INPUT AND STATE FUNCTIONAL OBSERVABILITY FOR DESCRIPTOR SYSTEMS

Taha Boukhobza, **Frédéric Hamelin**, **Cédric Join**, **and Dominique Sauter**

Centre de Recherche en Automatique de Nancy (CRAN),
UMR-CNRS 7039, Université Henri Poincaré (Nancy I),
BP-239, 54506 Vandœuvre-lès-Nancy, France.
E-mail: taha.boukhobza@cran.uhp-nancy.fr

Abstract: This paper deals with the right-hand side observability of a linear function of the state and the unknown inputs for linear systems in descriptor form. For such systems, necessary and sufficient conditions ensuring the ability of reconstructing any given functional of the state and unknown inputs are determined by means of geometrical tools like subspace sequences.

Keywords: Descriptor systems, unknown inputs, state and input functional observation.

1. INTRODUCTION

The state observation problem has been widely studied since the original works of Kalman (Kalman, 1960) and Luenberger (Luenberger, 1966). In this respect, many papers aim at designing reduced-order observers, which provide an estimate of a linear functional of the state as in (Murdoch, 1973; Fairman and Gupta, 1980; Moore and Ledwich, 1975; Hou *et al.*, 1999; Darouach, 2000; Tsui, 1996) for linear systems in standard form or in (Zhang, 1990; Verhaegen and Dooren, 1986) for systems in descriptor form. The problem of reconstructing a desired part of the state and the unknown inputs is of a great interest mainly in control law synthesis, fault detection and isolation, fault tolerant control, supervision and so on. Nevertheless, this problem is still open since in many works, the authors propose a solution based on a specific observer and do not focus on the observability property of the functional of the state to estimate. Furthermore, in most cases, only sufficient conditions for the existence of such observers are given, the systems considered are in standard form and are not driven by unknown inputs. Among the recent works, in (Darouach, 2000), necessary and sufficient conditions are given for the existence and stability of a functional observer of the same dimension than the state functional to be estimated. Note also that some works interest with the design of functional observers for linear systems in standard form with unknown inputs. We can cite among the most important works in this area, the approach developed in(Hautus, 1983) in the frequency domain. In the latter paper, the author gave definitions of strong detectability and strong observability and the conditions for existence of observers that estimated a functional of the state and unmeasured inputs. Other works such (Kudva *et al.*, 1980; Trinh and Ha, 2000; Tsui, 1996) present a reduced-order linear functional state observer under some decoupling conditions. After all, each of these works aims at designing an observer when some conditions are satisfied, but they do not give any information about the theoretical reconstructibility of the functional of the state to be estimated.

Our approach is quite different since on the one hand we do not interest to a specific observer but to the theoretical solvability of the problem of functional observation. On the other hand, our work concerns linear systems on descriptor form, which represent a more general class of systems than the ones in standard form. We interest to such systems because they result from a convenient and natural modelling process (Luenberger, 1977; Lewis, 1992). Moreover, applications of descriptor systems can be found in various fields (Müller, 2000) such as robotics, electrical circuit networks, biologic and economic systems. In this respect, classical features like solvability, controllability and observability are revisited for descriptor systems in (Aplevich, 1991; Cobb, 1984; Dai, 1989; Yip and Sincovec, 1981; Yamada and Luenberger, 1985; Hou and Müller, 1999*a*).

In fact, our work and the geometrical tools we use are close to the ones employed in (Willems, 1982), which deals with the solvability of disturbance decoupled estimation using invariant subspaces. Our main result consists of necessary and sufficient conditions to check a kind of right hand side observability (Hou and Müller, 1999*a*), called also detectability (Hou and Müller, 1999*b*), of any given functional of the state and unknown inputs. The right-hand side observability is equivalent to the *R*-observability (Dai, 1989; Yip and Sincovec, 1981) or to the finite observability (Verghese *et al.*, 1981) for regular descriptor systems. This property of right-hand side observability or detectability is a very important observability property for descriptor systems. Indeed, it is proved in (Hou and Müller, 1999*b*), that it is a necessary and sufficient condition for the existence of a generalized observer which allows to reconstruct the state. In fact, system (1) is right-hand side observable iff for $t > 0$, state $x(t)$ can be uniquely determined using the knowledge of the output and the input (Müller and Hou, 1993; Hou and Müller, 1999*b*).

