RATIONAL BEZOUT EQUATION AND INTERCONNECTION OF LINEAR SYSTEMS

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Abstract: The Bezout equation over the rings of proper stable rational functions and matrices is studied in this paper. First, a relationship between the rational Bezout equation and a combined serial/parallel interconnection of linear systems is established. The controllability and observability properties that this scheme has to fulfill in order the Bezout equation to be satisfied yield a numerical procedure for finding a particular solution of the concerned equation. This routine is usable for problems of small-to-medium size as demonstrated by numerical experiments.

Keywords: Rational Bezout equation, polynomial design methods, controllability, observability, numerical algorithms

1. INTRODUCTION

Consider the standard unit feedback loop shown in figure 1.

Let M/N be a stable coprime factorization of the plant and Y/X be a stable coprime factorization of the controller, with $M, N, X, Y \in \mathbf{S}$ where $\mathbf{S}$ denotes the set (ring) of proper stable rational functions. The closed loop is internally BIBO stable [2] if and only if the four transfer functions

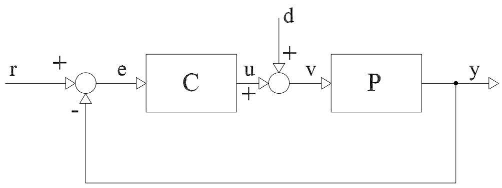

Fig. 1. Closed loop system

$$\frac{1}{1+PC}\begin{bmatrix} PC & P \\ C & -PC \end{bmatrix} =$$

$$= \frac{1}{NX+MY}\begin{bmatrix} NX & NY \\ MX & -NX \end{bmatrix}, \qquad (1)$$

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[2] bounded input results in bounded output

relating the reference and disturbance inputs to the plant and controller outputs, are stable (Kučera, 1991) (Kailath, 1980). We can see that this requirement is satisfied if and only if the term $NX + MY$ is a unit in $\mathbf{S}$. Hence the controllers that stabilize the given plant are generated by all solutions of the equation

$$NX + MY = 1, \tag{2}$$

known as the Bezout equation or Bezout identity. The set of all stabilizing controllers can be further expressed in the following form (the Youla-Kucera parameterization, (Kučera, 1991) (Kailath, 1980)):

$$C = \frac{\widehat{X} + MW}{\widehat{Y} - NW},\ \widehat{Y} - NW \neq 0, \tag{3}$$

where the pair $\widehat{X}, \widehat{Y}$ is a particular solution of (2) and $W \in \mathbf{S}$ is a free parameter.

This fundamental result remains valid for MIMO systems as well. However, one must distinguish between the left and right coprime factorization of the plant $P = NM^{-1} = \widetilde{M}^{-1}\widetilde{N}$, where $N, M, \widetilde{N}, \widetilde{M} \in \mathbf{M(S)}$ are matrices with entries in $\mathbf{S}$, and the situation gets slightly more involving. Refer to (Kučera, 1991) (Vidyasagar, 1987) for details.

A reader interested in further subtleties of the algebraic approach and the theory of stabilizing controllers is referred to the textbooks (Kučera, 1991) (Vidyasagar, 1987).

2. BEZOUT EQUATION AND SYSTEMS INTERCONNECTION

In the sequel the state-space descriptions of the rational arguments M, N, X, Y involved in the Bezout identity (2) shall be addressed. Let us consider the state-space representation of an LTI system in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{4}$$

State matrices A, B, C, D will be marked by the sub-indices M, N, X, Y to be clear which system they stand for.

The term $G(s) = N(s)X(s) + M(s)Y(s)$ represents a combined serial and parallel connection of four systems N, M, X, Y. State-space matrices of the overall system $G(s)$ are composed as follows (see (Kailath, 1980) for instance):

$$A_I = \begin{bmatrix} A_X & 0 & 0 & 0 \\ B_N C_X & A_N & 0 & 0 \\ 0 & 0 & A_Y & 0 \\ 0 & 0 & B_M C_Y & A_M \end{bmatrix}, \tag{5}$$

$$B_I = \begin{bmatrix} B_X \\ B_N D_X \\ B_Y \\ B_M D_Y \end{bmatrix}, \tag{6}$$

$$C_I = [\, D_N C_X \quad C_N \quad D_M C_Y \quad C_M \,], \tag{7}$$

$$D_I = D_N D_X + D_M D_Y, \tag{8}$$

In order to solve the Bezout equation (2) we require this overall system to have the transfer function $G(s)$ equal to identity: $G(s) = N(s)X(s) + M(s)Y(s) = I$. Therefore we must find systems $X(s), Y(s) \in \mathbf{S}$ such that the dynamics of $G(s)$ is hidden. It means that the poles of the connection are unobservable and/or uncontrollable and the product of the observability and controllability matrix must therefore be zero.

