STABILIZATION WITH J-DISSIPATIVE CONTROLLERS

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Abstract: Let $J = \mathrm{diag}(1, -1)$, and let $\mathfrak{B}$ be a controllable behavior. Let $\mathfrak{B}_{\mathrm{des}}$ be a stable, autonomous subspace of $\mathfrak{B}$ representing the desired behavior after feedback interconnection with some controller $\mathfrak{C}$. In this paper we address the following questions: does there exist a J-dissipative controller $\mathfrak{C}$ such that $\mathfrak{C} \cap \mathfrak{B} = \mathfrak{B}_{\mathrm{des}}$? How many unstable poles does the transfer function associated with the controllable part of $\mathfrak{C}$ have?

Keywords: Stabilization, dissipativity, interconnection, quadratic differential forms, metric interpolation problem

1. INTRODUCTION

In this paper we consider the following problem: *Let*

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and let $\mathfrak{B}$ be a controllable behavior. Let $\mathfrak{B}_{des}$ be a stable, autonomous subspace of $\mathfrak{B}$ representing the desired behavior after feedback interconnection with some controller $\mathfrak{C}$.

Does there exist a J-dissipative controller $\mathfrak{C}$ such that $\mathfrak{C} \cap \mathfrak{B} = \mathfrak{B}_{des}$?

Assuming such a controller exists, how many unstable poles does the transfer function associated with the controllable part of $\mathfrak{C}$ have?

We show that the solvability of this problem is closely related to the solvability of a metric interpolation problem associated with $\mathfrak{B}_{\mathrm{des}}$ and the signature matrix J. Consequently, the issue whether a stable, J-dissipative controller exists can be addressed by checking the sign-definiteness of a certain Pick-type matrix associated with J and with $\mathfrak{B}_{\mathrm{des}}$. The problem we consider in this paper brings together the theory of control-as-interconnection developed in (Willems, 1997), the characterization of stabilizing controllers developed in (Kuijper, 1995), and the theory of metric interpolation as exact modeling developed in (Rapisarda and Willems, 1997), (Kaneko and Rapisarda, 2003).

The papers (Kimura, 1984), (Kimura, 1989) and (Tannenbaum, 1980) introduced metric interpolation methods in the study of H_∞ control problems, and the paper (Cevik and Schumacher, 1997) took a metric interpolation approach to the regulation problem. This paper is much in the same spirit, and takes advantage of the polynomial framework developed in the behavioral approach. An essential role in the description of the problem and its solution is played by the notion of quadratic differential forms (QDFs), i.e. quadratic functionals of a system variables and their derivatives, introduced in (Willems and Trentelman, 1998).

In this paper we assume that the reader is familiar with the basics of the behavioral approach, with the framework for exact modeling, and with quadratic differential forms. The interested reader is referred to (Polderman and Willems, 1997), (Willems, 1986), and (Willems and Trentelman, 1998).

Notation. In this paper we denote the sets of real numbers with $\mathbb{R}$, and the set of complex numbers with $\mathbb{C}$. Let $\mathbb{C}_+$ ($\mathbb{C}_-$) denote the open right-half (left-half, respectively) plane. The space of $\mathtt{n}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and the space of $\mathtt{m}\times\mathtt{n}$ real matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. If $A \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, then $A^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$ denotes its transpose. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{C}^{\bullet\times\mathtt{n}}$ denotes the set of complex matrices with $\mathtt{n}$ columns and an unspecified number of rows. If $A_i \in \mathbb{R}^{\bullet\times\bullet}$, $i = 1, \dots, r$ have the same number of columns, $\mathrm{col}(A_i)_{i=1,\dots,r}$ denotes the matrix

$$\begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}$$

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$. The space of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$. Given a matrix $R \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, we define $R^*(\xi) := R^T(-\xi) \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$. If $R(\xi)$ has complex coefficients, then $R^*(\xi)$ denotes the matrix obtained from R by substituting $-\xi$ in place of ξ, transposing, and conjugating.

We denote with $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$.

