APPLICATION OF MOVING PARETO FRONTIER TECHNIQUE FOR EXPLORATION OF DYNAMIC CONTROLLED SYSTEMS

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Abstract: The paper describes a new approach to the exploration of dynamic models in the process of the design of technical systems. The approach combines the traditional concept of reachable sets for controlled dynamic systems with such modern decision support tool as interactive (animated) computer visualization of Pareto frontier in decision problems.
Combination of the approximating the reachable sets for a dynamic systems with Pareto frontier visualization results in the Moving Pareto Frontier (MPF) technique that helps a designer to select a preferred design of a technical system.

Keywords: optimal control, reachable sets, multiple criteria decision making, Pareto frontier, computer aided design.

The paper describes a new approach to the exploration of dynamic models in the process of the design of technical systems. The approach combines the traditional concept of reachable sets for controlled dynamic systems (see, for example, (Kurzhanski, A.B.,and I. Valyi, 1996)) with such modern decision support tool as interactive (animated) computer visualization of Pareto frontier in decision problems (Lotov and al. 2004). Combination of the approximating the reachable sets for a dynamic systems with Pareto frontier visualization results in the Moving Pareto Frontier (MPF) technique that helps a designer to select a preferred design of a technical system. The MPF technique was introduced in (Brusnikina, N.B., and A. V. Lotov, 2004) .

Let us start with a simple example that illustrates the idea. We consider the system of three heavy bodies that can move on the horizontal plane without friction (see Fig. 1). All the bodies are connected with springs, while the left-hand body is connected by the spring with the wall.

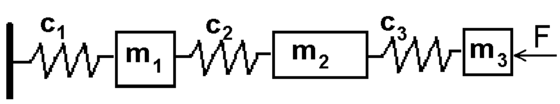

Fig. 1. The system under study.

The system is controlled by the external force that is applied to the right-hand body. The value of the external force is restricted from above. If all forces (including forces produced by springs) equal zero, there are equilibrium positions of the bodies. It is assumed that at the initial moment the system is out of the equilibrium state. The aim of the control is to bring the left-hand body into the vecinity of zero in its phase space, where it can be captured. The designer has to decide concerning the capacity of the capture, that is, the maximum values of deviation of the left-hand body from its equilibrium position and of its velocity at the moment of capturing.

It is assumed that the designer is interested in the decreasing the capacity of the capture and time required to capture the body, but is not able formulate the loss function, which describes the losses related to the capacity of the capture and time in an integrated form. Therefore, multiple criteria design problem must be studied. Since two first criteria (maximum distance from the equilibrium position of the left-hand body and its maximum velocity at the moment of capture) and the third criterion (time) are different in their nature, we propose to display the dynamics of the Pareto frontier for the first two criteria to the designer. This information must support his choice of the preferred design.

Now let us consider the mathematical formalization of the approach. Though the MPF technique was introduced in (Brusnikina, N.B., and A. V. Lotov, 2004) for non-linear dynamic systems, in this paper we restrict to a linear differential equation and convex constraints. Therefore, the following controlled system of differential equations is studied

$$\frac{dx}{dt} = A(t)x(t) + u(t),\ t \in [0,T], \qquad (1)$$

where $u(t) \in R^n$ are control variables, $x \in R^n$ are state variables, $A(t)$ is a given matrix.

Let us assume that constraints are imposed on values of the control variables

$$u(t) \in U(t),\ t \in [0,T], \qquad (2)$$

where $U(t)$ is a given convex compact set from R^n.

It is assumed that the initial state $x(0)$ belongs to a given compact convex set $X(0)$ from R^n:

$$x(0) \in X_0. \qquad (3)$$

By $\Gamma(\theta)$ we denote the reachable set for the time-moment $\theta \in [0,T]$, that is, the variety of the states, which can be reached by the system (1)-(3) precisely at a time-moment θ.

Let us consider a multi-criteria decision problem with the criterion vector z related to the state x by a mapping

$$z = F(x), \qquad (4)$$

which maps the states of the system into the linear criterion space R^m. We assume that the number of criteria is three to seven. The set $Z(\theta) = F(\Gamma(\theta))$ that describes criterion vectors, which are feasible at the time moment θ, is denoted as the feasible criterion set (FCS) for this time-moment. Note that, even in the case of a linear dynamic system (1)-(3), the mapping (4) can still be non-linear.

It is assumed that, the designer is interested to minimize the duration of the process, that is, he wants to select a time-moment $t^* \in [0,T]$ as small as possible. At the same time, it is assumed that, in addition to minimizing the duration, the designer is interested in minimization of all coordinates of the criterion vector z. To be precise, a criterion vector z^{**} is more preferable than z^* if $z_i^{**} \le z_i^*$ for all $i = 1,...,m$ and z^{**} is not equal z^*. A criterion vector $z^* \in Z(\theta)$ is said to be Pareto optimal (non-dominated) if there does not exist any criterion vector $z^{**} \in Z(\theta)$ that is better than z^*. The variety of Pareto optimal criterion vectors (Pareto frontier of the feasible criterion set $Z(\theta)$) is denoted by $P(Z(\theta))$. Since the feasible criterion set $Z(\theta)$ depends on time θ, the Pareto frontier $P(Z(\theta))$ depends on time, too.

