STABILIZATION OF LTI TIME-DELAYED PROCESSES USING ANALYTICAL PID CONTROLLERS

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Abstract: The analytical proportional-integral-derivative (PID) controllers, the tuning of which depends on simple formulas with one adjustable parameter, have been developed for processes with time delay. In this paper, using a simple method called dual-locus diagram, the solutions to the problems of stabilizing stable, integrating and unstable processes with time delay using analytical PID controllers are presented, respectively. For the stable and integrating process, the stabilizing range of the adjustable parameter is only related to the delay of the plant, while, for the unstable process, it depends on both the time constant of the plant and the ratio of the delay to the time constant. With the available stabilizing solution, the time-consuming stability check can be avoided in the controller application.

Keywords: stability, proportional-integral-derivative (PID) control, processes with time delay, analytical.

1. INTRODUCTION

Despite great development of advanced control strategies and continual improvement in control theory, the majority of control systems in the industrial process are operated by PID controllers (Åström, 1995). The popularity of PID controllers stems from their robust performance in a wide range of operating conditions and functional simplicity, which allow process engineers to operate them in a simple and straightforward manner (Ming, 2002). In terms of the fact that practical requirements on the design of control systems are usually specified in terms of time-domain or frequency-domain response, such as overshoot and stability margin, the analytical PID controllers that can meet these indexes have been developed based on optimal control theory for the stable, integrating and unstable processes with time delay, respectively (Zhang, 2002a). The analytical PID controller of this kind has only one adjustable parameter and can provide the quantitative time-domain and frequency-domain responses.

Although the analytical PID controllers can be operated more conveniently and provide good performance, they may fail to stabilize the original time-delayed plants in spite of their complete stabilization for the approximated plants since the rational approximation is employed in the controller design. Hence, certain constraint has to be imposed on the single adjustable parameter. However, it is not a trivial task to analyze the stability of the process with time delay since the corresponding closed-loop system has an infinite number of poles. Recently, by using the extended Hermite-Biehler Theorem, the complete set of the classical PID controllers for the first-order plants with time delay have been derived (Silva *et al.*, 2002). Whereas, the method on the basis of the extended Hermite-Biehler Theorem is complex and there are great difficulties to achieve the stabilizing range of the adjustable parameter by taking this method. The D-partition technique has been used to derive the complete set of stabilizing PID controller parameters for first-order plus time-delayed unstable processes (Hwang, 2004), but it is inapplicable to the case of the analytical PID controller with single parameter since the nature of the D-partition technique is to determine the boundaries of the stability region by three equations. Though the method developed by Xu (2003) can be used to compute the stabilizing region of the PID controller parameter, the analytical results and the relation between the stabilizing region and the plant parameters will not be derived. Compared with these above-mentioned approaches, the analytical stabilizing criterion called dual-locus diagram offers remarkable simplicity and ease of mathematical calculations. The dual-locus diagram was proposed by Satche (1949) and then developed by Smith (1958). Using the dual-locus, Olgac (1995) discussed the design of delayed resonators as tunable active vibration absorbers. Zhong (2003) applied it to the robust stability analysis of simple systems controlled over communication networks.

In this paper, the dual-locus diagram method is employed to analyze the problems of stabilizing processes with time delay using analytical PID controllers. The analytical ranges of the adjustable parameters for which the analytical PID controllers can stabilize stable, integrating and unstable processes with time delay are respectively presented. With the analytical stabilizing range of the adjustable parameter being available for a given process, it can avoid the time-consuming stability check in controller application and thereby to save the controller tuning time.

2. PRELIMINARY KNOWLEDGE FOR DUAL-LOCUS DIAGRAM METHOD

The dual-locus diagram method is based on the following argument principle.

Lemma 1 (Argument principle (Roger, 2000)): Let a function f be meromorphic in the domain interior to a positively oriented simple closed contour C, and suppose that f is analytic and nonzero on C. If, counting multiplicities, Z is the number of zeros and P is the number of poles inside C, then the number of times $f(s)$ winds around the origin is

$$n(f(s),0)=\frac{1}{2\pi}\Delta_C \arg f(s)=Z-P$$

where $\Delta_C \arg f(s)$ represents the variation of the argument of $f(s)$ along the contour C.