2. CONTROL AS INTERCONNECTION

In many control applications, the classic point of view of the controller as a signal processor accepting the plant output and deriving control inputs based on such outputs is unsuitable. Situations of this sort occur for example in many mechanical control systems, such as car dampers or operational amplifiers (see (Willems, 1997)), where the point of view of control as interconnection, and of the controller imposing new additional laws on the plant variables, is better suited. We introduce this idea first in the case in which all the external variables w of a plant are available for interconnection.

The *interconnection of* $\Sigma_1 = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}_1)$ and $\Sigma_2 = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}_2)$ is denoted by $\Sigma_1 \wedge \Sigma_2$ and defined as

$$\Sigma_1 \wedge \Sigma_2 := (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}_1 \cap \mathfrak{B}_2).$$

An interconnection is called a *feedback interconnection* if $\mathtt{p}(\Sigma_1 \wedge \Sigma_2) = \mathtt{p}(\Sigma_1) + \mathtt{p}(\Sigma_2)$, where $\mathtt{p}(\Sigma)$ is the number of outputs of the system (see (Polderman and Willems, 1997)). A feedback interconnection of two systems with $\mathtt{w}$ external variables is called *autonomous* if $\mathtt{p}(\Sigma_1 \wedge \Sigma_2) = \mathtt{w}$; in such case the behavior $\mathfrak{B}_1 \cap \mathfrak{B}_2$ is a finite-dimensional subspace, and consequently can be represented in kernel form by a square, nonsingular polynomial matrix. A feedback interconnection is called *asymptotically stable* if it is autonomous and all the trajectories of $\mathfrak{B}_1 \cap \mathfrak{B}_2$ tend to zero as $t \to \infty$.

Often, not all the external variables are available for interconnection, and the controller imposes new additional constraints on only a subset of the variables of the plant, the *interconnection variables* c. For the purposes of this paper, we can restrict ourselves to the case of a controllable plant represented in image form, in which all the latent variables ℓ are available for interconnection, i.e. $c = \ell$. Consider a controllable behavior $\mathfrak{B}$ represented in observable image form by a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$, i.e.

$$\begin{aligned} \mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \ \mid \ \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}}) \\ \text{such that } w = M(\tfrac{d}{dt})\ell\}. \end{aligned}$$

In many applications, and most notably in *LQ*-control, a certain desired subbehavior $\mathfrak{B}_{\mathrm{des}}$ is specified, which must be obtained from the plant $\mathfrak{B}$ by restricting the latent variable ℓ to satisfy certain differential equations. Formally,

$$\begin{aligned} w &= M(\frac{d}{dt})\ell \\ 0 &= D(\frac{d}{dt})\ell \end{aligned} \qquad (2)$$

and

$$\begin{aligned} \mathfrak{B}_{\mathrm{des}} = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \ \mid \ \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{l}}) \\ \text{such that (2) are satisfied}\}. \end{aligned} \qquad (3)$$

It is easy to see that $\mathfrak{B}_{\mathrm{des}}$ is finite-dimensional if and only if the matrix $D \in \mathbb{R}^{\bullet\times\mathtt{l}}[\xi]$ in (2) has full column rank $\mathtt{l}$, in which case it can be assumed without loss of generality that D is a nonsingular polynomial matrix.

In Lemma 3.2 p. 623 of (Kuijper, 1995) a parametrization of all controllers $\mathfrak{C}$ that give rise to the behavior $\mathfrak{B}_{\rm des}$ when interconnected with $\mathfrak{B}$ is given.

Theorem 1. Let $\mathfrak{B}$ be a controllable behavior, and let $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ induce a minimal kernel, respectively observable image, representation of $\mathfrak{B}$. Define $\mathfrak{B}_{\rm des}$ as in (3). Let $\mathfrak{B}' \in \mathcal{L}^{\mathtt{w}}$ be represented in kernel form by $C \in \mathbb{R}^{\mathtt{m}\times\mathtt{w}}[\xi]$ such that $\mathrm{col}(R, C)$ is nonsingular, and define $\mathfrak{B}_{\rm cl} := \mathfrak{B} \cap \mathfrak{B}' = \ker(\mathrm{col}(R(\frac{d}{dt}), C(\frac{d}{dt})))$.