$$\mathcal{O} = \begin{bmatrix} C_I \\ C_I A_I \\ C_I A_I^2 \\ \vdots \end{bmatrix},\ \mathcal{C} = [\, B_I \quad A_I B_I \quad A_I^2 B_I \quad \cdots \,],$$

$$\mathcal{OC} = \begin{bmatrix} C_I B_I & C_I A_I B_I & \cdots \\ C_I A_I B_I & C_I A_I^2 B_I & \cdots \\ C_I A_I^2 B_I & C_I A_I^3 B_I & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = 0.$$

Such a way we arrive at a set of matrix equations

$$C_I A_I^i B_I = 0. \tag{9}$$

The power of the matrix A_I can be rewritten as

$$A_I^n = \begin{bmatrix} A_X^n & 0 & 0 & 0 \\ \sum_{i=0}^{n-1} A_N^i B_N C_X A_X^{n-1-i} & A_N^n & 0 & 0 \\ 0 & 0 & A_Y^n & 0 \\ 0 & 0 & \sum_{i=0}^{n-1} A_M^i B_M C_Y A_Y^{n-1-i} & A_M^n \end{bmatrix},$$

and the product $C_I A_I^n B_I$ reads

$$\begin{aligned} C_I A_I^n B_I = {} & D_N C_X A_X^n B_X + \\ & + C_N \sum_{i=0}^{n-1} A_N^i B_N C_X A_X^{n-1-i} B_X + \\ & + C_N A_N^n B_N D_X + D_M C_Y A_Y^n B_Y + \\ & + C_M \sum_{i=0}^{n-1} A_M^i B_M C_Y A_Y^{n-1-i} B_Y + \\ & + C_M A_M^n B_M D_Y. \end{aligned} \tag{10}$$

In addition, we require $D_I = I$:

$$D_I = D_N D_X + D_M D_Y = I, \tag{11}$$

To proceed further let us consider without loss of generality that A_X, A_Y are formed by a single Jordan block (one multiple stable eigenvalue λ).

The powers of A_X and A_Y in (10) can thus be written out as

$$A_X^l = \sum_{j=0}^{m_X} \binom{l}{j} \lambda^{l-j} U^j, \quad A_Y^l = \sum_{j=0}^{m_Y} \binom{l}{j} \lambda^{i-j} U^j \quad (12)$$

where $m_X = \min(\delta_X - 1, l)$, $m_Y = \min(\delta_Y - 1, l)$ (here δ_X, δ_Y stand for the sizes of A_X, A_Y respectively), and U is a square matrix of the appropriate size with ones above the diagonal,

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

System of equations (10) is *nonlinear* in unknowns B_X, C_X, D_X, B_Y, C_Y. However, powers of system matrices $A_X, A_X^2 \ldots$ can be rewritten according to (12) to obtain a system of *linear* equations in unknowns $B_X C_X$, $B_X U C_X$, $B_X U^2 C_X, \ldots$ that can be solved using standard numerical linear algebra tools. Transfer functions of the resulting system X can be then composed according to the following formula

$$\begin{aligned} X(s) &= C_X(sI - A_X)^{-1} B_X + D_X = \quad (13) \\ &= C_X \frac{\text{adj }(sI - A_X)}{\det(sI - A_X)} B_X + D_X = \\ &= \frac{\sum_{i=0}^{\delta_X - 1} (s-\lambda)^{\delta_X - 1 - i} C_X U^i B_X}{(s-\lambda)^{\delta_X}} + D_X \end{aligned}$$

The function $Y(s)$ can be evaluated similarly.

3. ALGORITHM

Particular considerations of the previous section are summarized below, giving rise to the following numerical procedure for finding a particular solution of the rational Bezout identity.