The characteristic equation of the closed-loop system is usually written in the form

$$1+F(s)=0 \tag{1}$$

where $F(s)$ is the open-looped transfer function. Since the closed-loop system has to remain stable, the number of the right-half plane poles of the closed-loop transfer function is zero (i.e. $P=0$). Here, the closed contour C is the Nyquist contour. By appropriate separation of terms, the characteristic equation (1) can be rearranged as the following form:

$$F_1(s)=F_2(s) \tag{2}$$

Then, the dual-locus diagram with respect to $F_1(s)$ and $F_2(s)$ is obtained when s traverses the Nyquist contour. The argument of $F_1(s)-F_2(s)$ is the angle between the vector joining the corresponding points on the Nyquist plots of $F_1(s)$ and $F_2(s)$, and the positive real axis. According to the argument principle, the system is stable (i.e. $Z=0$) if and only if the variation of the argument of $F_1(s)-F_2(s)$ is zero.

Correspondingly, for the plants with time delay, the characteristic equation may be transformed into

$$H(s)=-e^{\theta s} \tag{3}$$

where $H(s)$ is the delay-free open-loop transfer function and θ is the time delay involved in the plants. Denote as m and n the degrees of the numerator and denominator of $H(s)$, respectively. Here, $H(s)$ and $-e^{\theta s}$ correspond to $F_1(s)$ and $F_2(s)$ in Eq. (2), respectively. Hence, for the plants with time delay, the closed-loop system is stable if and only if the variation of the argument of $H(s)+e^{\theta s}$ is zero. Equivalently, the criterion of the dual-locus diagram is simply illustrated as follows:

Corollary 1: If the following conditions are satisfied, the system is stable. Otherwise, it is unstable.
1) $n>m$, or, $|b_m/a_n|<1$ for $n=m$, where a_n and b_m are leading coefficients of the numerator and denominator of $H(s)$, respectively.
2) Either the loci of $H(s)$ and $-e^{\theta s}$ have no intersection or the locus of $H(s)$ arrives at the point of intersection earlier than that of $-e^{\theta s}$ if the two loci intersect.

It is seen that the dual-locus diagram method for the systems with time delay employs a counterclockwise unity circle prepared beforehand and only requires the plotting of a simple curve derived from the open-looped transfer function free of delay. Thus, the criterion presents remarkable simplicity and ease of mathematical calculations over other available conventional methods.

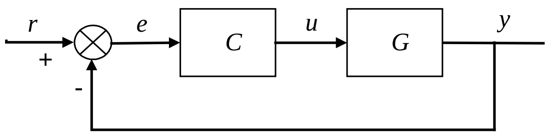

Fig. 1. Feedback control system equation

3. STABILIZATION USING ANALYTICAL PID CONTROLLERS

For the purpose of controller design and system analysis, practical industrial processes are frequently expressed as first-order and second-order time-delayed models. Since the stability analysis process of the analytical PID controller for second-order plants with time delay is similar to the case of first-order, only the first-order plants with time delay are considered in this paper. The feedback control system is shown in Fig.1 where $G(s)$ is the plant to be controlled, $C(s)$ is the analytical PID controller, r is the setpoint, and y is the output of the plant. The analytical PID controller is of the following type

$$C(s)=K_C\left(1+\frac{1}{T_I s}+T_D s\right)\frac{1}{T_F s+1} \tag{4}$$

where K_C, T_I, T_D and T_F are the functions with respect to λ if we denote as λ the adjustable positive parameter of the analytical PID controller.

The cases of stable, integrating and unstable processes with time delay are considered as follows.

3.1 Stable processes with time delay

For the first-order stable plant with the following transfer function

$$G(s)=\frac{k}{\tau s+1}e^{-\theta s} \tag{5}$$

K_C , T_I , T_D and T_F of the analytical PID controller $C(s)$ in (4) are respectively written as (Zhang, 1996):

$$T_F=\frac{\lambda^2}{2\lambda+\theta/2},\quad T_I=\tau+\frac{\theta}{2}$$

$$T_D=\frac{\theta\tau}{2T_I},\quad K_C=\frac{T_I}{K(2\lambda+\theta/2)}$$

The characteristic equation of the system in Fig.1 is given by

$$1+C(s)G(s)=1+\frac{(1+\theta s/2)}{\lambda^2s^2+(2\lambda+\theta/2)s}e^{-\theta s}=0 \tag{6}$$

