ONLINE TRAFFIC LIGHT CONTROL THROUGH GRADIENT ESTIMATION USING STOCHASTIC FLUID MODELS

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Abstract: In this paper, we consider the problem of dynamically regulating the timing of traffic light controllers in busy cities. We use a Stochastic Fluid Model (SFM) to model the dynamics of the queues formed at an intersection. Based on this model, we derive gradients of the queue lengths with respect to the green/red light lengths within a signal cycle. We report preliminary numerical results comparing the performance of the estimates with finite-difference and smoothed perturbation analysis estimates. Then all estimators are used to optimize the traffic system via Stochastic Approximation.

Keywords: Traffic control, perturbation analysis, stochastic approximation.

1. INTRODUCTION

In this paper we consider the problem of easing traffic congestion by dynamically adjusting the timing of the traffic light that regulates the vehicle flow at a single intersection. This is a problem that over the years has attracted the attention of several researchers and many approaches have been proposed. In (Moskowitz *et al.*, 1997) a model for estimating the traffic conditions based on measured information is proposed and in (Hoyer and Jumar, 1994) a fuzzy controller is developed. In (De Schutter, 1999) an Extended Linear Complementary Problem (ELCP) is formulated and in (Zhao and Chen, 2003) a hybrid systems formulation is presented. Finally, in (L. Head, 1996) a Perturbation Analysis framework (Ho and Cao, 1991) is used and in (Fu and Howell, 2003) a Smoothed Perturbation Analysis (SPA) is adopted. In this paper, we also use Infinitesimal Perturbation Analysis (IPA) but the model we use to derive the IPA estimators is a Stochastic Fluid Model (SFM). Subsequently, when we implement the estimators, we use observations from the actual Discrete-Event System (DES). Though SFMs might not be very accurate

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for *performance evaluation*, they have proven to be very robust with respect to *optimization*, because they seem to capture the salient features of the problem. Several authors have reported that use of SFM efficiently lead to optimal or near-optimal solutions (see (Cassandras *et al.*, 2002) and references therein). Using the SFM modeling framework, a new approach for resource management is being developed which is based on IPA (Cassandras *et al.*, 2002). In this approach, we derive estimators of the gradient of the performance measure of interest with respect to the control parameters based on SFMs. Then we evaluate them based on observations on the actual DES and use the resulting estimates with stochastic approximation algorithms to determine the optimal parameter setting. This approach has some important advantages.

- The gradient estimation is done *on-line*, thus the approach can be implemented on the traffic light controller; as operating conditions change, it will aim to *continuously* seek to optimize a generally time-varying performance metric (this holds for both SPA and SFM-based estimators).
- Unlike the SPA estimators, SFM-based estimators do *not* require any knowledge of the system's underlying stochastic processes.
- SFM-based IPA estimators are generally simpler to implement than SPA
- SPA estimators are generally more accurate than the SFM-based IPA estimators but simulation results indicated that in optimization problems they have comparable performance.

2. MODEL

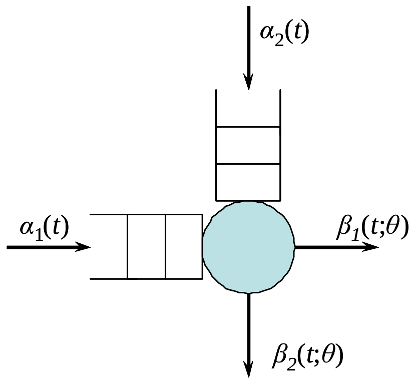

Fig. 1. Intersection model

For this paper, we model the traffic light with two buffers competing for the same server, as shown in Fig. 1. Vehicles arrive at queue $q \in \{1,2\}$ with rates $\alpha_q(t)$ and leave the intersection with rates $\beta_q(t;\theta) < \rho(t)$, where θ is a parameter that affects the traffic light operation (e.g., timing of red/green periods) and $\rho(t)$ is the maximum capacity of the intersection. For $q = 1,2$, we define $L_q(t;\theta)$ to be the number of vehicles waiting in queue q at time t and $\bar{L}_q(t;\theta)$ to be the average $L_q(t;\theta)$ up to time t, i.e.,

$$\bar{L}_q(t;\theta) = \frac{1}{t}\int_0^t L_q(x;\theta)dx.$$

Typically, one is interested in minimizing $\bar{L}_q(t;\theta)$, so we seek estimators of $\frac{dE[\bar{L}_q(t;\theta)]}{d\theta}$, $q = 1,2$.