Then $\mathfrak{B}_{\rm des} = \mathfrak{B}_{\rm cl}$ if and only if there exists a unimodular matrix $U \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$ such that

$$C \cdot M = U \cdot D$$

The necessary and sufficient condition of Theorem 1 will be instrumental in deriving necessary and sufficient conditions for a J-dissipative, stable controller to exist. In order to state such conditions, we need first to illustrate the basic features of metric interpolation problems.

3. THE TAKAGI INTERPOLATION PROBLEM

We now introduce the *Takagi interpolation problem* (see (Takagi, 1924)); in order to do this, we will expand on the results of (Rapisarda and Willems, 1997), where the subspace Nevanlinna interpolation problem has been solved using the concept of most powerful unfalsified model introduced in (Willems, 1986).

Let $w_i : \mathbb{R} \to \mathbb{C}^{\mathtt{w}}$, $i = 1, \dots, N$, be given functions; for the purposes of this paper, we assume that $w_i \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{C}^{\mathtt{w}})$ for all i. Let $\mathcal{M} \subseteq 2^{(\mathbb{C}^{\mathtt{w}})^{\mathbb{R}}}$ be a class of models, the choice of which reflects the assumptions that the modeler wishes to make on the structure of the phenomenon that produced the w_i's. In this paper, we choose $\mathcal{M} = \mathcal{L}^{\mathtt{w}} \subseteq 2^{\mathfrak{C}^\infty(\mathbb{R},\mathbb{C}^{\mathtt{w}})}$, the class of *linear differential behaviors*, i.e. those that are the kernel of a polynomial differential operator with constant coefficients.

$\mathfrak{B} \in \mathcal{M}$ is an *unfalsified model* for the *data set* $\{w_i\}_{i=1,\dots,N}$ if $w_i \in \mathfrak{B}$ for $i = 1, \dots, N$. We call $\mathfrak{B}^*$ the *Most Powerful Unfalsified Model* (*MPUM*) in $\mathcal{M}$ for the given data set, if it is unfalsified and moreover

$$[w_i \in \mathfrak{B}', i = 1, \dots, N,\ \mathfrak{B}' \in \mathcal{M}] \Longrightarrow [\mathfrak{B}^* \subseteq \mathfrak{B}']$$

i.e. if it is the smallest behavior in $\mathcal{M}$ containing the data. For the case of the model class $\mathcal{L}^{\mathtt{w}}$ it can be shown that the MPUM always exists and that it is unique (see (Willems, 1986)).

We proceed to state the version of the *Takagi interpolation problem* (*TIP*) which will be used in the rest of this paper. Let N distinct points λ_i in the open right-half plane be given, together with N values $b_i \in \mathbb{C}$, and let J be given as in (1). The TIP consists of finding the smallest $k \in \mathbb{N}$ and polynomials $u, y \in \mathbb{R}[\xi]$ such that

(a) u, y are coprime;
(b) y has k roots in $\mathbb{C}_+$;
(c) $\begin{bmatrix} u(\lambda_i) & -y(\lambda_i) \end{bmatrix} \begin{bmatrix} 1 \\ b_i \end{bmatrix} = 0,\ 1 \le i \le N;$
(d) $\|\frac{u}{y}\|_\infty < 1$

Following (Rapisarda and Willems, 1997), we show that this problem can be cast in the framework of exact modeling developed in (Willems, 1986) as follows. We define

$$v_i := \begin{bmatrix} 1 \\ b_i \end{bmatrix},$$

and associate to the data $\{(\lambda_i, v_i)\}_{i=1,\dots,N}$ the set of vector-exponential trajectories $v_i \exp_{\lambda_i}$; then it is easy to see that requirement (c) in the definition of solution to the SNIP is equivalent with

$$v_i \exp_{\lambda_i} \subseteq \ker \begin{bmatrix} u(\frac{d}{dt}) & -y(\frac{d}{dt}) \end{bmatrix},\ i = 1, \dots, N$$

The metric- and stability aspects of the solution to the SNIP (see requirements (d) and (b) above) can be accommodated in the MPUM framework, provided one constructs a special kernel representation for the MPUM associated to the "dualized data", which we now introduce.

