ROBUST EIGENSTRUCTURE ASSIGNMENT: NEW APPROACH

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Abstract: This paper considers the problem of robust eigenstructure assignment, which involves finding a feedback gain matrix that makes the closed-loop system insensitive to perturbations or parameter variations. The main contribution of the paper is to derive a new robustness measure using the matrix perturbation theory. The new measure is used to formulate a new method for robust control of the system via state or output feedback. The freedom provided by eigenvalue assignment of multivariable systems is applied to optimize the new index using genetic algorithms. The new results expressed in this paper confirm the previously derived results on robust eigenstructure assignment. In the case of state feedback design, the new robustness index verifies a previously proposed robustness measure. A numerical example is presented to illustrate the effectiveness of the proposed method.

Keywords: Eigenstructure assignment, Robust stability, Genetic algorithms, Multivariable control systems, Eigenvalues, Sensitivity.

1. INTRODUCTION

In many practical control design problems, there often exist perturbations or parameter variations in the system. Also, the presence of uncertainty in the system severely affects the performance and also the stability of closed-loop system, designed based on the nominal model of the system. If the closed-loop system is stable based on the model and also the sensitivity of eigenvalues to perturbations and parameter variations is minimized, then the possibility of instability will be reduced in the case of applying the designed controller to the real system. It is well known that feedback gain matrix, for a multivariable system is in general, non-unique for a given set of desired closed-loop poles. So, many methods have been proposed on the choice of the feedback gain matrix, such that the closed-loop system is robust (Kautsky, *et al*., 1985; Owens and O'Reilly, 1989; Duan, 1992; Liu and Patton, 1998; Ensor and Davies, 2000). In order to achieve the closed-loop system with low eigenvalue sensitivity, several measures of eigenvalue sensitivity has been introduced (Kautsky, *et al*., 1985; Liu and Patton, 1998). Most of the robust eigenstructure assignment methods try to minimize these measures. Particularly, the condition number of the eigenvector matrix of the closed-loop system is widely used. Some effective eigenstructure assignment algorithms are proposed by Kautsky, *et al.* (1985) to reduce the sensitivity indexes, which are essentially based on the iterative pole assignment procedures. Based on a well-known eigenstructure assignment approach, Duan (1992) has introduced a simple and effective algorithm for robust pole assignment, which minimizes the sensitivity measures. However, most of the proposed methods deal with the problems in which, full state feedback is permitted. In the case of output feedback design, there is not a full control over all the closed-loop eigenvalues and the problem becomes more complex.

There are some difficulties associated with using the spectral condition number as the measure of sensitivity (Duan 1992; Lam and Yan, 1996; Ichikawa, 1998). This leads to use of other robustness indexes, which are based on the Frobenius norm. For example, Frobenius condition number of the closed-loop eigenvector matrix is used, which is more conservative than the spectral condition number.

In this paper, a new robustness index is introduced based on the Frobenius norm. The new index is less conservative than the similar measures, in some cases. Also, an effective algorithm is suggested for the purpose of robust controller design, using an eigenstructure assignment approach with the aim of minimizing the new robustness measure. Genetic algorithm is applied to solve the optimization problem. Using the new method, the closed-loop stability will be guaranteed in the case of output feedback design, if the system is stabilizable.

The new results obtained in this paper are all matched with the previously derived results on robust eigenstructure assignment. The new algorithm tries to assign the eigenvectors of the closed-loop system such that they are orthogonal to each other. The closed-loop robust stability is achieved via normalizing the closed-loop matrix.

The paper is organized as follows: In section 2, the problem of robust eigenstructure assignment is presented. Section 3 provides the necessary background about the eigenstructure assignment. In sections 4, new results are achieved based on the theoretical analyses. In sections 5 an illustrative example is presented to show the effectiveness of the proposed method and concluding remarks follow in section 6.