It is equivalent to

$$H(s)=-e^{\theta s} \tag{7}$$

where

$$H(s)=\frac{2+\theta s}{2\lambda^2s^2+(4\lambda+\theta)s} \tag{8}$$

The loci of $H(s)$ and $-e^{\theta s}$ are shown in Fig.2. When ω increases from 0 to $+\infty$, the locus of $-e^{\theta s}$ is the counterclockwise unity circle starting at (-1,0) and the locus of $H(s)$ is a curve from the bottom up in Fig.2. The locus of $H(s)$ corresponding to $\omega=-\infty\sim 0$ is symmetric with respect to the real axis and is thus not taken into account. Since the transfer function $H(s)$ has one pole at the origin, its Nyquist plot shifts clockwise from $\pi/2$ to $-\pi/2$ with the infinite radius when s changes from $-j0$ to $j0$. Firstly, compute the critical frequency ω_c at which the locus of $H(s)$ intersects with the unity circle.

$$\left|\frac{\theta\omega_c j+2}{2\lambda^2(\omega_c j)^2+(4\lambda+\theta)\omega_c j}\right|=1 \tag{9}$$

Simplifying (9) yields

$$4\lambda^4\omega_c^{\,4}+(16\lambda^2+8\lambda\theta)\omega_c^{\,2}-4=0 \tag{10}$$

Eq.(10) has four analytical roots, of which the valid solution for ω_c is

$$\omega_c=\frac{\sqrt{\sqrt{5\beta^4+4\beta^3+\beta^2}-2\beta^2-\beta}}{\beta^2\theta} \tag{11}$$

where $\beta=\lambda/\theta$. The other three roots are either negative or complex values. The phase angle of $H(s)$ at ω_c is

$$\varphi_1=\arctan(\theta\omega_c/2)+\arctan[(4\lambda+\theta)/(2\lambda^2\omega)]+\pi \tag{12}$$

and the phase angle of $-e^{\theta s}$ at ω_c is

$$\varphi_2=\pi+\theta\omega_c \tag{13}$$

From Fig.2, it is seen that the stability condition that the locus of $H(s)$ arrives at the intersection earlier than that of $-e^{\theta s}$ can be satisfied only when the phase angle of $H(s)$ at ω_c is larger than that of $-e^{\theta s}$ (i.e. $\varphi_1-\varphi_2>0$). Taking $\varphi_1-\varphi_2=0$ and substituting (11)-(13) into it yield

$$\varphi_1-\varphi_2=\arctan(\theta\omega_c/2)+\arctan[(4\lambda+\theta)/(2\lambda^2\omega_c)]-\theta\omega_c \tag{14}$$

It is clear that Eq.(14) is only related to β (i.e. λ/θ). The solution $\lambda/\theta=0.0735$ of Eq.(14) can be obtained by using the function *fzero* of the Matlab. Fig.3 shows that the requirement $\varphi_1-\varphi_2>0$ can just be satisfied only if $\lambda/\theta>0.0735$. Moreover, from (8), it is known that the denominator degree of $H(s)$ is larger than its numerator. Therefore, in terms of the criterion of dual-locus diagram in

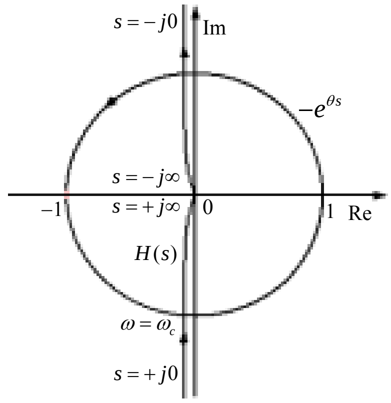

Fig. 2 The loci of $H(s)$ in (8) and $-e^{j\omega\theta}$

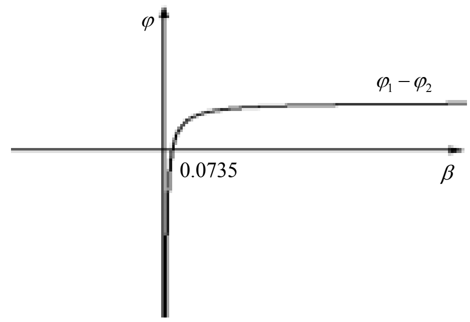

Fig.3 Plot of $\varphi_1-\varphi_2$ in Eq.(14) as a function of β

Corollary 1, the range of the adjustable parameter λ for which the analytical PID controller can stabilize the first-order stable plant with time delay is that $\lambda>0.0735\theta$.

