{q}_1 \\ \bar{q}_1' & \bar{q}_2 \end{bmatrix} \qquad (43)$$

$$\bar{\sigma}(z) = \bar{\eta}_1\,\bar{Q}_{[n-1]}^{-1}\,\bar{\eta}_1' - 2\bar{q}_1'\,\bar{Q}_{[n-1]}^{-1}\,\delta_n^{-1}\,\bar{\eta}' + 2\delta_n^{-1}\bar{\eta}_2 \qquad (44)$$

the sufficient condition (45) for $\bar{r}(z)$ is obtained which makes $\bar{q}(z)$ positive.

$$\bar{r}^{-1}(z) \ge 1 + \bar{\sigma}(z) \qquad (45)$$

(46) introduces a function for $\bar{r}(z)$ which satisfies the sufficient conditions,($\bar{r}(0) = 1$, $\bar{r}^{-1} \ge 1$ and (45)).

$$\bar{r}(z) = \begin{cases} (1 + \bar{\sigma}(z))^{-1}, & \bar{\sigma}(z) \ge 0 \\ 1\;, & -1 \le \bar{\sigma}(z) < 0 \\ \bar{\varepsilon}(\bar{\sigma})\;, & \bar{\sigma}(z) < -1 \end{cases} \qquad (46)$$

where $\bar{\varepsilon}(\bar{\sigma}) \ge 1$, $\bar{\varepsilon}(-1) = 1$.

By this choice for $\bar{r}(z)$, $\dot{\bar{V}}$ satisfies all conditions of (Teel, 1999), thus the global stability and disturbance attenuating properties are met. Since $\bar{r}(0) = 1$, the control law (37) in the neighborhoods of the origin converts to (26), so all local properties (optimality and α-stability) are met. ■

6. NUMERICAL EXAMPLE

Suppose nonlinear system (47),

$$\begin{cases} \dot{x}_1 = x_1^2 + x_2 + w \\ \dot{x}_2 = u \end{cases} \qquad (47)$$

and local H_∞ cost functional (48),

$$J_l = \int_0^\infty [x_1^2 + x_2^2 + u_l^2 - 5^2 w_l^2]\,dt \qquad (48)$$

and α-stability constraint with $\alpha = 4$.

Four different controllers are designed for this system.

- *Linear* H_∞: $u_1 = -1.06x_1 - 1.78x_2$ which places the linearized closed-loop poles at -0.89 ± j0.52.
- *Linear constrained (α-stabilizing) H_∞*: $u_2 = -20x_1 - 8x_2$ which places the linearized closed-loop poles at -4 ± j 0.91. -4 ± j 0.91
- *Nonlinear with local optimality*: u_3 see (Ezal, *et al.*, 2000).
- *Nonlinear with local constrained optimality*:

$$u_4 = \begin{cases} -(20x_1 + 8x_2 + 8x_1^2)(1 + \sigma);\ \sigma \ge 0 \\ -(20x_1 + 8x_2 + 8x_1^2) \qquad ;\ \sigma < 0 \end{cases}$$

Figures 1 and 2 show the phase trajectories and Regions of Attraction (RA) (hatched domain) of closed-loop systems with u_1 and u_2, respectively.

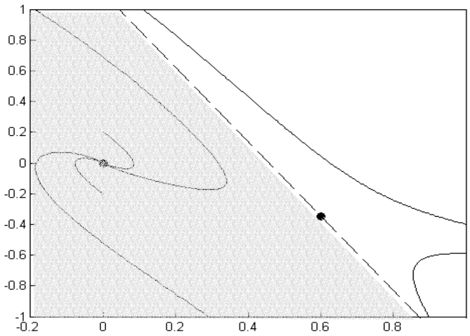

Fig. 1. Phase plane for linear optimal design

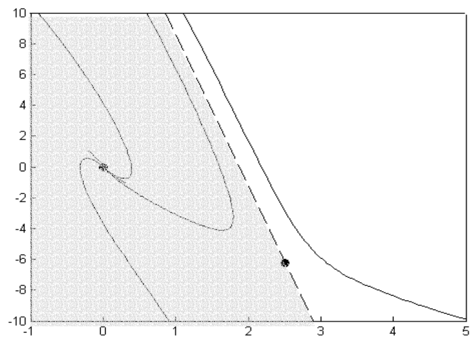

Fig. 2. Phase plane for linear constrained optimal design.

Obviously, u_2 leads to larger RA. It means that by using α -stability constraint in design procedure, the RA (where the linear part of system is dominant) becomes larger, thus the desired local performance (e.g. disturbance attenuation and rate of regulation) are aimed on the larger domain.

Figure 3 compares the responses of closed-loop systems with nonlinear controller u_3 and constrained nonlinear controller u_4, with no disturbance.

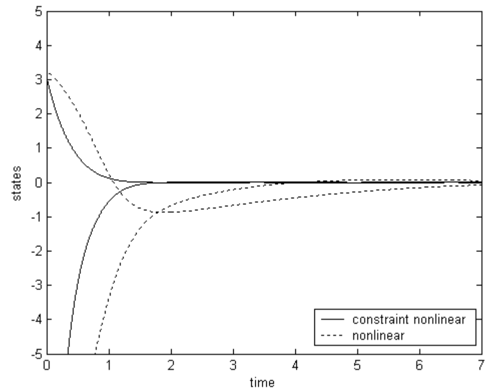

Fig. 3. states regulation for nonlinear designs without disturbance.

Note that, setting $\alpha = 4$ leads to $t_s \cong 1\sec$ (settling time) for linear systems, but the settling time in figure 3 (with constrained nonlinear controller) is little more than expected value, because the initial states values are not chosen on the RA of linear constrained and unconstrained controllers and the nonlinear term of (47) is excited.
Figure 4 shows the states behaviors in the presence of the L_2 disturbance, (49).

$$w(t) = \begin{cases} 0.5 & 3 \le t \le 5 \\ 0 & otherwise \end{cases} \qquad (49)$$

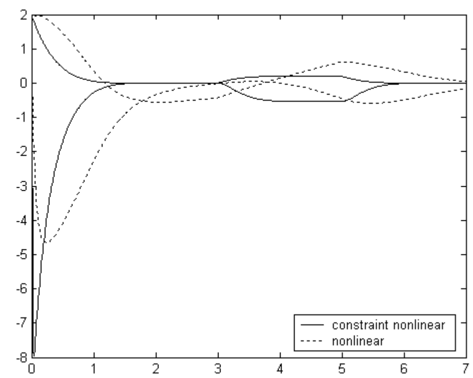

Fig. 4. States regulation for nonlinear designs with the energy-bounded disturbance.

The last two figures clearly illustrate the desired effect of entering α -stability constraints in the local optimality.

7. CONCLUDING REMARKS

Based on (Ezal, *et al*., 2000), the design procedure is introduced to achieve three goals: local optimality in the sense of H_∞, local α -stability and global stability. The linear backstepping procedure is employed to find linear transformation for the local goals and nonlinear backstepping procedure completes Diffeomorphism transformation. Finally through a theorem, the desired nonlinear control law for transformed system is obtained and the achievement of the goals is proved. Numerical example to illustrate the advantages of locally constrained optimal and globally stable design is employed.

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