We assume that a complete cycle (T) constitutes the completion of a green and red light period. For queue 1, we denote the green light period with T_1 and the red light period with T_2, $T = T_1 + T_2$ (For queue 2, T_1 and T_2 denote the red and green light periods respectively). Furthermore, we assume that T is fixed, and the control parameter θ is either T_1 or T_2.

For the SFM approximation, we let $x_q(t;\theta)$, $q \in \{1,2\}$ denote the *fluid* buffer content at $t \in [0,S]$ (where S is the observation interval) and define

$$Q_q(t;\theta) = \frac{1}{t}\int_0^t x_q(\tau;\theta)d\tau. \tag{1}$$

We derive sample derivatives of $Q_q(t;\theta)$ with respect to θ using two different SFM approximations. Recall that $\bar{L}_q(t;\theta)$ and $Q_q(t;\theta)$ correspond to the average queue levels up to t at street $q \in \{1,2\}$ of the discrete-event and stochastic fluid models, respectively. We take $Q_q(t;\theta)$ as an approximation of $\bar{L}_q(t;\theta)$. Using IPA, $\frac{dQ_q(t;\theta)}{d\theta}$ is derived, which is used to approximate $\frac{dE[\bar{L}_q(t;\theta)]}{d\theta}$.

3. SIMULTANEOUS MODEL

In this section we assume a rather unrealistic SFM where both queues are served simultaneously. Buffer 1 receives a fraction θ of the intersection capacity and buffer 2 gets the remaining $1-\theta$. Thus, $\beta_1(t;\theta) = \theta\rho(t)$ and $\beta_2(t;\theta) = (1-\theta)\rho(t)$. In the DES, this approach would be implemented as $T_1 = \theta T$ and $T_2 = (1-\theta)T$.

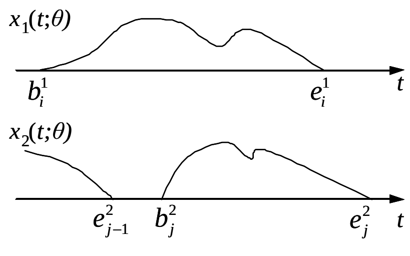

Fig. 2. Typical sample path for the simultaneous model

3.1 Sample Path Partition

For this model we partition the sample path into empty and non-empty periods. Empty periods are maximal intervals where $x_q(t;\theta) = 0$ while non-empty intervals indicate the intervals such that $x_q(t;\theta) > 0$, $q \in \{1,2\}$. Let $\bar{E}_i^q = (b_i^q, e_i^q)$ indicate the ith non-empty period, where b_i^q indicates the beginning and e_i^q the end of the ith non-empty period at queue $q \in \{1,2\}$. Using this notation, the sample functions (1) can be written as

$$Q_q(S;\theta) = \frac{1}{S}\sum_{j=1}^{N_q} q_j(S;\theta) \equiv \frac{1}{S}\sum_{j=1}^{N_q}\int_{b_j^q}^{e_j^q} x_q(t;\theta)dt \qquad (2)$$

where N_q denotes the random number of non-empty periods in the interval $[0,S]$. Differentiating with respect to θ we get

$$\frac{dQ_q(\theta)}{d\theta} = \frac{1}{S}\sum_{j=1}^{N_q}\frac{dq_j(\theta)}{d\theta} = \frac{1}{S}\sum_{j=1}^{N_q}\int_{b_j^q}^{e_j^q}\frac{dx_q(t;\theta)}{d\theta}dt \qquad (3)$$

Lemma 3.1. The derivative of $x_q(t;\theta)$, $q \in \{1,2\}$ with respect to θ is given by

$$\frac{dx_q(t;\theta)}{d\theta} = s(q)\int_{b_j^q}^{t}\rho(\tau)d\tau \qquad (4)$$

where

$$s(q) = \begin{cases} -1 & \text{if } q = 1 \\ 1 & \text{if } q = 2 \end{cases} \qquad (5)$$

All proofs are omitted due to space limitations. Substituting (4) in (3) we get the following result.