Algorithm
Input: matrices $N, M \in \mathbf{M}(\mathbf{S})$ of appropriate dimensions
Output: matrices $X, Y \in \mathbf{M}(\mathbf{S})$ such that $NX + MY = I$

(1) Transform matrices N, M into state-space form (compose their realizations).
(2) Choose system matrices A_X, A_Y as a single Jordan block with its eigenvalue in the stability area.
(3) Solve the system of linear matrix equations (10), (11) using (12), with the terms $C_X B_X, C_X U B_X, C_X U^2 B_X, \ldots, C_X U^{\delta_X - 1} B_X$, $C_Y B_Y, C_Y U B_Y, \ldots, C_Y U^{\delta_Y - 1} B_Y$, D_X, D_Y as unknowns.
(4) Compute transfer matrices of X, Y according to (13)

□

Performance of the proposed algorithm is illustrated by two simple examples.

Example 1. Let

$$M(s) = \frac{s^2}{(s+1)^2}, N(s) = \frac{1}{(s+1)^2}.$$

State-space representation of the systems $N(s), M(s)$ reads

$$A_N = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, B_N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_N = [1 \quad 0], D_N = 0,$$

$$A_M = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, B_M = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, C_M = [1/2 \quad -1], D_M = 1.$$

Now the system matrices A_X, A_Y are to be chosen. System matrices must be stable, minimum size, and of the Jordan structure. The choice

$$A_X = -1, \; A_Y = -1$$

satisfies these restrictions. From equation (8) we obtain

$$D_I = D_N D_X + D_M D_Y = 0 D_X + 1 D_Y = 1 \Rightarrow D_Y = 1.$$

Next the system of equations (10) shall be solved

$$C_I B_I = C_Y B_Y - 2 = 0 \Rightarrow C_Y B_Y = 2,$$
$$C_I A_I B_I = -3 C_Y B_Y + D_X + 3 = D_X - 3 = 0 \Rightarrow D_X = 3,$$
$$C_I A_I^2 B_I = C_X B_X + 6 C_Y B_Y - 2 D_X - 4 = 0 \Rightarrow C_X B_X = -2.$$

So we have arrived at the matrix $D_X = 3$, and the products $C_X B_X = -2, C_Y B_Y = 2$. This information is sufficient to compose the desired rational functions $X(s), Y(s)$ as

$$X(s) = \frac{C_X B_X}{s+1} + D_X = \frac{3s+1}{s+1},$$
$$Y(s) = \frac{C_Y B_Y}{s+1} + D_Y = \frac{s+3}{s+1}.$$

It is easy to verify that this is a particular solution to the Bezout identity.

The whole calculation was very simple because the system matrices A_X, A_Y were scalar. Let us investigate a bit more complicated situation in the example to follow.

Example 2. Solve the Bezout identity $NX + MY = 1$ with

$$N(s) = \frac{1}{(s+1)(s+2)}, \; M(s) = \frac{s^2}{(s+1)^2}.$$

Let us consider the state-space realizations of M and N in the Jordan canonical form for instance (it is not crucial however),

$$A_N = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, B_N = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, C_N = [2 \quad -2], D_N = 0,$$

$$A_M = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, B_M = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, C_M = [1/2 \quad -1], D_M = 1.$$

The matrices A_X, A_Y have to be chosen properly - in the Jordan canonical form with a single stable multiple eigenvalue λ (we take $\lambda = -1$):

$$A_X = A_Y = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

The equation (11) reads

$$D_I = 0D_X + 1D_Y = 1 \Rightarrow D_Y = 1.$$

Next the system of linear matrix equations (10) needs to be solved,

$$\begin{aligned}
C_I B_I &= C_Y B_Y - 2 = 0, \\
C_I A_I B_I &= D_X + C_Y A_Y B_Y - 2C_Y B_Y + 3 = \\
&= D_X - 3C_Y B_Y + C_Y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B_Y + 3 = 0, \\
C_I A_I^2 B_I &= C_X B_X - 3D_X + C_Y A_Y^2 B_Y - 2C_Y A_Y B_Y + 3C_Y B_Y - 4 = \\
&= C_X B_X - 3D_X + 6C_Y B_Y - 4C_Y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B_Y - 4 = 0, \\
C_I A_I^3 B_I &= -4C_X B_X + C_X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B_X + 7D_X - 10C_Y B_Y + \\
&+ 10C_Y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B_Y + 5 = 0, \\
C_I A_I^4 B_I &= 11C_X B_X - 5C_X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B_X - 15D_X + \\
&+ 15C_Y B_Y - 20C_Y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B_Y - 6 = 0.
\end{aligned}$$