2. PROBLEM FORMULATION

Consider the linear multivariable system with the state-space description:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (1)$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$ and $x \in R^n$, is the state vector, $u \in R^m$ is the input vector and $y \in R^p$ is the output vector. Also, it is assumed that (A, B) is controllable and (A, C) is observable. Consider the static output feedback of the form:

$$u = Ky \quad (2)$$

Then the closed-loop system representation is given by:

$$\dot{x}(t) = (A + BKC)x(t) \quad (3)$$

Assume that the set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is composed of the desired closed-loop eigenvalues. In practical applications, the system parameters are subjected to perturbations and so the eigenstructure of the closed-loop system varies from the nominal designed eigenstructure. The robust eigenstructure assignment problem is to find K such that the closed-loop eigenstructure is as insensitive as possible to the variations of system parameters. The closed-lop eigenvalues must be assigned at the desired stable places, which are all distinct and different from the open-loop poles. These conditions are necessary to have a closed-loop system with minimum eigenstructure sensitivity to the perturbations (Wilkinson, 1965). In addition, the closed-loop system must be designed to be as robust as possible under a sensitivity index. The specified closed-loop performance will be achieved by assigning the set of eigenvectors in a special manner. It leads to minimize the index and the freedom provided by the eigenstructure assignment is used for the purpose of robustness.

3. EIGENSTRUCTURE ASSIGNMENT

In this section, a simple and effective eigenstructure assignment is described, which is used in section 4. Suppose that the nominal closed-loop eigenvalues are arranged in descending order with respect to their real parts, that is:

$$\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2) \geq \cdots \geq \mathrm{Re}(\lambda_n) \quad (4)$$

It is well-known that *Max*(*m,p*) self-conjugate eigenvalues and their corresponding eigenvectors can be assigned by K (Andry, *et al.*, 1983). So the aim is to choose K such that *Max*(*m,p*) number of dominant eigenvalues of the closed-loop system are assigned in the desired places. Also the corresponding eigenvectors $v_i \in \Re^n$ must be assigned to satisfy the desired specifications. For any pair of desired closed-loop eigenvalues and their associated closed-loop eigenvectors the following equation holds:

$$(A + BKC)v_i = \lambda_i v_i \qquad i = 1, \ldots, n \quad (5)$$

So, it is simple to show that:

$$v_i = L_i m_i \qquad i = 1, \ldots, n \quad (6)$$

where:

$$\begin{aligned} m_i &= KCv_i \\ L_i &= (\lambda_i I - A)^{-1} B \end{aligned} \quad (7)$$

$m_i \in \Re^m$ are the parameter vectors. Also, the eigenvector matrix and parameter matrix are as follows:

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \quad (8)$$
$$M = \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}$$

In the case when the state feedback is permitted, K can be easily computed as follows:

$$K = MV^{-1} \quad (9)$$

Now, for the case in which output feedback is required, consider a linear transformation matrix T as follows:

$$T = \begin{bmatrix} B & P \end{bmatrix} \in \Re^{n\times n} \quad (10)$$

where $P \in \Re^{n\times(n-m)}$ is an arbitrary matrix such that *rank(T)=n*. The corresponding eigenvectors under such a transformation are:

$$\bar{v}_i = T^{-1} v_i \quad (11)$$

The feedback gain matrix which solves the eigenstructure assignment problem can be computed as follows (Jiang, 1994):

$$K = (S - \bar{A}\bar{V})(\bar{C}\bar{V})^{-1} \quad (12)$$

where:

$$S = \begin{bmatrix} \lambda_1 s_1 & \lambda_2 s_2 & \cdots & \lambda_q s_q \end{bmatrix},$$
$$\bar{V} = \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_q \end{bmatrix} \quad (13)$$
$$\bar{A} = T^{-1}AT, \qquad \bar{C} = CT, \qquad \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix},$$
$$\bar{A}_1 = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \end{bmatrix}$$

Note that $q = Max(m, p)$ and $s_i \in \Re^m$ can be obtained from $\bar{v}_i = \begin{bmatrix} s_i \\ g_i \end{bmatrix}$.

It is shown that if $m + p > n$, the whole set of desired eigenvalues can be assigned with some restrictions on eigenvector selection (Liu and Patton ,1998). Other approaches for output feedback eigenstructure assignment can be used in these cases to assign the whole spectrum (Liu and Patton, 1998; Duan, 1992).

4. ROBUST POLE ASSIGNMENT

The spectral condition number of the closed-loop eigenvector matrix (modal matrix) still remains as the most widely accepted measure of sensitivity. This is because of the Bauer-Fike Theorem, which is derived from the matrix perturbation theory (Wilkinson, 1965; Golub and Van Loan, 1989; Stewart and Sun, 1990). The main question of matrix perturbation theory is: How does the eigenstructure of a matrix changes when its elements are subjected to a perturbation.