For a given linear system in descriptor form driven by both known and unknown inputs, we propose to answer the question whether or not a desired functional of the state and the unknown input can be reconstructed using the knowledge of the known inputs and the outputs. If it can not, we can affirm that no observer of any form and any order can estimate the desired functional of the state and the unknown inputs. Note that unlike many works, the regularity assumption (*i.e.* A, E square and $|\ sE - A\ | \not\equiv 0$) on descriptor systems is dropped through this paper. That means, that we consider the most general case of descriptor systems *i.e.* nonregular even non-square descriptor systems. Note that our results can be linked easily to these obtained in (Hautus, 1983) for standard systems but using geometrical tools.

The paper is organised as follows: after section 2, which is devoted to the problem formulation, necessary and sufficient conditions for the reconstruction of a functional of the unknown inputs and the state are given in section 3. These conditions are illustrated with an example in section 4. Finally, some concluding remarks are made.

2. PROBLEM STATEMENT

Consider the descriptor system:

$$\begin{aligned} E\dot{x} &= Ax + Bu + Fv \\ y &= Cx + Dv \end{aligned} \qquad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^v$ and $y \in \mathbb{R}^p$ are respectively the state vector, the known input vector, the unknown input vector and the output vector. $E \in \mathbb{R}^{q\times n}$, $A \in \mathbb{R}^{q\times n}$, $B \in \mathbb{R}^{q\times m}$, $F \in \mathbb{R}^{q\times v}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times v}$ are constant matrices.

In fact, it is clear that if E is full column rank, then we can put system (1) in standard form. We assume only, without loss of generality, that E is full row rank ($rank\, E = q \leq n$).

Our aim is to check the ability of state reconstruction or in other terms the reconstructibility of a functional of the state and the unknown input:

$$w = L_x x + L_v v \qquad (2)$$

using the knowledge of u, y and its derivatives. w is reconstructible if and only if we can express it using the known inputs, the outputs and their derivatives. In this case, and only in this case, there can exist a generalized observer which estimates w (Hou and Müller, 1999*b*).

In other words, we focus on the conditions which ensure that a chosen part of both the state and the unknown inputs is reconstructible using the known variables. Unknown input vector v may represent any kind of faults, disturbances, ... The knowledge of them can be beneficial for robust control or fault tolerant control objectives. The problem of estimating unknown inputs is motivated in part by certain applications where it is either too expensive or perhaps not possible to measure some of the system's inputs. Thus, there have been numerous studies investigating the problem of input observability and reconstruction particularly for linear time-invariant systems in standard form (Hautus, 1983; Hou and Patton, 1998). Note that lot of works are totally dedicated to inputs estimation and some of them use state functional observers to achieve this (Xiong and Saif, 2003).

In this paper, we consider a descriptor system driven by known as well as unknown and possibly time-varying inputs. We propose to answer the question whether or not the problem of the reconstructibility of a functional of all the unmeasured variables ($L_x x + L_v v$) has a solution.

Note that since we assume that matrix E is full row rank, the system (1) can be transformed into a state-space of the form:

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + A_2 x_2 + Bu + Fv \\ y &= C_1 x_1 + C_2 x_2 + Dv \end{aligned} \qquad (3)$$

By regarding (x_2, v) as unknown input. Therefore, the problem we study is equivalent to the functional observability of $L_{1,x}x_1 + (L_{2,x}x_2 + L_v v)$. We choose in the sequel to work with systems in descriptor form (1) for two main reasons. The first one is that form (3) is a special case of (1). Secondly, we think that using geometrical tools, like subspace sequences, there is no more difficulty to work with a system of the form (1) instead of a system of the form (3). Indeed, the main complexity of the problem lies in the fact that we consider systems with unknown inputs for which we are interested by observing only a part of the state and the unknown inputs.

