DESIGN OF NONLINEAR OBSERVERS USING POPOV'S CRITERION

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Abstract: Recently there has been an intense activity in designing observers by means of the circle criterion. However, it is known that the use of Popov's criterion reduces the conservativeness of stability analysis based on the circle criterion, if further restrictions are satisfied by the nonlinearity. It is shown in this paper that Popov's criterion can indeed be used with advantage for observer design, reducing the restrictions of previous designs. Although only the scalar case will be considered here, it is possible to extend the results to a multivariable setting.

Keywords: Nonlinear Observers; Detectability; Nonlinear observer design methods; Popov's Criterion.

1. INTRODUCTION

An intensive research activity has been done in the last years aiming at developing design strategies for nonlinear observers, and different methods have been proposed (Misawa and Hedrick, 1989; Nijmeijer and Fossen, 1999). Recently (Arcak and Kokotovic, 1999; Fan and Arcak, 2003) have proposed the use of the circle criterion for the design of nonlinear observers, when the estimation error of the observer can be decomposed in a linear dynamical subsystem and a nonlinear static feedback, i.e. the class of Lur'e systems. The influence of the plant's dynamics on the error equation enters through the nonlinear term, that has to be considered as time-varying, and the time variance is dependent on the plant's trajectory. The basic design idea consists in proposing a quadratic Lyapunov function for the error equation, and to check if this can be so selected that the nonlinear part does not destroy the stability of the equilibrium point. This is the idea of the classical circle criterion (Khalil, 2002). This basic idea can be further generalized in different directions (Moreno, 2004*a*; Moreno, 2004*b*).

It is a classical result, that if the nonlinearity is time invariant the conditions of the Popov criterion are much weaker than those of the circle criterion, leading to less restrictive conditions. From a Lyapunov function perspective the novelty of the Popov criterion is the use of a Lur'e type of Lyapunov functions that consist of a quadratic term plus the integral of system's nonlinearity. This extended class of Lyapunov functions allows a weakening of the stability conditions.

However, the use of the classical Popov criterion for observer design is impossibilited by the fact that the nonlinearity in the error equation is always time-varying, whereas Popov criterion is only valid for time invariant nonlinearities. This is maybe the reason why it has not been yet used for this objective.

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The objective of this work is to show that, despite of the difficulty, it is possible and advantageous to use an extended Popov criterion for the design of nonlinear observers for a class of nonlinear systems. The basic idea consists in proposing Lyapunov function candidates of Lur'e type for the error dynamics of the observation problem. Since the the nonlinearity is time-varying in this case, ideas similar to the classical results on extensions of Popov's criterion to time-varying nonlinearities (Willems, 1967; Narendra and Taylor, 1973; Willems, 1970; Bliman, 1999) can be used. Basically, all these results rely on restricting not only the sector of the nonlinearity, but also its time-variation. A new aspect here is the fact that the time-variation of the nonlinearity of the error system depends on the plant's state dynamics. And so the use of Popov's criterion for the observer design will require to use information on the plant's dynamics. This is a very interesting feature, that cannot be reached by the circle criterion, for which the plant's dynamics is not taken into account.

In fact a major motivation for this work is the possibility of considering Lyapunov functions that are not only functions of the error dynamics, but also that are dependent on the plant's dynamics. This allows the use of the particular dynamic characteristics of the plant for the design of the observer. Recall that most observer design methods rely on Lyapunov functions that are functions exclusively of the error system state. So for example, the high-gain method (Gauthier *et al.*, 1992; Gauthier and Kupka, 2001) uses quadratic Lyapunov functions in the error state.

To simplify the presentation only the case of a scalar nonlinearity will be considered, although the same ideas can be extended easily to the multivariable one. The rest of the paper is organized in the following form. In the next Section the class of systems and the problem to be solved in this work is introduced. The proposed method is described in Section 3. The geometric interpretation of the design conditions is given in Section 4, and in Section 5 an illustrative example is presented.