3.2 Integrating process with time delay

The model of the integrating process with time delay is described as

$$G(s)=\frac{k}{s}e^{-\theta s} \tag{15}$$

If the approximation $e^{-\theta s}=(1-\theta s/2)/(1+\theta s)$ is adopted in Zhang (1999), the analytical PID controller designed for the plant with transfer function (15) has the following parameters

$$T_F = \frac{4\lambda^3}{12\lambda^2 + 6\lambda\theta + \theta^2}, \quad T_I = 3\lambda + \theta$$

$$T_D = \frac{6\lambda\theta + \theta^2}{4T_I}, \quad K_C = \frac{1}{K}\frac{4T_I}{12\lambda^2 + 6\lambda\theta + \theta^2}$$

The characteristic equation of the system in Fig.1 is given by

$$1 + \frac{(6\lambda\theta + \theta^2)s^2 + (12\lambda + 4\theta)s + 4}{4\lambda^3 s^3 + (12\lambda^2 + 6\lambda\theta + \theta^2)s^2} e^{-\theta s} = 0 \quad (16)$$

In terms of Eq. (3), we have

$$H(s) = \frac{(6\lambda\theta + \theta^2)s^2 + (12\lambda + 4\theta)s + 4}{4\lambda^3 s^3 + (12\lambda^2 + 6\lambda\theta + \theta^2)s^2} \quad (17)$$

The loci of $H(s)$ and $-e^{\theta s}$ are shown in Fig.4. Since the transfer function $H(s)$ has two poles at the origin, its Nyquist plot shifts clockwise from π to $-\pi$ with the infinite radius when s changes from

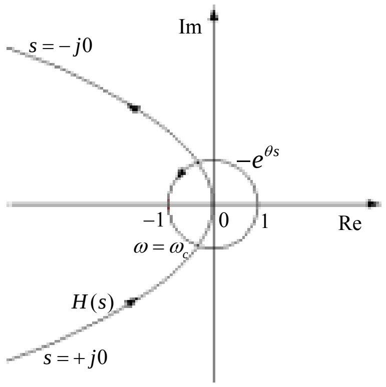

Fig. 4. The loci of $H(s)$ in(18) and $-e^{j\omega\theta}$

$-j0$ to $j0$. Similar to the case of the stable process, only the positive values of ω are considered .The frequency ω_c satisfies the equation

$$|H(j\omega_c)| = \left| \frac{(6\lambda\theta + \theta^2)(j\omega_c)^2 + j\omega(12\lambda + 4\theta) + 4}{4\lambda^3 (j\omega)^3 + (12\lambda^2 + 6\lambda\theta + \theta^2)(j\omega)^2} \right| = 1 \quad (18)$$

Simplifying (18), we have

$$a\omega_c^6 + b\omega_c^4 + c\omega_c^2 - 1 = 0 \quad (19)$$

where

$$a = \lambda^6$$
$$b = 3(6\lambda^2 + 6\lambda\theta + \theta^2)/2$$
$$c = -(18\lambda^2 + 6\lambda\theta + \theta^2)/2$$

Eq. (19) has six analytical roots, among which the valid solution for ω_c is

$$\omega_c = \sqrt{\sqrt[3]{d+e} + \sqrt[3]{d-e} - \frac{b}{3a}} \quad (20)$$

where

$$d = -\frac{q}{2} \text{ and } e = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \quad (21)$$

p and q in (21) are given by the following equations, respectively.

$$p = \frac{c}{a} - \frac{b^2}{3a^2} \quad \text{and} \quad q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} - \frac{1}{a} \quad (22)$$

The other five roots of (19) are either negative or complex values. Assume that $\beta = \lambda/\theta$, then the Eq.(20) can be rewritten as

$$\omega_c = f(\beta)/\theta \quad (23)$$

where $f(\beta)$ denotes the function with respect to β. The phase angle of $H(s)$ at ω_c is

$$\varphi_1 = \arctan\left[\frac{(12\lambda + 4\theta)\omega_c}{4 - (6\lambda\theta + \theta^2)\omega_c^2}\right] - \arctan\left(\frac{4\lambda^3\omega_c}{12\lambda^2 + 6\lambda\theta + \theta^2}\right) + \pi \quad (24)$$

Following the similar lines as in the case of the stable process, the system is stable if and only if the phase angle of $H(j\omega)$ at ω_c is larger than that of $-e^{j\omega\theta}$, i.e. $\varphi_1 - \pi - \theta\omega_c > 0$. Substituting (20), (24) and $\lambda = \beta\theta$ into the equation $\varphi_1 - \pi - \theta\omega_c = 0$, the following equation is obtained.

$$\arctan\left\{\frac{(12\beta + 4)f(\beta)}{[4 - (6\beta + 1)][f(\beta)]^2}\right\} - \arctan\left(\frac{4\beta^3 f(\beta)}{12\beta^2 + 6\beta + 1}\right) - f(\beta) = 0 \quad (25)$$

It is clear that Eq.(25) is only related to β (i.e. λ/θ). The solution of Eq.(25) is that $\lambda/\theta = 0.3614$, so the requirement $\varphi_1 - \varphi_2 > 0$ can be satisfied only if $\lambda/\theta > 0.3614$. Thus, in order to stabilize the integrating plant with the transfer function in (15) using the analytical PID controller, the value of the adjustable parameter λ must be larger than 0.3614θ.