3. MAIN RESULTS

This section is subdivided into two parts. In the first one, we consider unperturbed descriptor systems ($v = 0$) for which the presentation is quite simple. In a second part, using an extended state-space technique, we generalize the obtained results to the descriptor systems driven by unknown inputs.

3.1 Observation of a functional of the state for a system without unknown inputs

For the sake of simplicity, we will first consider a system without unknown input ($v = 0$):

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{4}$$

Recall that the derivative of a functional Hx can be generically expressed in terms of the state and the inputs iff $\mathrm{Im}(H^T) \subseteq Im(E^T)$. In this case and only in this case, the derivative of Hx can be exploited for state reconstruction and then we say that Hx has a permitted derivative. This important point is not tackled for linear systems in standard form. Indeed, $E = I$ implies that $H\dot{x}$ can be expressed in function of the state and the inputs for all H: $H\dot{x} = HAx + HBu$. We precise hereafter the notion of permitted derivatives:

Definition 1. Consider system (4), we say that:
• $\Gamma y = \Gamma C x$ has generically a permitted derivative iff $\mathrm{Im}\left((\Gamma C)^T\right) \subseteq \mathrm{Im}\left(E^T\right)$;
• $\Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(r)} \end{pmatrix}$ has a permitted derivative iff $\exists$ a matrix H such that $\Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(r)} \end{pmatrix} = Hx$ with $\mathrm{Im}\left(H^T\right) \subseteq \mathrm{Im}\left(E^T\right)$. □

To answer the question if $w = L_x x$ is reconstructible or not, we define for matrices E, A, C the following sequence of subspaces :

$$\begin{cases} \Delta^0_{E,A,C} = \mathrm{Im}\left(C^T\right) \\ \Delta^{i+1}_{E,A,C} = \Delta^i_{E,A,C} + A^T(EE^T)^{-1}E\left(\Delta^i_{E,A,C} \cap \mathrm{Im}\left(E^T\right)\right) \end{cases} \tag{5}$$

This sequence is a bounded non-decreasing sequence of subspaces. Thus there exists an integer κ_0 such that $\Delta^{\kappa_0+1}_{E_e,A_e,C_e} = \Delta^{\kappa_0}_{E_e,A_e,C_e}$ which represents the maximal element of this sequence. We denote this element $\Delta^*_{E,A,C}$. The following proposition provides necessary and sufficient conditions for w reconstructibility.

Proposition 2. For system (4), $w = L_x x$ is reconstructible using the knowledge of input u, measured variable y and its derivatives iff

$$\mathrm{Im}(L_x^T) \subseteq \Delta^*_{E,A,C}$$

□

Proof:
Necessity: Firstly, we demonstrate property P_1:

P_1**:** *A functional of the state Hx can be expressed using the inputs, the outputs and their i^{th} first derivatives* i.e. $Hx = \Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i)} \end{pmatrix} + \beta u$ *only if* $\mathrm{Im}(H^T) \subseteq \Delta^i_{E,A,C}$.