Given the interpolation data $\{(\lambda_i, v_i)\}_{i=1,\dots,N}$, we consider the "mirror image" (see also (Antoulas and Anderson, 1989)) of v_i, defined as

$$v_i^\perp := \begin{bmatrix} \bar{b}_i \\ 1 \end{bmatrix}$$

and we define the *dual* of $v_i \exp_{\lambda_i}$ to be

$$v_i^\perp \exp_{-\bar{\lambda}_i}$$

We also define the *dualized data* $\mathcal{D}$ as

$$\mathcal{D} := \cup_{i=1,\dots,N}\{v_i \exp_{\lambda_i}, v_i^\perp \exp_{-\bar{\lambda}_i}\} \qquad (4)$$

Now consider the following procedure for the construction of a kernel representation of the MPUM for the data $\{(\lambda_i, v_i)\}_{1\le i\le N}$.

Algorithm

Input: $\{(\lambda_i, v_i)\}_{i=1,\dots,N}$
Output: Kernel representation of MPUM for $\mathcal{D}$

Define $R_0 := I_2$;
For $i = 1, \dots, N$
$\epsilon_i := R_{i-1}(\lambda_i) v_i$;
$R_i(\xi) := \left[(\xi + \bar{\lambda}_i) I_2 - \epsilon_i (\frac{\epsilon_i^* J \epsilon_i}{\lambda_i + \bar{\lambda}_i})^{-1} \epsilon_i^* J\right] R_{i-1}(\xi)$;
end;

Proof of the correctness of this algorithm can be found in (Rapisarda and Willems, 1997).

We now relate the properties of the representation of the $MPUM$ for the dualized data $\mathcal{D}$ to those of the Pick matrix of the data.

Theorem 2. Assume that the Hermitian matrix $T_N := \left[\frac{1-\bar{b}_i b_j}{\lambda_i+\lambda_j}\right]_{i,j=1,\dots,N}$ (Pick matrix) is invertible. Then the following statements are equivalent:

(1) The Hermitian matrix $T_N := \left[\frac{1-\bar{b}_i b_j}{\lambda_i+\lambda_j}\right]_{i,j=1,\dots,N}$ has k negative eigenvalues;
(2) The algorithm above produces a kernel representation of the MPUM for the dualized data set $\mathcal{D}$ defined in (4) induced by a matrix of the form

$$R := \begin{bmatrix} -d^* & n^* \\ n & -d \end{bmatrix} \quad (5)$$

where $n, d \in \mathbb{R}[\xi]$ satisfy the following properties:
(a) $d \neq 0$;
(b) d has k roots in $\mathbb{C}_+$;
(c) $RJR^* = R^*JR = pp^*J$ with $p(\xi) = \Pi_{i=1}^N(\xi + \bar{\lambda}_i)$;
(d) $\|\frac{n}{d}\|_\infty < 1$;
(e) $\|\frac{n^*}{d}\|_\infty < 1$.

Proof. The proof of this Theorem follows an analogous line to that of the main result of (Rapisarda and Willems, 1997). ■

The special kernel representation of the MPUM for $\mathcal{D}$ described in Theorem 2 allows us to characterize the solutions of the TIP as follows.

Theorem 3. Assume that the Hermitian matrix $T_N := \left[\frac{1-\bar{b}_i b_j}{\lambda_i+\lambda_j}\right]_{i,j=1,\dots,N}$ is invertible and has k negative eigenvalues, and let (5) be the representation of the MPUM for $\mathcal{D}$ computed with the algorithm above.