The results are

$$\begin{aligned}
&C_X B_X = 1,\ C_X U B_X = -2,\ D_X = 3, \\
&C_Y B_Y = 2,\ C_Y U B_Y = 0.
\end{aligned}$$

and the transfer functions X and Y read

$$\begin{aligned}
X(s) &= \frac{(s+1)C_X B_X + C_X U B_X}{(s+1)^2} + D_X = \\
&= \frac{3s^2 + 7s + 2}{(s+1)^2}, \\
Y(s) &= \frac{s^2 + 4s + 3}{(s+1)^2}.
\end{aligned}$$

3.1 *Degree estimation of system matrices A_X, A_Y*

Concerning the sizes of A_X and A_Y (in other words, the degrees of denominators of X and Y), they can be deduced from the theory of stabilization. As indicated in the introductory section, the solution of rational Bezout identity (2) relates to a controller stabilizing the LTI system $G(s) = M(s)N^{-1}(s)$. It is well known (Kailath, 1980) that in the SISO case there always exists such a controller of order $n-1$ where n is the order of G. Hence $\delta_X = \delta_Y = n-1$ is a reasonable choice for M and N scalars. For MIMO case, the recommendation reads $\delta_X = \delta_Y = n - \text{rank } C_G$ where C_G is the state-to-output matrix of the system G.

Table 1. Results of the numerical test

d	R
2	0
3	3.9e-15
4	2.4e-13
5	3.7e-14
6	1.5e-13
7	3.3e-10
8	6.3e-08
9	8.1e-07
10	1.1e-07
11	1.7e-03
12	> 1
13	> 1

4. IMPLEMENTATION AND A NUMERICAL EXPERIMENT

The method was programmed in MATLAB (Mathworks, 1999*b*) using the pre-release version 3.0 of the Polynomial Toolbox (H. Kwakernaak, n.d.). New objects for rational functions and matrices implemented in the new version of the Polynomial Toolbox were used.

We will show numerical properties of the presented algorithm by a simple benchmark experiment. First, random stable systems $G(s)$ of order d are generated with two inputs and two outputs by the `rss` command of the Control Systems Toolbox (Mathworks, 1999*a*). These systems are then factorized as $G(s) = NM^{-1}$, where $N, M \in \mathbf{M}(\mathbf{S})$ are coprime. In Matlab:

```
[N,M] = dcf(G);
```

where `dcf` is our doubly-coprime factorization function, interested readers are referred to (J. Lidinsky, 2004) for details. Such a way, a pair of coprime proper and stable rational matrices is obtained as an input for our procedure. Subsequently, the Bezout identity is solved by the function `axby1` that is the implementation of the algorithm described in this report:

```
[X,Y] = axby1(M,N);
```

The results for orders $d = 2, \ldots, 13$ are presented in the Table 1.

As a measure of numerical performance the infinity norm $R = \|NX + MY - I\|_\infty$ was evaluated. The method is well usable for matrices of degree up to ten.

5. CONCLUSIONS AND DISCUSSION

The Bezout identity over the rings of proper stable rational functions and matrices has been studied in this report. Revealed relations between the Bezout equation and controllability/observability properties of a related systems interconnection led to a numerical procedure for finding a particular solution to the Bezout equation problem.

The proposed algorithm obviously does not feature superior numerical properties. The procedure works nicely for medium size problems, however, higher degrees are not handled properly, as illustrated in the Table 1. Similar results can be easily achieved directly by composing the virtual plant $G(s) = M(s)N^{-1}(s)$ and computing a stabilizing controller $C(s)$ in a numerically reliable manner, see (Varga, 1981) for instance. X and Y then follow from the coprime factorization of C over $\mathbf{S}$ (or $\mathbf{M}(\mathbf{S})$, respectively). A numerically reliable state-space routine for this task has been presented in (Varga, 1998). The limitations are probably due to the Jordan structure of the A_X and A_Y matrices. Jordan canonical form is known to be difficult to handle in the floating point arithmetics for various reasons, see (G.H. Golub, 1990) (Higham, 1996) for instance. On the other hand, this choice of structure is essential to make the main idea of the algorithm computationally tractable. For this reason we do not see a simple way how to improve the numerical properties significantly.

Nevertheless, the proposed method has been proved to work and appears practically usable for SISO as well as MIMO problems of reasonable size. Moreover, some well known results of the matrix theory, theory of stabilizing controllers, and dynamic systems analysis have been exploited and put in a new and, hopefully, interesting enough context in this paper while elaborating the basic ideas into a workable routine.

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