For $i = 0$, it is obvious that if $Hx = \Gamma y$ then $\mathrm{Im}(H^T) \subseteq \mathrm{Im}(C^T) = \Delta^0_{E,A,C}$.
For $i = 1$, let us assume that there are matrices Γ and β such that $Hx = \Gamma\begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \beta u$. Then, we can find two matrices Γ_0 and Γ_1 such that

$$Hx = \Gamma\begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \beta u = \Gamma_0 y + \frac{d(\Gamma_1 y)}{dt} + \beta u$$

where from definition 1, $\Gamma_1 y = \Gamma_1 C x$ with $\mathrm{Im}((\Gamma_1 C)^T) \subseteq \mathrm{Im}(E^T)$. The latter inclusion implies that there exists a matrix K such that $\Gamma_1 C = KE$. So,

$$\frac{d(\Gamma_1 y)}{dt} = \frac{d(\Gamma_1 C x)}{dt} = KE\dot{x} = KAx + KBu \tag{6}$$

Since E is full row rank, (EE^T) is invertible, matrix K is unique and is given by

$$K = \Gamma_1 C E^T (EE^T)^{-1}$$

Let $H_1 x = \frac{d(\Gamma_1 y)}{dt} + \beta u$, then equation (6) implies $H_1 = KA$ and $\beta = -KB$. Moreover, since $\mathrm{Im}\left((\Gamma_1 C)^T\right) \subseteq \mathrm{Im}(C^T) = \Delta^0_{E,A,C}$ and $\mathrm{Im}((\Gamma_1 C)^T) \subseteq \mathrm{Im}(E^T)$, it follows that $\mathrm{Im}\left((\Gamma_1 C)^T\right) \subseteq \Delta^0_{E,A,C} \cap \mathrm{Im}(E^T)$. Consequently, $\mathrm{Im}(H_1^T) = \mathrm{Im}\left(A^T K^T\right) \subseteq A^T(EE^T)^{-1}E(\Delta^0_{E,A,C} \cap \mathrm{Im}(E^T)) \subseteq \Delta^1_{E,A,C}$.
By denoting $H_0 x = \Gamma_0 C x$, we have $H = H_0 + H_1$ and according to the first step of this proof (for $i = 0$), $\mathrm{Im}(H_0^T) \subseteq \Delta^0_{E,A,C} \subseteq \Delta^1_{E,A,C}$. Thus, $\mathrm{Im}(H^T) = \mathrm{Im}(H_0^T + H_1^T) \subseteq \Delta^1_{E,A,C}$.

Assume now that property P_1 is valid until $i = i_0$, we prove hereafter that it remains true for $i = i_0 + 1$.

If we can define Γ and β such that $Hx = \Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i_0)} \\ y^{(i_0+1)} \end{pmatrix} + \beta u$, then there must exist two matrices Γ_0 and Γ_1 such that

$$Hx = \Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i_0)} \\ y^{(i_0+1)} \end{pmatrix} + \beta u = \Gamma_0 \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i_0)} \end{pmatrix} + \frac{d}{dt}\Gamma_1 \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i_0)} \end{pmatrix} + \beta u$$

where from definition 1, $\Gamma_1 \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i_0)} \end{pmatrix} = \Gamma_1 Mx$ with $\mathrm{Im}\left((\Gamma_1 M)^T\right) \subseteq \mathrm{Im}(E^T)$. The latter inclusion implies that there is a matrix K such that $\Gamma_1 M = KE$. Then, we can write

$$\frac{d(\Gamma_1 Mx)}{dt} = KE\dot{x} = KAx + KBu \qquad (7)$$

Since E is full row rank, (EE^T) is invertible and so matrix K is unique and is given by

$$K = \Gamma_1 ME^T(EE^T)^{-1}$$

Let $H_1 x = \frac{d\Gamma_1 Mx}{dt} + \beta_1 u$, then equation (7) implies that $H_1 = KA$ and $\beta_1 = -KB$. Moreover, since property P_1 is verified until $i = i_0$, we have $\mathrm{Im}\left((\Gamma_1 M)^T\right) \subseteq \Delta^{i_0}_{E,A,C}$. Furthermore, as $\mathrm{Im}\left((\Gamma_1 M)^T\right) \subseteq \mathrm{Im}(E^T)$, then $\mathrm{Im}\left((\Gamma_1 M)^T\right) \subseteq \Delta^{i_0}_{E,A,C} \cap \mathrm{Im}(E^T)$. Thus, $\mathrm{Im}(H_1^T) = \mathrm{Im}\left(A^T K^T\right) \subseteq A^T(EE^T)^{-1}E(\Delta^{i_0}_{E,A,C} \cap \mathrm{Im}(E^T)) \subseteq \Delta^{i_0+1}_{E,A,C}$.