2. PROBLEM FORMULATION

Consider a plant that can be brought to the form

$$\Sigma : \begin{cases} \dot{x} = Ax + G\psi(\sigma) + \varphi(t, y, u) \ , \quad x(0) = x_0 \\ y = Cx \ , \\ \sigma = Hx \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is a known input, $y \in \mathbb{R}^p$ is the measured output, and $\sigma \in \mathbb{R}$ is a (not necessarily measured) linear function of the state. $\varphi(t, y, u)$ is an arbitrary nonlinear function of the time, the input and the output. $\psi(\sigma)$ is a scalar function that depends on σ. ψ and φ are assumed to be locally Lipschitz in σ or y, continuous in u, and piecewise continuous in t. Since the plant Σ is not assumed globally Lipschitz the global existence of solutions is not guaranteed, i.e. for some initial conditions and inputs finite escape time is possible. This is a not desirable situation and will be excluded by assuming that Σ (1) is either *complete*, i.e. the state trajectories $x(t)$ are defined for every $t \geq 0$, every initial condition $x_0 \in \mathbb{R}^n$ and every input $u(\cdot) \in \mathcal{U}$, or that the initial states and/or inputs are so restricted that the state trajectory is locally bounded, i.e. $x(t) \in \mathcal{L}_{\infty e}$.

An observer for Σ of the form

$$\Omega : \begin{cases} \dot{\hat{x}} = A\hat{x} + L(\hat{y} - y) + G\psi(\hat{\sigma} + N(\hat{y} - y)) + \\ \qquad +\varphi(t, y, u) \ , \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y} = C\hat{x} \ , \\ \hat{\sigma} = H\hat{x} \end{cases} \tag{2}$$

is proposed, where matrices $L \in \mathbb{R}^{n \times p}$, and $N \in \mathbb{R}^{1 \times p}$ have to be designed. Defining the state estimation error $e \triangleq \hat{x} - x$, the output estimation error $\tilde{y} \triangleq \hat{y} - y$, and the function estimation error $\tilde{\sigma} \triangleq \hat{\sigma} - \sigma$, the dynamics of e is given by

$$\begin{aligned} &\dot{e} = (A + LC)e - G[\psi(\sigma) - \psi(\hat{\sigma} + N\tilde{y})] \ , \\ &\tilde{y} = Ce \ , \qquad e(0) = e_0 = \hat{x}_0 - x_0 \\ &\tilde{\sigma} = He \ . \end{aligned}$$

Note that $\hat{\sigma} + N\tilde{y} = H\hat{x} + NCe = Hx + He + NCe = \sigma + (H + NC)e$. Defining

$$z \triangleq (H + NC)e = \tilde{\sigma} + N\tilde{y}$$
$$\phi(z, \sigma) \triangleq \psi(\sigma) - \psi(\sigma + z) \ , \tag{3}$$

the dynamics of the error can be written as

$$\Xi : \begin{cases} \dot{e} = A_L e + G\nu \ , \quad e(0) = e_0 \\ z = H_N e \ , \\ \nu = -\phi(z, \sigma) \end{cases} \tag{4}$$

where $A_L \triangleq A + LC$, and $H_N \triangleq H + NC$. Note that $\phi(0, \sigma) = 0$ for all σ and u. The error dynamics (4) is not autonomous, as in the linear case, but it is driven by the plant (1) through the linear function of the state $\sigma = Hx$. ϕ is therefore a time varying nonlinearity, whose time variation depends on the state trajectory of the plant. In fact the error dynamics Ξ is driven by the plant Σ.

The aim is to find matrices L and N such that for every initial state of the error system e_0 and every initial state of the plant x_0, and any input u the state of the error equation $e \to 0$ as $t \to \infty$. This corresponds for the composite system $\Sigma - \Xi$ to the globally asymptotically stability of e uniformly in x_0 and u, i.e. to a concept of partial stability (Chellaboina and Haddad, 2002).

It will be assumed that the memoryless function $\phi(z, \sigma)$, that is given by the problem data, belongs to a sector $[0, k]$ with respect to z. This means that (see (Khalil, 2002)) for some $k \in \mathbb{R}$, $k > 0$.

$$\phi(z, \sigma) \left[\frac{1}{k} \phi(z, \sigma) - z \right] \leq 0 \ , \tag{5}$$

for all (z, σ). In case $k = \infty$, i.e. for a non Lipschitz nonlinearity, the sector condition (5) becomes $z\phi(z, \sigma) \geq 0$.