According to the Bauer-Fike Theorem, eigenvalue perturbation due to the perturbation E in matrix A is bounded by $\kappa_2(V)\|E\|_2$, where $\kappa_2(.)$ denotes the spectral condition number of $(.)$. Also V is the modal matrix of A, as defined in (7). So, the spectral condition number of the closed-loop modal matrix provides a meaningful measure on the sensitivity of the closed-loop eigenvalues due to the perturbations in system. But there are some difficulties in using the spectral condition number as the measure of sensitivity (Duan 1992; Lam and Yan, 1996; Ichikawa, 1998). It is difficult to handle and in practical applications, it may be replaced by Frobenius condition number, $\kappa_F(V) = \|V\|_F \|V^{-1}\|_F$, where $\|.\|_F$ denotes the Frobenius norm of $(.)$. This is a more conservative measure than the spectral condition number. But the non-smoothness of the spectral norm is removed (Lam and Yan, 1996). In this section, a new sensitivity index is proposed based on another Theorem derived from the matrix perturbation theory.

Before deriving the main results, several basic Theorems are presented. In the following Theorems, the matrix A which satisfies the property $A^H A = AA^H$ is called normal (H denotes the conjugate transpose of the matrix). Also Schur decomposition (Golub and Van Loan, 1989) is used to give some results on matrix perturbation theory in the following Theorem.

Theorem 1: Let

$$Q^H AQ = D + N \quad (14)$$

be a Schur decomposition of $A \in C^{n\times n}$ where $D = diag(\lambda_1, \ldots, \lambda_n)$, $N \in C^{n\times n}$ is a strictly upper triangular matrix and Q is an appropriate unitary matrix. Suppose that $E \in C^{n\times n}$ is an arbitrary matrix. If $\mu \in \lambda(A+E)$ and p is the smallest positive integer such that $|N|^p = 0$, then

$$\min_{\lambda} |\mu - \lambda| \le \max(\theta, \theta^{1/p}) \quad (15)$$

where:

$$\theta = \|E\|_2 \sum_{k=0}^{p-1} \|N\|_2^k \quad (16)$$

$\lambda(.)$ means the eigenvalue of (.), λ is the eigenvalue of the matrix A and $|N| = \begin{bmatrix} |n_{ij}| \end{bmatrix}$.

Proof: see (Golub and Van Loan, 1989).

It is shown that Theorem 1 is less conservative than Bauer-Fike Theorem, in some cases (Golub and Van Loan, 1989).

Corollary 1: When the closed-loop system matrix in (3) is faced with a perturbation matrix *E*, if *K* is designed such that *N* is equal to zero, the upper bound on variations of closed-loop eigenvalues will be minimized. *N* is the corresponding matrix, obtained from Schur decomposition of the closed-loop system matrix.

Lemma 1: Matrix $A \in C^{n\times n}$ is normal if and only if the matrix *N* is equal to zero in the Schur decomposition of matrix *A*.

Proof: see (Golub and Van Loan, 1989).

Theorem 2: Matrix $A \in C^{n\times n}$ is normal if and only if

$$\|A\|_F^2 = \sum_{i=1}^{n} |\lambda_i(A)|^2 \tag{17}$$

where $\|A\|_F$ is the Frobenius norm of *A*.

Proof: It is shown that unitary similarity transformations do not affect on the Frobenius norm of a matrix. So, it can be concluded that:

$$\|Q^H AQ\|_F = \|A\|_F \tag{18}$$

If the matrix *A* is normal, then based on lemma 1, in the equation (14) the matrix *N=0*, and

$$\|A\|_F^2 = \|D\|_F^2 = \sum_{i=1}^{n} |\lambda_i(A)|^2 \tag{19}$$

and the sufficient condition is proved. In order to show the necessary part of the Theorem, since the Schur decomposition of the matrix *A* gives the following:

$$\|A\|_F = \|D+N\|_F = \sqrt{\sum_{i=1}^{n} |\lambda_i(A)|^2 + \sum_{j=1}^{n}\sum_{i=1}^{n} |n_{ij}|^2} \tag{20}$$

and it is assumed that:

$$\|A\|_F^2 = \sum_{i=1}^{n} |\lambda_i(A)|^2 \tag{21}$$

it can be concluded that $n_{ij} = 0$ and *N=0*, therefore the matrix is normal, and the proof is complete. □