In the book (Lotov and al. 2004), it is shown how the Pareto frontier can be visualized for static models with three to seven criteria with the help of the Interactive Decision Maps (IDM) technique. In the framework of the IDM technique, the *Edgeworth-Pareto Hull* (EPH) of the FCS is approximated in advance, before the interactive visualization of the Pareto frontier can start. In the case of multi-criteria minimization problem, the EPH that is defined as $Z^* = Z + R_+^m$, where R_+^m is the non-negative orthant of the criterion space R^m. It is important that the Pareto frontiers of the sets Z and Z^* coincide. The IDM technique provides an opportunity to display the Pareto frontier on-line as frontiers of various collections of two-criterion slices of the EPH (decision maps). After the exploration of the Pareto frontier is completed, the user can identify a preferred criterion point goal. The associated decision is found then by the computer automatically. The IDM technique proved to be effective in both linear and non-linear static problems. Moreover, it is shown in the book (Lotov and al. 2004) how the technique can be applied to study dynamic multi-criteria problems if time is not a decision criterion. Here, we propose to develop the concept of the IDM technique for the case of dynamic decision problems with a decision criterion that is associated with time. This development is based on transformation of the IDM technique into the MFP technique.

In the framework of the MPF technique, the reachable sets $\Gamma(t_k)$ are constructed numerically for a sufficiently large number N of time moments $t_k = k\frac{T}{N}$, where $k = 0,1,..., N$. To be precise, the sets $\Gamma(t_k)$ are approximated by polyhedral sets Γ_k with a sufficient precision. Then, the sets Z_k^* are approximated for all moments $k = 0,1,..., N$. The designer can specify a desired combination of two-criterion slices of the sets Z_k^*, which are displayed in consecutive order. This results in an animated movement of the Pareto frontier on the computer display. The designer studies the movement of the frontier (perhaps, many times) and selects a time-moment $t^* \in [0,T]$ of the end of the process (actually, he stops movement of the frontier at a certain moment). Then, the designer has to identify a preferred goal point z^* at the Pareto frontier for the selected time-moment. As in the IDM technique (Lotov and al. 2004), the associated control $u(t)$, $t \in [0,T]$ is computed automatically. To implement the MPF technique in the form of a numerical procedure, a multi-step linear approximation of (1)-(3) is considered. In the framework of such an approximation, the differential equation is substituted for a multi-step equation and the compact convex sets that describe constraints are approximated by polytopes. Approximations Γ_k of the reachable sets $\Gamma(t_k)$ are constructed using optimal methods for polyhedral approximation of convex reachable sets described in Chapter 7 of the book (Lotov and al. 2004). Methods for approximation of the sets Z_k^* for the given sets Γ_k for linear and non-linear criteria are provided in the same book (Lotov and al. 2004).

The MPF technique was applied to the illustrative problem described above. We denote by x_1, x_2, x_3 the deviations from the equilibrium points for the left-hand, the central and the right-hand bodies. Let m_1, m_2, m_3 be the masses of the bodies, and c_1, c_2, c_3 be the resilience coefficients of the springs. Then, the dynamics of the system is described by the equations

$$\begin{cases} m_1\ddot{x}_1 + (c_1 + c_2)x_1 - c_2 x_2 = 0 \\ m_2\ddot{x}_2 + (c_2 + c_3)x_2 - c_2 x_1 - c_3 x_3 = 0, \\ m_3\ddot{x}_3 + c_3 x_3 - c_3 x_2 = -F \end{cases}$$

$$|F| \le 1.$$

The system was studied for the following particular values of parameters:

$c_1 = 16, \quad c_2 = 2, \quad c_3 = 2,$

$m_1 = 2, \quad m_2 = 4, \quad m_3 = 1.$

By denoting by v_1, v_2, v_3 the velocities of the bodies, we obtain

$$\begin{cases} \dot{x}_1 = v_1 \\ \dot{v}_1 = -9x_1 + x_2 \\ \dot{x}_2 = v_2 \\ \dot{v}_2 = 0.5x_1 - x_2 + 0.5x_3 \\ \dot{x}_3 = v_3 \\ \dot{v}_3 = 2x_2 - 2x_3 + u, \qquad -1 \le u \le 1 \end{cases}$$

Let us assume that at the moment $t = 0$ the system was in the state

$x_1 = x_2 = x_3 = 2$, $v_1 = v_2 = v_3 = 0$.

In addition to time of capture, two decision criteria were considered -- the deviation of the first body x_1 and its speed in the time-moment of capture.