3.3 Unstable process with time delay

For the unstable process with the following transfer function

$$G(s) = \frac{k}{\tau s - 1} e^{-\theta s} \quad (26)$$

the values of K_C, T_I, T_D and T_F in the analytical PID controller (4) are presented by (Zhang, 2002b).

$$T_F = 0, \quad T_I = \frac{\lambda^2 + 2\lambda\tau + \theta\tau}{\tau - \theta}$$

$$T_D = 0, \quad K_C = \frac{\lambda^2 + 2\lambda\tau + \theta\tau}{K(\lambda + \theta)^2}$$

The closed-loop charact eristic equation is written as:

$$1 + \frac{[(\lambda^2 + 2\lambda\tau + \theta\tau)s + \tau - \theta]e^{-\theta s}}{(\lambda + \theta)^2(\tau s - 1)s} = 0 \quad (27)$$

It can be rearranged as

$$H(s) = \frac{(\lambda^2 + 2\lambda\tau + \theta\tau)s + \tau - \theta}{(\lambda + \theta)^2(\tau s - 1)s} \quad (28)$$

In terms of $|H(j\omega_c)| = 1$, we have

$$(\lambda + \theta)^4\tau^2\omega_c^4 + [(\lambda + \theta)^4 - (\lambda^2 + 2\lambda\tau + \theta\tau)^2]\omega_c^2 - (\tau - \theta)^2 = 0 \quad (29)$$

Taking $\beta = \lambda / \tau$ and $m = \theta / \tau$, the valid solution for ω_c can be obtained from Eq. (29).

$$\omega_c = \frac{1}{\tau}\sqrt{\frac{-b+\sqrt{b^2-4ac}}{2a}} \tag{30}$$

where

$$a = (\beta + m)^4$$
$$b = (\beta + m)^4 - (\beta^2 + 2\beta m + m)^2$$
$$c = -(1-m)^2$$

The phase angle of $H(s)$ at ω_c is

$$\varphi_1 = \arctan\left[\frac{(\beta^2+2\beta+m)\tau\omega_c}{1-m}\right] + \arctan(\tau\omega_c) + \frac{\pi}{2} \tag{31}$$

and the phase angle of $-e^{\theta s}$ at ω_c is given in (13). Assume that $\varphi_1 - \varphi_2 = 0$ for $\beta = \beta_0$. Then, from (30) and (31), we have

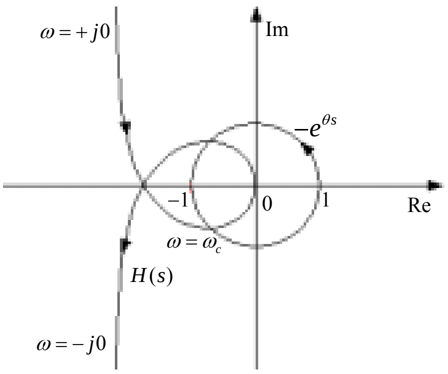

Fig. 5. The loci of $H(s)$ in (28) and $-e^{j\omega\theta}$

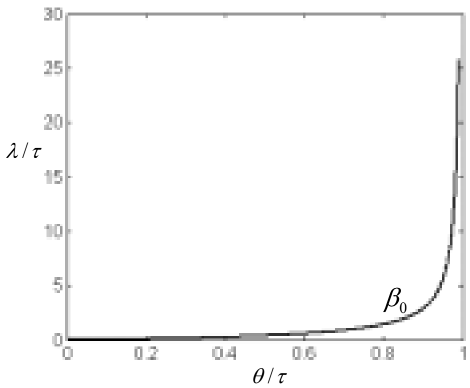

Fig.6 the relation between λ and θ / τ

$$\arctan\left[\frac{({\beta_0}^2+2\beta_0+m)\sqrt{-b_0+\sqrt{{b_0}^2-4a_0c_0}}}{\sqrt{2a_0}(1-m)}\right] + \arctan\left(\sqrt{\frac{-b_0+\sqrt{{b_0}^2-4a_0c_0}}{2a_0}}\right) - \frac{m\sqrt{-b_0+\sqrt{{b_0}^2-4a_0c_0}}}{\sqrt{2a_0}} - \frac{\pi}{2} = 0 \tag{32}$$