3. OBSERVER DESIGN METHOD

To design the observer consider the (partial) Lyapunov function candidate of Lur'e type for the composite system $\Sigma - \Xi$

$$V(e, \sigma) = e^T P e + 2m_l \int_0^z \phi(\xi, \sigma) \, d\xi + \tag{6}$$
$$+ 2m_u \int_0^z \left[\xi - \frac{1}{k} \phi(\xi, \sigma) \right] d\xi \ ,$$

with $P = P^T > 0$ a positive definite matrix and $m_l, m_u \geq 0$. Since $P > 0$ and $\phi(z, \sigma)$ satisfies (5) it follows easily that V is positive definite and radially unbounded with respect to e, uniformly in σ. To assure that this function is decrescent in e uniformly in σ consider the Jacobian

$$\frac{\partial V(e, \sigma)}{\partial e} = 2e^T P + 2m_l \phi(z, \sigma) H_N +$$
$$+ 2m_u \left[z - \frac{1}{k} \phi(z, \sigma) \right] H_N \ .$$

If this is a bounded function of σ for every fixed (with finite norm) e, then V is decrescent (Narendra and Taylor, 1973, Ch. 3, Lemma A).

The time derivative of V along the solutions of the error system Ξ is given by

$$\dot{V} = \begin{bmatrix} e \\ \phi \end{bmatrix}^T \begin{bmatrix} S & R^T \\ R & Q \end{bmatrix} \begin{bmatrix} e \\ \phi \end{bmatrix} + 2m \int_0^z \frac{\partial \phi(\xi, \sigma)}{\partial \sigma} d\xi \dot{\sigma} \ ,$$

with

$$S \triangleq \Pi A_L + A_L^T \Pi \ , \ Q \triangleq -m \left(H_N G + G^T H_N^T \right) \ ,$$
$$R^T \triangleq -\Pi G + m A_L^T H_N^T \ ,$$
$$\Pi \triangleq P + m_u H_N^T H_N \ , \ m \triangleq m_l - \frac{m_u}{k} \ .$$

Note that the sector condition of the nonlinearity (5) can be written as

$$\begin{bmatrix} e \\ \phi \end{bmatrix}^T \begin{bmatrix} 0 & H_N^T \\ H_N & -\frac{2}{k} \end{bmatrix} \begin{bmatrix} e \\ \phi \end{bmatrix} \geq 0 \ .$$

Adding these two expressions one obtains

$$\dot{V} \leq \begin{bmatrix} e \\ \phi \end{bmatrix}^T \begin{bmatrix} S & R^T + H_N^T \\ R + H_N \ , & -\frac{2}{k} + Q \end{bmatrix} \begin{bmatrix} e \\ \phi \end{bmatrix} + \tag{7}$$
$$+ 2m \int_0^z \frac{\partial \phi(\xi, \sigma)}{\partial \sigma} d\xi \ \dot{\sigma}$$

Now, define

$$\varrho(z, \sigma) \triangleq \frac{\partial \phi(z, \sigma)}{\partial \sigma} = \psi'(\sigma) - \psi'(\sigma + z) \ , \tag{8}$$

that satisfies $\varrho(0, \sigma) = 0$, for every σ. The nonlinearity satisfies

$$\int_0^z \frac{\partial \phi(\xi, \sigma)}{\partial \sigma} d\xi = \left[\psi'(\sigma) H_N \ , \ 1 \right] \begin{bmatrix} e \\ \phi \end{bmatrix} \ .$$

Note that this integral is "linear" in (e, ϕ). However, this is difficult to compensate in the derivative of the Lyapunov function, since it is not sign defined, and since a linear term dominates the quadratic ones in the Lyapunov expression.