Corollary 2: When the closed-loop system matrix in (3) is subjected to the perturbation matrix *E*, the upper bound on variations of closed-loop eigenvalues will be minimized if *K* is designed such that:

$$\|A+BKC\|_F^2 = \sum_{i=1}^{n} |\lambda_i(A+BKC)|^2 \tag{22}$$

In this case, the closed-loop system matrix is a normal matrix.

It is shown that if the eigenvectors of a system are assigned to match exactly a set of mutually orthogonal vectors, then the corresponding eigenvalues will have the minimum sensitivity to the perturbations and parameter variations (Liu and Patton, 1998). There are some sensitivity measures, proposed by the authors (Kautsky, *et al.*, 1985; Liu and Patton, 1998); when they are minimized, the set of closed-loop eigenvectors becomes near orthogonal. Also, it is important to note that normal matrices are exactly those that possess a complete set of orthogonal eigenvectors (Strang, 1986). Therefore, the new results are matched with the previously derived results.

The mentioned results shows that the following cost function can be used as an eigenvalue sensitivity index for matrix *A*:

$$\left| \|A\|_F^2 - \sum_{i=1}^{n} |\lambda_i(A)|^2 \right| \tag{23}$$

Therefore, to solve the problem of robust pole assignment, the output feedback *K* can be designed such that the following index is minimized:

$$\left| \|A+BKC\|_F^2 - \sum_{i=1}^{n} |\lambda_i(A+BKC)|^2 \right| \tag{24}$$

Based on the above discussions, an algorithm for eigenstructure assignment with minimum sensitivity of closed-loop eigenvalues can be given now.

Algorithm:

1) Select the set of desired closed-loop eigenvalues, according to the stability and dynamic response characteristic requirements of the system:

$$\{\lambda_1, \lambda_2, \ldots, \lambda_q\} \qquad q = Max(m,p) \tag{25}$$

2) For each eigenvalue, calculate the corresponding design matrices as follows:

$$L_i = (\lambda_i I - A)^{-1} B \tag{26}$$

3) Minimize the following cost function, in order to solve the robust eigenstructure assignment problem:

$$J = \left| \|A+BKC\|_F^2 - \sum_{i=1}^{n} |\lambda_i(A+BKC)|^2 \right| \tag{27}$$

The optimization variables are m_i, the parameter vectors in (6) corresponding to each closed-loop eigenvalue. So, the output feedback matrix K can be estimated according to (12) and using the obtained parameter vectors.

To solve the optimization problem, Genetic

Algorithm is used, which is previously used for the purpose of eigenstructure assignment (Patton and Liu, 1994; Esna Ashari and Khaki Sedigh, 2004). Then at each step, the possible solutions for the parameter vectors (chromosomes) will be produced using the random search operations. After that, K is estimated according to the selected parameter vectors. Then the cost function (27) must be computed and a new generation of chromosomes should be created.

If the algorithm is trying to design an output feedback, there will not be a full control over all the closed-loop eigenvalues. So the chromosomes that lead to an unstable closed-loop system must be deleted in the process of genetic algorithms.

Remark 1: In the cases in which, all the closed-loop eigenvalues can be assigned, especially when a state feedback is designed, the term $\sum_{i=1}^{n} |\lambda_i(A+BK)|^2$ is constant for a desirable set of closed-loop poles. Therefore (24) is reduced to:

$$J = \|A + BKC\|_F^2 \tag{28}$$

The index is previously derived by Dickman (1987) in a different way.

Remark 2: It is obviously possible to replace the (27) with another robustness index (for example, the spectral condition number of closed-loop eigenvector matrix).

5. ILLUSTRATIVE EXAMPLE

Consider the following dynamical system (Liu and Patton, 1998):

$$A = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.3060 & -2.1320 & 0.9807 & 0 & 0 \\ 0 & 1.5950 & -3.1490 & 1.5470 & 0 \\ 0 & 0.0355 & 2.6320 & -4.2570 & 1.8550 \\ 0 & 0.0023 & 0 & 0.1636 & -0.1625 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0.0638 & 0.0838 & 0.1004 & 0.0063 \\ 0 & 0 & -0.1396 & -0.2060 & -0.0128 \end{bmatrix}^T$$

$$C = I_{5\times 5}$$

The desired eigenvalues of the closed-loop system are chosen to be: {-0.5, -1, -2, -3, -4}.