Then $\begin{bmatrix} u & -y \end{bmatrix}$ is a solution to the TIP with y having k roots in $\mathbb{C}_+$ if and only if there exist $\pi, \phi, f \in \mathbb{R}[\xi]$, with ϕ, f Hurwitz, and $\|\frac{\pi}{\phi}\|_\infty < 1$, such that

$$f \begin{bmatrix} u & -y \end{bmatrix} = \begin{bmatrix} \pi & -\phi \end{bmatrix} \begin{bmatrix} -d^* & n^* \\ n & -d \end{bmatrix} \quad (6)$$

Proof. The proof of this Theorem is analogous to that of the main result of (Rapisarda and Willems, 1997). ■

4. MAIN RESULT

Recall from section 2 that a behavior $\mathfrak{C} \in \mathcal{L}^2$ with a minimal kernel representation induced by $C \in \mathbb{R}^{2\times 1}[\xi]$ yields a desired behavior $\mathfrak{B}_{\text{des}} \in \mathcal{L}^2$ defined as in (3) and represented as in (2) when interconnected with the behavior $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2) \mid w = M(\frac{d}{dt})\ell\}$ if and only if

$$C \cdot M = \alpha \cdot d \quad (7)$$

with $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The following Proposition is instrumental in deriving the main result of this paper.

Proposition 4. Let $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2 \mid w = M(\frac{d}{dt})\ell\} \in \mathcal{L}^{\mathtt{w}}$, and let $\mathfrak{B}_{\text{des}}$ defined as in (3) and represented as in (2). Assume that $D \in \mathbb{R}[\xi]$ in (2) has only simple roots. Let $\mathfrak{C} \in \mathcal{L}^2$ be represented in minimal kernel form by a matrix $C \in \mathbb{R}^{1\times 2}[\xi]$. Then the following two statements are equivalent:

(1) $\mathfrak{C}$ is a controller yielding $\mathfrak{B}_{\text{des}}$ when interconnected with $\mathfrak{B}$;
(2) $\ker C(\frac{d}{dt}) \cap \mathrm{im} M(\frac{d}{dt}) = \mathit{span}\{M(\lambda_i) \mid \lambda_i \text{ such that } d(\lambda_i) = 0\}$.

Proof. Follows from Theorem 1. ■

The result of Proposition 4 connects the problem of unfalsified modeling of vector exponential time series, with the problem of computing a controller which, when interconnected with a given plant, yields a desired autonomous behavior. Indeed, observe that the kernel representation induced by C in statement 2 of Proposition 4 is an interpolant for the data $\{M(\lambda_i) \mid \lambda_i \text{ is such that } d(\lambda_i) = 0\}$ with the additional property (2) in Proposition 4. The following equivalent conditions follow easily from this observation, and from restating the conditions of Theorem 2 for the case of $\lambda_i \in \mathbb{C}_-$, $1 \leq i \leq N$.

Corollary 5. Let $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2) \mid w = M(\frac{d}{dt})\ell\} \in \mathcal{L}^{\mathtt{w}}$, and let $\mathfrak{B}_{\text{des}}$ defined as in (3) and represented as in (2). Assume that D in (2) has only simple roots λ_i, $i = 1, \dots, N$. Then the following two statements are equivalent:

(1) There exists a J dissipative controller yielding $\mathfrak{B}_{\text{des}}$ when interconnected with $\mathfrak{B}$ and whose transfer function has k unstable poles;
(2) There exists a J dissipative behavior $\mathfrak{C}$ with a transfer function having k unstable poles, such that

$$\mathfrak{C} \cap \mathfrak{B} = \mathfrak{B}^{\text{des}}$$

(3) There exists a solution $\begin{bmatrix} u & -y \end{bmatrix}$ to the TIP with data $\{(\lambda_i, M(\lambda_i))\}$, such that

$$\ker \begin{bmatrix} u(\frac{d}{dt}) & -y(\frac{d}{dt}) \end{bmatrix} \cap \mathfrak{B} = \mathfrak{B}^{\text{des}}$$

and y has k roots in $\mathbb{C}_+$;
(4) There exists a behavior $\mathfrak{C}$ with controllable part $\mathfrak{C}^{\text{contr}}$ such that
(a) $\mathfrak{C} \supset$ lin span $\mathcal{D}$, with $\mathcal{D}$ the dualized data set (4);
(b) $\mathfrak{C}^{\text{contr}}$ is dissipative;

(c) $\mathfrak{C}^{\text{contr}} \cap \mathfrak{B} = \mathfrak{B}^{\text{des}}$;
(d) The transfer function associated with $\mathfrak{C}^{\text{contr}}$ has k poles in $\mathbb{C}_+$.