We look therefore for a quadratic representation of the integral term. This is possible under some assumptions, as the following paragraphs clarify. First, a "linear" representation of the integrand is obtained. By the mean value theorem, if $\varrho(z, \sigma)$ is a continuously differentiable function, it follows (Vidyasagar, 1993) that there exists a continuous $F(z, \sigma)$ such that

$$\varrho(z, \sigma) = F(z, \sigma) z \ ,$$

and

$$F(z, \sigma) = \int_0^1 \frac{\partial \varrho(\lambda z, \sigma)}{\partial z} d\lambda \ , \tag{9}$$

is an explicit representation of the function. Since

$$\frac{\partial \varrho(z, \sigma)}{\partial z} = -\psi''(\sigma + z) \ ,$$

it follows that

$$F(z, \sigma) = \begin{cases} \frac{\varrho(z, \sigma)}{z} \ , & \text{for } z \neq 0 \\ -\psi''(\sigma) \ , & \text{for } z = 0 \end{cases} \ .$$

If $F(z, \sigma)$ is upper and lower bounded, i.e. $\exists \ \alpha, \beta \in \mathbb{R}$ so that

$$\alpha \leq F(z, \sigma) \leq \beta \ , \ \forall \ z, \sigma \ , \tag{10}$$

then the searched quadratic representation for the integral term is given by

$$\frac{1}{2} \alpha z^2 \leq \int_0^z \varrho(\xi, \sigma) \, d\xi \leq \frac{1}{2} \beta z^2 \ .$$

Furthermore, if the time derivative of σ is also upper and lower bounded, i.e. $\exists \ a, b \in \mathbb{R}$ so that

$$a \leq \dot{\sigma} \leq b \ , \tag{11}$$

then there exist $\gamma, \delta \in \mathbb{R}$ so that

$$\frac{1}{2} \gamma z^2 \leq \int_0^z \varrho(\xi, \sigma) \, d\xi \ \dot{\sigma} \leq \frac{1}{2} \delta z^2 \ . \tag{12}$$

So, for example, if $\alpha = -\beta$, and $a = -b$, then $\gamma = -\delta = -b\beta$.

Since

$$\left(\gamma m_l - \delta \frac{m_u}{k} \right) z^2 \leq 2m \int_0^z \varrho(\xi, \sigma) \, d\xi \ \dot{\sigma} \leq \mu z^2 \ ,$$

with $\mu \triangleq \left(\delta m_l - \gamma \frac{m_u}{k} \right)$, then (7) becomes

$$\dot{V} \leq \begin{bmatrix} e \\ \phi \end{bmatrix}^T \begin{bmatrix} S + \mu H_N^T H_N \ , & R^T + H_N^T \\ R + H_N & -\frac{2}{k} + Q \end{bmatrix} \begin{bmatrix} e \\ \phi \end{bmatrix} \ .$$

The main result of the paper is the following

Theorem 1. Consider the plant Σ (1). Suppose that $F(z,\sigma)$ (9) is bounded (10), and that $d\sigma/dt$ is also bounded (11). If there exist matrices $P = P^T > 0$, L, N, non negative scalars m_l, m_u, and a positive scalar $\epsilon > 0$ such that

$$\begin{bmatrix} S + \mu H_N^T H_N + \epsilon P & R^T + H_N^T \\ R + H_N & -\frac{2}{k} + Q \end{bmatrix} \leq 0 \ , \qquad (13)$$

with $A_L = A + LC$, $H_N = H + NC$, $S = \Pi A_L + A_L^T \Pi$, $Q = -m\left(H_N G + G^T H_N^T\right)$, $R^T = -\Pi G + m A_L^T H_N^T$, $\mu = \left(\delta m_l - \gamma \frac{m_u}{k}\right)$, $\Pi = P + m_u H_N^T H_N$, $m = m_l - \frac{m_u}{k}$, is satisfied, then the error system is globally asymptotically stable with respect to e uniformly in x_0, i.e. there exist a KL function η so that

$$\|e(t)\| \leq \eta\left(\|e(0)\|, t\right) \ , \ \forall t \geq 0 \ , \ \forall x_0 \in \mathbb{R}^n \ .$$

If $k = \infty$ then $m_u = 0$, and $m = m_l \geq 0$, and for the uniformity of the convergence it has to be assumed that V in (6) is decrescent. Moreover, if k is finite then the error system is globally exponentially stable with respect to e uniformly in x_0, i.e. there exist constants $\kappa, \lambda > 0$ such that for Ξ (4)

$$\|e(t)\| \leq \kappa \|e(0)\| \exp(-\lambda t) \ , \ \forall t \geq 0 \ , \ \forall x_0 \in \mathbb{R}^n \ .$$

for every locally bounded trajectory of the plant. I.e. system Ω (2) is an observer for the plant.