Using the method proposed in section 4, the following controllers are designed by minimizing cost function (26) and also the spectral condition number of closed-loop eigenvector matrix, respectively:

$$K_1 = \begin{bmatrix} -150.7102 & 27.9662 & -46.1743 & 54.7426 & 56.7101 \\ -58.3445 & 16.8117 & -4.7370 & 20.5022 & 70.0154 \end{bmatrix}$$

and:

$$K_2 = \begin{bmatrix} -245.1060 & -2.6951 & -35.4448 & 31.6681 & 66.4664 \\ -131.4648 & -10.8048 & 9.3406 & -5.2734 & 72.5363 \end{bmatrix}$$

At the end of design procedures, the cost functions will be $\|A + BK_1\|_F^2 = 16.2867$ in the first design and $\kappa_2(V) = 25.3665$ in the second design.

For the purpose of comparison between the designed controllers and a feedback control matrix that leads to sensitive closed-loop system, consider K_3:

$$K_3 = 10^5 \times \begin{bmatrix} -1.6734 & -0.0798 & 0.0267 & -0.0434 & 0.2376 \\ -4.3434 & -0.2075 & 0.0706 & -0.1139 & 0.6171 \end{bmatrix}$$

Table 1 Comparison of perturbation effect on eigenvalues of closed-loop system

Changes in all elements of the matrix A	Feedback matrix	closed-loop eigenvalues
+1%	K_1	-0.5003, -1.0003, -1.9998, -2.9995, -4.0098
	K_2	-0.5004, -1.0000, -1.9987, -3.0038, -4.0069
	K_3	-0.2863, -0.9473, -2.9514 ± 1.6005i, -3.3733
-1%	K_1	-0.4997, -0.9997, -2.0002, -3.0005, -3.9901
	K_2	-0.4996, -1.0000, -2.0013, -2.9962, -3.9931
	K_3	-0.8783, -0.7157 ± 0.6801i -3.2207, -4.9597
+10%	K_1	-0.5028, -1.0035, -1.9980, -2.9979, -4.0959
	K_2	-0.5038, -0.9998, -1.9878, -3.0376, -4.0691
	K_3	0.0437, -0.9375, -3.1911 ± 5.3831i, -3.3221
-10%	K_1	-0.4971, -0.9965, -2.0016, -3.0084, -3.8982
	K_2	-0.4962, -1.0002, -2.0141, -2.9608, -3.9306
	K_3	1.4827, 0.7596, -0.9132, -3.2422, -8.4888

In the last design, the sensitivity measures are as follows:

$$\|A+BKC\|_F^2 = 8.8124\times10^4, \quad \kappa_2(V) = 1.2310\times10^6$$

So the closed-loop system is not well-conditioned referring to each of the measures.

The effects of perturbations, on the closed-loop eigenvalues of the systems are shown in Table 1. The closed-loop systems are subjected to the variation of the elements of matrix A in different cases. It is evident from the results that the closed-loop systems under the feedback matrices obtained by the proposed approach are more robust to variations in the matrix A. It is interesting to note that in the case in which, the additive perturbation is +10% or -10%, the closed-loop system under K_3 will be unstable. But the other closed-loop systems will have small variation in their eigenvalues.

6. CONCLUSION

This paper has introduced a new approach to design robust controllers for multivariable systems. A new robustness measure is proposed, which can be minimized by assigning the eigenstructure of the closed-loop system, appropriately. The new index is less conservative than the similar measures in some cases. Also, it is used to propose an effective method of designing the robust controllers. The proposed method can be implemented in the case of output feedback design. The algorithm tries to make the set of closed-loop eigenvectors orthogonal, and the closed-loop system will have minimum sensitivity to the perturbations of the system parameters. The new results expressed in this paper confirm the previously derived results on robust eigenstructure assignment. In the case of state feedback design, the new robustness index verifies a previously proposed robustness measure, which was derived in a different way.

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