The following necessary condition for the solvability of the stabilization with dissipative controller problem derives easily from Corollary 5.

Corollary 6. Assume that $D \in \mathbb{R}[\xi]$ in (2) is Hurwitz and has only simple roots. If the plant represented in observable image form by M can be interconnected with a dissipative controller having a transfer function with k unstable poles in order to achieve $\mathfrak{B}_{des}$ described in (3), then the *Pick matrix associated to the closed-loop behavior*:

$$\left[\frac{M(\lambda_i)^* J M(\lambda_j)}{\bar{\lambda}_i + \lambda_j}\right]_{i,j=1,\ldots,N} \tag{8}$$

must have k negative eigenvalues.

We now proceed to establish another equivalent condition for the solvability of the stabilization problem with a dissipative controller.

We consider plants represented in observable image form by a polynomial vector

$$M = \begin{bmatrix} p \\ q \end{bmatrix} \tag{9}$$

with $p, q \in \mathbb{R}[\xi]$, $GCD(p,q) = 1$, $\deg(p) \geq \deg(q)$. Note that the latter assumption implies that the variables w are partitioned as $\text{col}(u, y)$ with y an input variable and u an output variable.

Assume that the Pick matrix of the data is nonsingular. It follows then from the fact that (5) is a representation for the MPUM for $\mathcal{D}$, that there exist $e_i \in \mathbb{R}[\xi]$, $i = 1, 2$ such that

$$\begin{bmatrix} d(-\xi) & -n(-\xi) \\ n(\xi) & -d(\xi) \end{bmatrix} M(\xi) = \begin{bmatrix} e_1(\xi) \\ e_2(\xi) \end{bmatrix} D(\xi) \tag{10}$$

Consequently, $\text{col}(e_1, e_2)D$ represents how the model represented by (5) fails to explain the trajectories $M(\lambda)\exp_\lambda$, $\lambda \neq \lambda_i$, $i = 1, \ldots, N$. For this reason, we call

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

an image representation of the *error system of the MPUM (5) on* $\text{im}M(\frac{d}{dt})$, defined as

$$\mathfrak{E} := \text{im} \begin{bmatrix} e_1(\frac{d}{dt}) \\ e_2(\frac{d}{dt}) \end{bmatrix} \tag{11}$$

Observe that the error system depends on the particular representation (5) of the MPUM.

We now state one additional equivalent condition for the solvability of the stabilization problem with dissipative controller.

Theorem 7. Let $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2) \mid w = M(\frac{d}{dt})\ell\} \in \mathcal{L}^2$, and let $\mathfrak{B}_{\text{des}}$ defined as in (3) and represented as in (2). Assume that $D \in \mathbb{R}[\xi]$ has only simple roots. Then the following statements are equivalent:

(1) There exists a J dissipative controller yielding $\mathfrak{B}_{\text{des}}$ when interconnected with $\mathfrak{B}$, and whose controllable behavior has a transfer function with k poles in $\mathbb{C}_+$;
(2) There exists a representation of the MPUM for $\mathcal{D}$ such that d has k roots in $\mathbb{C}_+$. Let the associated error system be represented by (11). Then there exist $\pi, \phi \in \mathbb{R}[\xi]$ with ϕ Hurwitz and $\|\frac{\pi}{\phi}\|_\infty < 1$, and $\alpha \in \mathbb{R}$ such that
$$\begin{bmatrix} \pi & -\phi \end{bmatrix} \begin{bmatrix} d^* & n^* \\ n & -d \end{bmatrix} = \alpha(\pi e_1 - \phi e_2) \begin{bmatrix} u & -y \end{bmatrix}$$
and $GCD(u, y) = 1$.