By denoting $H_0 x = \Gamma_0 \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i_0)} \end{pmatrix} + \beta_0 u$, such that $\beta_0 + \beta_1 = \beta$, we have then $H = H_0 + H_1$.
Since property P_1 is true until $i = i_0$, we have $\mathrm{Im}(H_0^T) \subseteq \Delta^{i_0}_{E,A,C} \subseteq \Delta^{i_0+1}_{E,A,C}$. Therefore, we obtain $\mathrm{Im}(H^T) = \mathrm{Im}(H_0^T + H_1^T) \subseteq \Delta^{i_0+1}_{E,A,C}$. P_1 is then proved.

$w = L_x x$ is reconstructible using the knowledge of inputs u, measured variables y and their derivatives means that there exist an integer i, matrices Γ and β such that $w = \Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i)} \end{pmatrix} + \beta u$. In this case, from P_1, $\mathrm{Im}(L_x^T) \subseteq \Delta^{i}_{E,A,C}$. Consequently, according to the fact that $\forall i \geq 0$, $\Delta^{i}_{E,A,C} \subseteq \Delta^{*}_{E,A,C}$, it follows that $\mathrm{Im}(L_x^T) \subseteq \Delta^{*}_{E,A,C}$. We prove then the necessity of proposition 2.

Sufficiency: Firstly, we demonstrate property P_2:

P_2**:** *for all* $i \geq 0$, *if* $\mathrm{Im}(H^T) \subseteq \Delta^{i}_{E,A,C}$, *then we can find matrices* Γ *and* β *such that* $Hx = \Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i)} \end{pmatrix} + \beta u$

For $i = 0$, it is obvious that if $\mathrm{Im}(H^T) \subseteq \Delta^{0}_{E,A,C} = \mathrm{Im}(C^T)$ then there is a matrix Γ such that $H = \Gamma C$ and so $Hx = \Gamma y$.
For $i = 1$, if $\mathrm{Im}(H^T) \subseteq \Delta^{1}_{E,A,C} = \Delta^{0}_{E,A,C} + A^T(EE^T)^{-1}E(\Delta^{0}_{E,A,C} \cap \mathrm{Im}(E^T))$, then there exist matrices H_0 and H_1 such that $H = H_0 + H_1$ with $\mathrm{Im}(H_0^T) \subseteq \Delta^{0}_{E,A,C}$ and $\mathrm{Im}(H_1^T) \subseteq A^T(EE^T)^{-1}E(\Delta^{0}_{E,A,C} \cap \mathrm{Im}(E^T))$. Since P_2 is true for $i = 0$, there is a matrix Γ_0 such that $H_0 x = \Gamma_0 y$.
Moreover, $\Delta^{0}_{E,A,C} \cap \mathrm{Im}(E^T)$ represents the permitted derivable part of $\Delta^{0}_{E,A,C}$. This implies that there exist two matrices K and Γ_1 with $KE = \Gamma_1 C$ such that $H_1 x = KEE^T(EE^T)^{-1}Ax = KAx$. According to the system dynamics (4), we can write $H_1 x = KE\dot{x} - KBu$ and so $H_1 x = \Gamma_1 C\dot{x} - KBu = \Gamma_1 \dot{y} + \beta u$. Therefore, we can find matrices Γ and β such that $Hx = \Gamma\begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \beta u$.