Theorem 3.1. For $q \in \{1,2\}$, the sample derivatives for the workload are given by

$$\frac{dQ_q(\theta)}{d\theta} = s(q)\frac{1}{S}\sum_{j=1}^{N_q}\int_{b_j^q}^{e_j^q}\int_{b_j^q}^{t}\rho(\tau)d\tau dt \qquad (6)$$

where $s(q)$ is given by (5).

Example: Let us consider the case where $\rho(t) = \rho$ (constant). In this case, it is straightforward to show that

$$\frac{dQ_q(\theta)}{d\theta} = s(q)\frac{\rho}{2S}\sum_{j=1}^{N_q}\left(e_j^q - b_j^q\right)^2. \qquad (7)$$

Note that these estimators are extremely simple to implement. We just accumulate the squares of the duration of each non-empty period.

4. ON/OFF MODEL

In this section we study a more relevant SFM where the *entire* server capacity is allocated to the first queue for a period $0 < \theta < T$ and to the second queue for a period $0 < T - \theta < T$. Fig. 3 shows a typical sample path due to this model.

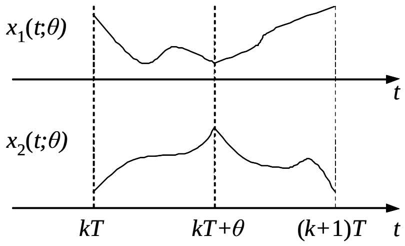

Fig. 3. Sample path for the ON/OFF model

4.1 Sample Path Partition

In this case, the sample path is divided into intervals of length T, and the dynamics of the two queues are given by

$$\frac{dx_1(t;\theta)}{dt} = \begin{cases} \alpha_1(t) - \beta_1(t;\theta), \\ \quad \text{if } kT \le t < kT + \theta \\ \alpha_1(t), \\ \quad \text{if } kT + \theta \le t < (k+1)T \end{cases} \qquad (8)$$

$$\frac{dx_2(t;\theta)}{dt} = \begin{cases} \alpha_2(t), \\ \quad \text{if } kT \le t < kT + \theta \\ \alpha_2(t) - \beta_2(t;\theta), \\ \quad \text{if } kT + \theta \le t < (k+1)T \end{cases} \qquad (9)$$

$k = 1, 2, \cdots$. The outflow rates are given by

$$\beta_1(t;\theta) = \begin{cases} \rho_1(t), & \text{if } kT \le t < kT + \theta \\ & \text{and } x_1(t;\theta) > 0 \\ \alpha_1(t), & \text{if } kT \le t < kT + \theta \\ & \text{and } x_1(t;\theta) = 0 \\ 0, & \text{otherwise} \end{cases} \qquad (10)$$

$$\beta_2(t;\theta) = \begin{cases} \rho_2(t), \\ \quad \text{if } kT + \theta \le t < (k+1)T \\ \quad \text{and } x_2(t;\theta) > 0 \\ \alpha_2(t), \\ \quad \text{if } kT + \theta \le t < (k+1)T \\ \quad \text{and } x_2(t;\theta) = 0 \\ 0, \quad \text{otherwise} \end{cases} \qquad (11)$$

where $\rho_1(t)$ and $\rho_2(t)$ are the maximum possible outflows from queues 1 and 2, respectively. The sample functions of (1) can be written as