Proof. The result can be proved using the characterization of all interpolants given in Theorem 3. ■

5. EXAMPLE

Consider the plant represented in image form by

$$M(\xi) = \begin{bmatrix} \frac{2}{3} + \frac{\xi}{6} \\ 1 \end{bmatrix}$$

Let $\mathfrak{B}_{\text{des}} = \text{span}\,\{M(-1)\exp_{-1}, M(-2)\exp_{-2}\}$; is it possible to obtain such closed-loop behavior connecting M to a stable, J-dissipative controller?

Using the necessary condition of Corollary 6, we first check the sign of the Pick matrix associated with the data $\{M(-1)\exp_{-1}, M(-2)\exp_{-2}\}$. It can be verified that this matrix

$$\begin{bmatrix} \frac{3}{8} & \frac{5}{18} \\ \frac{5}{18} & \frac{2}{9} \end{bmatrix}$$

is positive definite. We conclude that there exists a representation (5) of the MPUM for the dualized set of data, with d stable. It can be verified that one such representation is

$$\begin{bmatrix} \frac{17}{4} - \frac{15}{4}\xi + \xi^2 & \frac{3}{4}\xi - \frac{15}{4} \\ -\frac{3}{4}\xi - \frac{15}{4} & \frac{17}{4} + \frac{15}{4}\xi + \xi^2 \end{bmatrix} \tag{12}$$

An image representation of the error system corresponding to (12) is

$$\begin{bmatrix} \frac{1}{24}(-11 + 4\xi) \\ \frac{7}{8} \end{bmatrix}$$

Define $\mathfrak{C} := \ker \begin{bmatrix} -\frac{3}{4}\frac{d}{dt} - \frac{15}{4} & \frac{17}{4} + \frac{15}{4}\frac{d}{dt} + \frac{d^2}{dt^2} \end{bmatrix}$, the behavior represented by the second row of the MPUM representation (12). Observe that the transfer function associated with $\mathfrak{C}$ is stable; moreover, it can be proved that its infinity norm equals 0.8824.

Now since

$$\begin{bmatrix} -\frac{3}{4}\xi - \frac{15}{4} & \frac{17}{4} + \frac{15}{4}\xi + \xi^2 \end{bmatrix} M(\xi) = \frac{7}{8}(\xi+1)(\xi+2)$$

the closed-loop behavior $\mathfrak{B} \cap \mathfrak{C}$, represented by

$$\begin{bmatrix} -1 & \frac{2}{3} + \frac{\xi}{6} \\ -\frac{3}{4}\xi - \frac{15}{4} & \frac{17}{4} + \frac{15}{4}\xi + \xi^2 \end{bmatrix}$$

is exactly equal to $\mathfrak{B}_{\mathrm{des}}$. It can be verified easily that the characterization given by Theorem 7 holds true in this case, with $\alpha = \frac{8}{7}$.

6. CONCLUSIONS AND FURTHER WORK

The main theme of this paper is the connections existing between stabilization, metric interpolation, and exact identification. The research presented in this paper is being extended in several directions, most notably the following ones.

The multivariable case The multivariable case ($\mathtt{w} > 2$) presents no conceptual difficulty, although the technical difficulties involved in the extension of our results are not negligible. Research in this direction is already under way.

Efficient algorithms In view of the extension of the results presented in this communication to the multivariable case, it is especially important to develop efficient algorithms that take as inputs the polynomial matrix representations of the plant and of the error system, and compute a kernel representation of the controller. The result of Theorem 7 can be considered as a first step in this direction.

State-space formulas The state-space case is a special case of the results presented in this paper; however, deriving explicit state-space formulas is a task deserving interest in its own right; in this respect, see also (Gohberg and Olshevsky, 1994).

Generalizations The most pressing generalization of the results presented in this paper is a discussion of the problem of *stabilizability*, in which the closed-loop behavior $\mathfrak{B}_{\mathrm{des}}$ is not explicitly given, but is required only to be stable. In such case, the question whether the plant is stabilizable at all should find adequate treatment in the framework proposed in this paper.

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