PROOF. Consider (6) as Lyapunov function candidate. If (13) is satisfied it follows easily that

$$\dot{V} \leq -\epsilon e^T P e$$

and the composite system $\Sigma - \Theta$ is globally asymptotically stable with respect to e uniformly in x_0, as far as V is decrescent (Chellaboina and Haddad, 2002). If k is finite, then the stability is exponential, but for $k = \infty$ only asymptotic stability can be assured, since the Lyapunov function can grow faster than a quadratic term. □

Remark 2. The inequality (13) is an intermediate result between the circle criterion and the classical Popov criterion, and contains them as special cases. If $m = m_u = m_l = 0$, then the circle criterion is obtained, whereas the classical Popov criterion with time-invariant nonlinearities is recovered setting $\delta = \gamma = 0$. Note, however, that the Popov criterion with time-invariant nonlinearity is not applicable in our case.

Remark 3. The classical Lyapunov function of Lur'e type includes only the first integral term. The inclusion of the second term allows the consideration of positive or negative values of m, when k is finite.

Remark 4. The boundedness of $F(z,\sigma)$ in Theorem 1 means that the growth of the nonlinearity $\phi(z,\sigma)$ is slower than quadratic in z. In fact, if k is finite, then $F(z,\sigma)$ is automatically bounded. Although this condition limits the growth of the nonlinearity it does not require it to be Lipschitz.

Remark 5. The boundedness of $\dot{\sigma}$ in Theorem 1 is a requirement on the growth velocity of the trajectories of the plant. This is usually a reasonable assumption. Note that this restriction does not imply that the plant's trajectories cannot grow unboundedly, but they have to grow slowly enough. However, if plant's trajectories are bounded and stay in a compact set, then this condition is satisfied. It is also possible to show, that if $\dot{\sigma}$ converges to a bounded signal, then the convergence of the observer is assured. This is for example the case, if plant's trajectories converge to a bounded set, in particular if the plant has a globally stable equilibrium point, or set.

4. GEOMETRIC INTERPRETATION

The classical Popov criterion is usually stated as a frequency domain condition, and this was very useful because of its geometric interpretation in the complex plane (Khalil, 2002). In a similar manner the design inequality (13) can also be converted into a frequency domain condition, from which a geometric interpretation can be derived.

The frequency domain condition is derived using the classical Kalman-Yakubovich-Popov(KYP) Lemma, that we recall here, in its following form (Willems, 1971; Rantzer, 1996)

Proposition 6. Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(j\omega I - A) \neq 0$ for $\omega \in \mathbb{R}$ and (A, B) controllable, the following two statements are equivalent:

(i) for $\forall\omega \in \mathbb{R} \cup \{\infty\}$

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \leq 0 \ .$$

(ii) There exists a matrix $P \in \mathbb{R}^{n\times n}$ such that $P = P^T$ and

$$M + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \leq 0 \ .$$

The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable.
When the upper left corner of M is positive semidefinite, and when A is Hurwitz stable it follows that $P \geq 0$.

The application of Proposition 6 to (13) delivers the frequency domain condition.

Proposition 7. Inequality (13) is satisfied if and only if $\forall\omega \in \mathbb{R} \cup \{\infty\}$

$$(1 + m_u H_N G)\,\mathrm{Re}\{G(j\omega)\} - m_u \frac{(\delta-\gamma)}{2k}|G(j\omega)|^2$$
$$-m\left(\omega\,\mathrm{Im}\{G(j\omega)\} - \frac{\delta}{2}|G(j\omega)|^2\right) + \frac{1}{k} > 0 \quad (14)$$

where $G(s) = H_N (sI - A_L)^{-1} G$.