Assume now that property P_2 is valid until $i = i_0$, we prove hereafter that it remains true for $i = i_0 + 1$.
Indeed, $\mathrm{Im}(H^T) \subseteq \Delta^{i_0+1}_{E,A,C}$ implies that there exist matrices H_0 and H_1 such that $H = H_0 + H_1$ with $\mathrm{Im}(H_0^T) \subseteq \Delta^{i_0}_{E,A,C}$ and $\mathrm{Im}(H_1^T) \subseteq A^T(EE^T)^{-1}E(\Delta^{i_0}_{E,A,C} \cap \mathrm{Im}(E^T))$. It follows that there are also two matrices K and Γ_1 verifying $KE = \Gamma_1 M$ with $\mathrm{Im}(M^T) = \Delta^{i_0}_{E,A,C}$ such that $H_1 x = KEE^T(EE^T)^{-1}Ax = KAx$. According to the system dynamics (4), we can write $H_1 x = KE\dot{x} - KBu$ and so $H_1 x = \Gamma_1 M\dot{x} - KBu$. Furthermore, since we have supposed that P_2 is true until $i = i_0$, then $H_1 x = \Gamma_1' \begin{pmatrix} \dot{y} \\ y^{(2)} \\ \vdots \\ y^{(i_0)} \\ y^{(i_0+1)} \end{pmatrix} + \beta_1 u$. Also, we can find matrices Γ_0 and β_0 such that $H_0 x = \Gamma_0 \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i_0)} \end{pmatrix} + \beta_0 u$.
Consequently, there exist matrices Γ and β such that $Hx = H_0 x + H_1 x = \Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(i_0)} \\ y^{(i_0+1)} \end{pmatrix} + \beta u$. P_2 is then proved.
Assume now that $\mathrm{Im}(L_x^T) \subseteq \Delta^{*}_{E,A,C}$, then according to the previous results, we can find matrices Γ and β such that $L_x x = \Gamma \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(\kappa_0)} \end{pmatrix} + \beta u$ where κ_0 is such that $\Delta^{\kappa_0}_{E,A,C} = \Delta^{\kappa_0+1}_{E,A,C} = \Delta^{*}_{E,A,C}$. The sufficiency is then also proved and the proposition follows. △

We can do some comments about the previous result. The first one is that if E equals the identity matrix, $\Delta^{*}_{E,A,C}$ represents the classical well-known observability subspace. Secondly, when L_x equals the identity matrix, we can prove easily that proposition 2 is equivalent to $rank \begin{bmatrix} sE - A \\ C \end{bmatrix} = n$ for all $s \in \mathbb{C}$, which is the necessary and sufficient condition to ensure system (4) is right-hand side observable and consequently that vector x can be uniquely determined from equations (4) (Hou and Müller, 1999*b*). Moreover, when L_x is different from the identity matrix, our conditions are evidently less restrictive than the observability of whole the state vector.

Finally, since we assume that $rankE \leq n$, the state x may not be unique. The conditions of proposition 2 do not discuss about theuniqueness or the solvability of system (1). Therefore, readers interested by the solvability of descriptor systems can refer to (Geerts, 1993).

3.2 Observation of a functional of the state and the unknown inputs

In this part, we extend previous results to the case where the system is driven by unknown inputs, which have to be reconstructed.

In order to generalise previous results, it is useful to represent descriptor system (1) in the same form than descriptor system (4). At this aim, let us define extended state $x_e = \begin{pmatrix} x \\ v \end{pmatrix}$. Using this notation, system (1) becomes:

$$\begin{aligned} E_e \dot{x}_e &= A_e x_e + Bu \\ y &= C_e x_e \end{aligned} \qquad (8)$$

with $E_e = \begin{pmatrix} E \mid 0 \end{pmatrix}$, $A_e = \begin{pmatrix} A \mid F \end{pmatrix}$, $C_e = \begin{pmatrix} C \mid D \end{pmatrix}$.
Moreover, functional $w = L_x x + L_v v$ to be reconstructed can be written $w = L_{x_e} x_e$ where $L_{x_e} = \begin{pmatrix} L_x \mid L_v \end{pmatrix}$. According to proposition 2 and equation (8), we deduce the following proposition.