$$Q_q(\theta) = \frac{1}{S}\sum_{k=1}^{K}\phi_k^q(\theta) \equiv \frac{1}{S}\sum_{k=1}^{K}\int_{kT}^{(k+1)T} x_q(t;\theta)dt \qquad (12)$$

where K is the number of periods included in the interval $[0,S]$ and the index $q \in \{1,2\}$. Differentiating with respect to θ we get

$$\frac{dQ_q(\theta)}{d\theta} = \frac{1}{S}\sum_{k=1}^{K}\frac{d\phi_k^q(\theta)}{d\theta} = \frac{1}{S}\sum_{k=1}^{K}\int\limits_{kT}^{(k+1)T}\frac{dx_q(t;\theta)}{d\theta}dt \quad (13)$$

For simplicity, let us first try to evaluate a single term from the summation for $q = 1$, i.e.,

$$\frac{d\phi_k^1(\theta)}{d\theta} = \int\limits_{kT}^{(k+1)T}\frac{dx_1(t;\theta)}{d\theta}dt.$$

Given the queue dynamics of (8), we determine the queue content

$$x_1(t;\theta) = \begin{cases} x_1(kT;\theta) + \int\limits_{kT}^{t}\alpha_1(t) - \rho_1(t)dt, \\ \qquad \text{if } kT \le t < e^1_{j_k} \\ 0, \quad \text{if } e^1_{j_k} \le t < kT + \theta \\ x_1(kT+\theta;\theta) + \int\limits_{kT+\theta}^{t}\alpha_1(t)dt, \\ \qquad \text{if } kT+\theta \le t < (k+1)T \end{cases} \quad (14)$$

where $e^1_{j_k}$ indicates the time when the buffer empties during the kth period. If no such event occurs, then we set $e^1_{j_k} = (kT+\theta)$, thus the second case does not occur. Next, differentiating (14)

$$\frac{dx_1(t;\theta)}{d\theta} = \begin{cases} \frac{dx_1(kT;\theta)}{d\theta}, \quad \text{if } kT \le t < e^1_{j_k} \\ 0, \quad \text{if } e^1_{j_k} \le t < kT+\theta \\ \frac{dx_1(kT+\theta;\theta)}{d\theta} - \alpha_1(kT+\theta), \\ \qquad \text{if } kT+\theta \le t < (k+1)T \end{cases} \quad (15)$$

In other words, the derivative $\frac{dx_1(t;\theta)}{d\theta}$ is a piecewise constant function. Even though it may look complicated, this function is very easy to implement iteratively using a single accumulator! As a result, the derivative $\frac{dQ_1(\theta)}{d\theta}$ is also very easy to evaluate; it is just the derivative times the corresponding intervals.

$$\begin{aligned}\frac{dQ_1(\theta)}{d\theta} &= \frac{1}{S}\sum_{k=1}^{K}(e^1_{j_k} - kT)\frac{dx_1(kT;\theta)}{d\theta} \\ &+(T-\theta) \\ &\times\left(\frac{dx_1(kT+\theta;\theta)}{d\theta} - \alpha_1(kT+\theta)\right) \quad (16)\end{aligned}$$

where as mentioned earlier, $e^1_{j_k}$ is the time that buffer 1 empties during the interval $[kT, kT+\theta)$ and if no such event occurs, then $e^1_{j_k} = kT + \theta$. $\frac{dQ_2(\theta)}{d\theta}$ can be derived in a similar fashion.

5. SIMULATION RESULTS

In this section, we use simulation to acquire numerical results for the SFM estimators derived in this paper. These estimators along with the SPA estimators are then compared to finite difference estimates. All simulation results are based on the scenarios presented in Table 1. In all cases, the interarrival and service time distributions were exponentially distributed with rates λ and μ, respectively, the number of cycles was 10,000, and the number of replications was also 10,000. Estimators were simulated for all 4 cases; however, the optimization was carried out for C1, C2 and C3.