A geometric interpretation of the frequency domain inequality (14) is easily given for $m_u = 0$ (in this case $m \geq 0$): If one plots $\mathrm{Re}\{G(j\omega)\}$ versus $\omega\,\mathrm{Im}\{G(j\omega)\} - \frac{\delta}{2}|G(j\omega)|^2$, then condition (14) is satisfied if the plot lies to the right of the line that intercepts the point $-1/k + j0$ with a (non negative) slope $1/m$. Note that this plot is neither Nyquist plot (that is the plot of $\mathrm{Re}\{G(j\omega)\}$ versus $\mathrm{Im}\{G(j\omega)\}$) nor Popov plot (that is the plot of $\mathrm{Re}\{G(j\omega)\}$ versus $\omega\,\mathrm{Im}\{G(j\omega)\}$).

If, furthermore, $m = 0$ the condition reduces to the one for the circle criterion, i.e.

$$\mathrm{Re}\{G(j\omega)\} + \frac{1}{k} > 0\ ,\ \forall\omega \in \mathbb{R}\cup\{\infty\}\ .$$

This shows that the conditions of (14) are weaker than those of the circle criterion, and convergence of the observer can be obtained under less stringent conditions.

5. EXAMPLE

For illustration consider the system

$$\Sigma : \begin{cases} \dot{x}_1 = x_2\ , \\ \dot{x}_2 = x_3\ , \\ \dot{x}_3 = -\kappa x_3 + g(x_2) + u \\ y = x_1\ , \end{cases}$$

with $g(x_2) = -x_2|x_2|$, and $\kappa > 0$. This system can be written in the form (1), and the error equation (4) is given with

$$A_L = \begin{bmatrix} l_1 & 1 & 0 \\ l_2 & 0 & 1 \\ l_3 & 0 & -\kappa \end{bmatrix}\ ,\ G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\ ,\ C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\ ,$$
$$\sigma = x_2\ ,\ H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\ ,$$
$$\phi(z,\sigma) = g(\sigma) - g(\sigma+z) = -\sigma|\sigma| + (\sigma+z)|\sigma+z|\ .$$

The nonlinearity belongs to the sector $\phi \in [0,\infty]$ for all values of σ, since $z\phi(z,\sigma) \geq 0$ for all σ and z, and $k = \infty$ since for $\sigma = 0$, $\phi(z,0) = -g(z)$ is not globally Lipschitz, i.e. it grows faster than any linear function. The continuous function $F(z,\sigma)$, given by

$$F(z,\sigma) = \begin{cases} 2 & \text{if } \sigma \geq 0, z+\sigma \geq 0 \\ -2(1+2\sigma/z) & \text{if } \sigma \geq 0, z+\sigma \leq 0 \\ 2(1+2\sigma/z) & \text{if } \sigma \leq 0, z+\sigma \geq 0 \\ -2 & \text{if } \sigma \leq 0, z+\sigma \leq 0 \end{cases}\ ,$$

is bounded by $\alpha = -2$, and $\beta = 2$ (10). Note that in this example $\varrho(z,\sigma)$ is not continuously differentiable (as assumed in the general development) but however the same ideas can be applied.

Now it is required to determine the bounds of $\dot{\sigma}$(11). Note that $\dot{\sigma} = x_3$, and take the (partial) Lyapunov function

$$W(x_2,x_3) = \frac{1}{4}\begin{bmatrix} x_2 & x_3 \end{bmatrix}\begin{bmatrix} \kappa^2 & \kappa \\ \kappa & 2 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \int_0^{x_2} g(z)\,dz$$

and calculate its time derivative along the solutions of the plant Σ

$$\dot{W} = -\frac{1}{2}\kappa x_3^2 + \frac{1}{2}\kappa x_2 g(x_2) + u\left(x_3 + \frac{1}{2}\kappa x_2\right)\ .$$

If u is bounded this shows that (x_2, x_3) converges globally and uniformly to a ball containing the origin. In this case $\dot{\sigma}$ is bounded after some finite time. So in (12) the case with $\delta = -\gamma > 0$ will be considered, where δ is a positive constant depending on the bound of u.