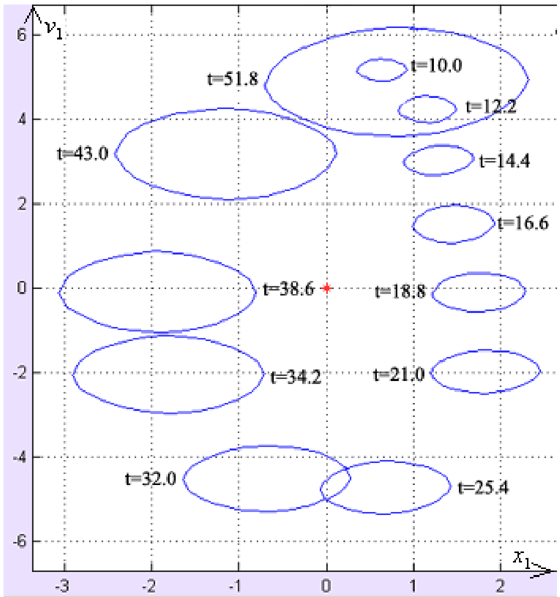

Fig. 2. Positions of attainable values of phase variables of the left-hand body for different time-moments. Part1.

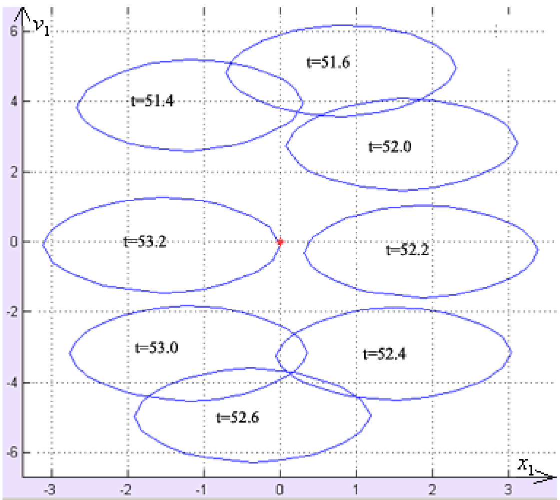

Fig. 3. Positions of attainable values of phase variables of the left-hand body for different time-moments. Part2.

The approximations of 6-dimensional reachable sets for the system under study were constructed first. Then, the related EPH's were approximated, too. Since we have got only two criteria (in addition to time), we simply display the sequence of the sets Z_k^*. The moving Pareto frontier is displayed as the frontier of the EPH. Unfortunately, we cannot display animation in the paper. For this reason, we provide two kinds of alternative pictures. In Figs. 2 and 3, the projection of the reachable set into the phase space of the first body is given. One can see a sequence of projections until $t = 53.2$, that is, the time-moment, at which the point $x_1 = 0$, $v_1 = 0$ is practically getting attainable. Figs. 2 and 3 give some idea about how the animation can look: the set of attainable values of x_1, v_1 rotates around zero and approaches it at $t = 53.2$. One must note that in animation the number of displayed sets is much greater, it is about 430. Due to it, animation effect is provided.

In this particular case, the designer could use the animation of attainable values of x_1, v_1, but in general animation of the Pareto frontier turns out to be more informative. Since its movement is too sophisticated to be displayed by a simple static picture, we prepared Fig. 4 that contains unions of Z_k^* for several time periods: for $t \in [0,12]$, $t \in [0,20]$, $t \in [0,30]$, $t \in [0,38]$, $t \in [0,43]$, $t \in [0,53.2]$. Since every next time period includes the previous, such unions expand.

Due to it, their frontiers do not intersect, and one can easily see how the Pareto frontier approaches the goal. This information can help the designer to identify preferred time-moment and criterion values. The control that brings the system into the identified state can be found automatically.

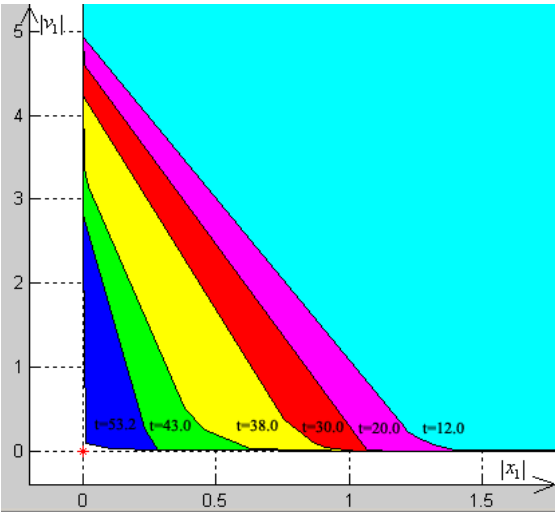

Fig. 4. Superimposed Pareto frontiers for several time-moments.

Such an animation supports human exploration of the criterion tradeoffs. Once again, the visualization of the movement of the Pareto frontier is made possible by the preliminary approximation of the reachable sets for the dynamic system under study.

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REFERENCES

Kurzhanski, A.B.,and I. Valyi (1996) *Ellipsoidal Calculus for Estimation and Control,* Birkhaeuser, Boston.

Lotov, A.V., V. A. Bushenkov and G. K. Kamenev (2004) *Interactive Decision Maps. Approximation and Visualization of Pareto frontier*, Kluwer Academic Publishers, Boston.

Brusnikina, N.B., and A. V. Lotov (2004) Moving Pareto Frontier for Dynamic Models. In: *Proc. of the 4th Moscow International Conference on Operations Research (ORM2004)*, 47-50. Maks Press, Moscow, 2004.