where a_0, b_0 and c_0 represent the corresponding values of a, b and c for $\beta = \beta_0$, respectively. From Eq.(32), the plot of β_0 as the function of θ / τ can be obtained, which is shown in Fig.6.it is known that the solution of β only depends on m (i.e. θ / τ). In order to satisfy the requirement that the locus of $H(s)$ arrive at the point of intersection earlier than $-e^{\theta s}$, the phase angle φ_1 must be larger than φ_2, which is shown in Fig.5. Thus, the analytical PID controller is able to stabilize the unstable process with time delay only if the value of λ / τ is larger than the lower boundary β_0. The stabilizing region of the adjustable parameter for which the analytical PID controller can stabilize the first-order unstable process with time delay is related both θ and τ. For a fixed value of θ / τ, the stabilizing range of the adjustable parameter λ is given by $\lambda > \tau\beta_0$, where β_0 is the solution of Eq.(32). The stabilizing solution denominates not all the unstable plants with time delay can be stabilized by the analytical PID controller and only the unstable plant with $\theta / \tau < 1$ can do so, which is shown in Fig.6.

4. CONCLUSIONS

In this paper, the problems of stabilizing processes with time delay using analytical PID controllers have been considered. By employing the dual-locus diagram method, the stabilizing range of the adjustable parameter in the analytical PID controller is respectively presented for three different cases: stable, integrating and unstable time-delayed processes. For the stable and integrating processes, the stabilizing range depends on the delay of the plant, and for the unstable processes, it is related to both the delay and time constant of the plant. The presented stabilizing solutions provide convenience for the tuning of the analytical PID controller.

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REFERENCES

Åström, K. J. and Haägglund, T (1995), *PID controllers: theory, design, and tuning*, Research Triangle Park, NC, USA.

Hwang, C. and Hwang, J. H. (2004). Stabilization of first-order plus dead-time unstable processes using PID controllers. *Control Theory and Applications, IEE Proceedings-*, **151**, 89–94.

Ming, T. H. and Chia, Y. L. (2002). PID controller design for robust performance. *Proceedings of the 41st IEEE Conference on Decision and Control*, **1**, 1062-1067.

Olgac, N. and Holm-Hansen, B. (1995). Design considerations for delayed-resonator vibration absorbers. *Journal of Engineering Mechanics*, **121**, 80-89.

Roger, L, *Lecture note on complex variables*, available at http://pc53. math.ntnu.edu.tw/~yclin/02a/cx/cx20.pdf.

Sathe, M. (1949). Discussion on 'stability of linear oscillating systems with constant time lag'. *Journal of Applied Mechanics, Transactions of the ASME*, **16**, 419-420.

Silva, G. J. Datta, A. and Bhattacharyya, S. P. (2002). New results on the synthesis of PID controllers. *IEEE Transactions on Automatic Control*, **47**, 241-252.

Smith, O. J. M. (1958). *Feedback Control Systems*. New York: McGraw-Hill Book Company, Inc.

Xu, H., Datta, A. and Bhattacharyya, S. P. (2003). PID stabilization of LTI plants with time-delay. *Proceedings of the IEEE Conference on Decision and Control*, **4**, 4038-4043.

Zhang, W. D., Sun, Y. X. and Xu, X. M. (1996). Modified PID controller based on H_∞ theory. *Journal of Engineering and Applied Science*, 9-12.

Zhang,W. D., Xu ,X. M. and Sun, Y. X. (1999). Quantitative control of integrating processes with time delay. *Automatica*, **35**, 719-723.

Zhang, W. D., Xi, Y. G. and Xu, X. M. (2002). Design PID Controllers for Desired Time Domain and Frequency Domain Response. *ISA Transactions*, **41**, 511-520.

Zhang, W. D. and Xu, X. M. (2002). H Infinity PID Controller Design for Runaway Processes with Time Delay. *ISA Transactions*, **41,** 317-322.

Zhong, Q. C. (2003). Robust stability analysis of simple control algorithms in communication networks using dual-locus diagram. *Automatica*, **39**, 1309-1312.