For this simple SISO case the satisfaction of the circle criterion can be interpreted in the frequency domain. The transfer function of the linear part of the error equation is given by

$$G(s) = \frac{s - l_1 + N}{s^3 - (l_1 - \kappa)s^2 - (l_2 + l_1\kappa)s - (l_3 + l_2\kappa)}\ .$$

Note that by choosing appropriately the parameters (N, l_1, l_2, l_3) the numerator and denominator coefficients of this transfer function can be arbitrarily assigned. In this case the satisfaction of the circle criterion for the sector $[0,\infty]$ corresponds to the selection of the parameters (N, l_1, l_2, l_3) so that the transfer matrix $G(s)$ is SPR (Khalil, 2002). However, this is not possible, since the relative degree of $G(s)$ is 2 for every selection of the parameters, and the relative degree of an SPR transfer function has to be 0 or 1. Therefore, the circle criterion cannot be satisfied in this case. Moreover, it is not possible to design a high-gain observer, since the nonlinearity is not globally Lipschitz.

The satisfaction of the Popov criterion corresponds to the selection of the parameters (N, l_1, l_2, l_3) such that the inequality (14) $\forall\omega \in \mathbb{R}\cup\{\infty\}$, $Z(j\omega) > 0$, where $Z(j\omega) \triangleq \mathrm{Re}\{G(j\omega)\} - m\left(\omega\,\mathrm{Im}\{G(j\omega)\} - \frac{\delta}{2}|G(j\omega)|^2\right)$ (recall that $H_N G = 0$ and $k = \infty$) is satisfied for some $m \geq 0$. Since for $\omega = \pm\infty$ the left-hand side is zero, it is necessary to check additionally that

$$\lim_{\omega\to\infty} \omega^2 Z(j\omega) > 0\ .$$

By selecting the parameters (N, l_1, l_2, l_3) such that

$$G(s) = \frac{s+1}{(s+2)(s+3)(s+4)}\ ,$$

the modified Popov plot for three different values of δ is given in Figure 1. It is clear that it is possible to select the value of $m > 0$ such that the Popov plot lies to the right of the line of slope $1/m$, for every value of δ. Moreover, the limit

$$\lim_{\omega\to\infty} \omega^2 Z(j\omega) = 8m + 1 > 0\ ,$$

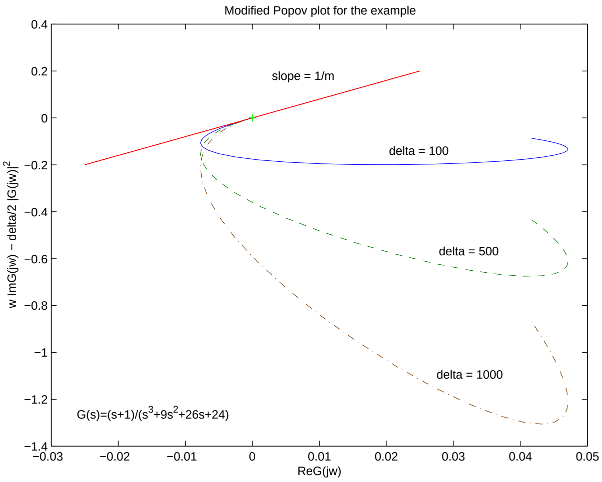

Fig. 1. Modified Popov Plot for $G(s)$ in example, for $\delta = \{100, 500, 1000\}$.

for every $m > 0$ independently of δ. The slope of the curve for $\omega \to \infty$ is given by

$$\lim_{\omega \to \infty} \frac{\omega \operatorname{Im}\{G(j\omega)\} - \frac{\delta}{2}|G(j\omega)|^2}{\operatorname{Re}\{G(j\omega)\}} = 8 \ ,$$

that is independent of δ. This means that for $m \geq 1/8$ the generalized Popov conditions are satisfied for every δ. The observer is therefore globally convergent, if the input u is bounded, although the plant's state is not bounded.

6. CONCLUSIONS

Popov-like criteria can be used with advantage for the design of nonlinear observers, if they are conveniently modified. This has the advantage that the requirements for the assertion of the convergence of the observer are weaker than when using circle-like criteria proposed previously. Moreover, the dynamics of the plant has to be taken into account to assert the convergence properties of the observer. This seems to be a new feature in the design of nonlinear observers. Usually the Lyapunov functions considered in the design of nonlinear observers is independent of the dynamics of the plant. The method proposed here introduces a family of Lyapunov functions parameterized by the trajectories of the plant. The extension of this ideas to the multivariable case and to more general classes of nonlinear systems is also possible and will be reported elsewhere.

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