Proposition 3. For system (1), w is reconstructible using the knowledge of input u, measured variable y and its derivatives iff $\mathrm{Im}(L_{x_e}^T) \subseteq \Delta^*_{E_e,A_e,C_e}$. □

Proof:
The proof is an immediate consequence of proposition 2 by replacing matrices E, A, C and L_x by respectively E_e, A_e, C_e and L_{x_e}.

4. EXAMPLE

In this section, we will illustrate the results presented above on a simple example. Nevertheless, it is clear that the proposed method is very well-adapted to more complex systems.

Consider the linear system in descriptor form defined by:

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } D = 0.$$

Suppose that we want to reconstruct $w = (x_1,\ x_2,\ x_3,\ x_4,\ x_6,\ v_1\ )^T$.

The extended state vector is equal to $x_e = (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ v_1,\ v_2\ )^T$. The matrices of extended system form (8) are such that:

$$E_e = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix},$$

$$A_e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}$$

and $C_e = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Moreover, in the extended state space, the functional of the state to reconstruct is $w = L_{x_e} x_e$ with

$$L_{x_e} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We must now compute $\Delta^*_{E_e,A_e,C_e}$ according to the subspace sequence defined by relations (5) in the extended state space:

$\Delta^0_{E_e,A_e,C_e} = Im(C_e^T)$;

$$\Delta^1_{E_e,A_e,C_e} = \Delta^0_{E_e,A_e,C_e} + \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 3 \\ -2 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ -2 \\ 0 \\ -7 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\};$$

$\Delta^2_{E_e,A_e,C_e} = \Delta^1_{E_e,A_e,C_e} \Rightarrow \Delta^*_{E_e,A_e,C_e} = \Delta^1_{E_e,A_e,C_e}$.

We can also write for convenience $\Delta^*_{E_e,A_e,C_e} =$

$$\mathrm{Im}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ Thus, we have $\mathrm{Im}(L_{x_e}^T) \nsubseteq \Delta^*_{E_e,A_e,C_e}$.

Consequently, it is not possible to reconstruct w. Nevertheless, if we want to reconstruct $w' = L'_{x_e} x_e = \begin{pmatrix} x_1,\ x_2,\ x_3,\ x_4,\ x_6,\ v_1 + v_2 \end{pmatrix}^T$, it is easy to see that $\mathrm{Im}({L'_{x_e}}^T) \subseteq \Delta^*_{E_e,A_e,C_e}$. So w' and all the combinations of its components are reconstructible.

Comparatively with the most recent works on the functional state observers, our approach allows us to answer the question whether or not a desired linear function of the state and unknown input can be reconstructed while other works deal with the design of functional observer when the system verifies some conditions. It results that, in these works, it is impossible to conclude about the reconstructibility because the design conditions are linked to the proposed observer structure and not to the system structure.

To illustrate this fact, it appears that it is possible to reconstruct $w'' = x_3$ in the previous example. However, if we refer to (Darouach, 2000), it can be seen that design conditions of a first order observer estimating w'' are not satisfied. That is why, the design method proposed in this latter work fails. On other hand, many works dealing with the synthesis of functional observers fail to design an observer for the system defined in this example since it is not completely observable.

5. CONCLUSION

In this paper, we use a subspace sequence to provide necessary and sufficient conditions to check the total or partial right-hand side observability of the state and the unknown inputs for linear systems on descriptor form. These conditions do not depend on a specific observer form. Furthermore, contrary to many works on functional observation, our conditions do not require the complete observability of the system. Finally, our approach is mainly an analysis one and it can be greatly improved by proposing in further works an observer design method to achieve the estimation of a desired functional of unmeasured variables.

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