Table 1. Simulation Cases

Case	$1/\lambda$	$1/\mu$	T_1	T
C1	4.5	2.0	30	60
C2	5.0	1.5	35	110
C3	3.5	0.5	20	40
C4	10.5	5.0	20	40

5.1 Stochastic Fluid Model Estimators

The simultaneous model estimators (7) were implemented on the underlying stochastic DES model, where for ρ we used the mean service rate of the server. As indicated in the simulation results (labelled SFM1), these estimators did not work well partly because the service rate of the buffer was not fixed and deterministic (as assumed by this SFM model) and partly because this model did not capture the fact that at any queue, during the "red" light period, the service rate is 0.

The ON/OFF model was also implemented on the underlying DES stochastic model, where for the instantaneous arrival rates $\alpha_q(t)$ in (16), we simply used the average arrival rate. The results for this model are labelled SFM2. In addition, we noticed that the assumption that the queue stays empty once it empties causes the estimator to be low, because in general the queue can become nonempty again during the same green-light cycle. We made a slight modification to the estimator to allow for arrivals to the queue during a green light phase while the system is empty. Instead of resetting $\frac{dx(kT;\theta)}{d\theta}$ whenever the queue empties, we only reset when the queue is empty at the epoch of the light change. This makes intuitive sense because the perturbation can only propagate through the cycle if the system is nonempty. The results of the modified estimators are labelled SFM2mod.

5.2 Smoothed Perturbation Analysis (SPA)

Following the framework of (Fu and Hu, 1997), the general SPA estimator consists of an IPA term and a conditional term, the latter due to possible critical event order change. The SPA estimator is

$$\left(\frac{dE[\bar{L}]}{d\theta}\right)_{SPA} = \frac{d\bar{L}}{d\theta} + \lim_{\Delta\theta\to 0}\frac{P(\beta(\Delta\theta))}{\Delta\theta}\delta\bar{L}(\beta), \quad (17)$$

where $\beta(\Delta\theta)$ denotes a critical event change due to a perturbation of $\Delta\theta$, and $\delta\bar{L}(\beta)$ denotes the corresponding expected change in the performance measure $\bar{L}$.

Using this general form we derived four estimators, left and right-hand estimators for both queues. Here we state two of the SPA estimators. The right-hand estimator for queue 1 is

$$\left(\frac{dE[\bar{L}_1]}{d\theta}\right)_{SPA,r} = \frac{1}{NT}\sum_{i=1}^{N}\frac{f_1(\alpha_i)}{1-F_1(\alpha_i)}\left[-R_{q_i}^{(1)}\right]$$

where α_i is the time until light change (from green to red) from last entry to service during ith cycle, $R_n^{(i)}$ is the expected time to empty queue i, given n cars in the queue, q_i is the number in queue at the epoch of the ith light change from green to red, f_i is the service time density and F_i is the service time distribution. The left-hand estimator for queue 2 is

$$\left(\frac{dE[\bar{L}_2]}{d\theta}\right)_{SPA,l} = \frac{1}{NT}\sum_{i=1}^{N}\left(H_i + \frac{f_2(\alpha_i)\left[R_{q_i}^{(2)}\right]}{1-F_2(\alpha_i)}\right),$$

where H_i is the number of "critical" departures in cycle i. In the results, we denote SPA(RH) and SPA(LH) as the right hand and left hand SPA estimators respectively. Finally, FD denotes the finite difference results.

5.3 Stochastic Approximation (SA)

SA is a gradient-based stochastic optimization algorithm, where the "best guess" of the optimal parameter is updated iteratively based on the estimate of the gradient of the performance measure with respect to the parameter (Fu and Hu, 1997). The general form of SA is

$$t_{n+1}^* = \prod_{(0,t_p)}(t_n^* - a_n\nabla J_n), \tag{18}$$

where t_n^* is the parameter value at the beginning of iteration n, a_n is a positive sequence of step sizes, ∇J_n is the estimate of $\nabla J_n(t_n^*)$, the gradient of J_n at parameter value t_n^* and, $\prod_\Omega$ is the projection onto Ω. $\prod_\Omega$ keeps the parameter within the valid range of values. In this implementation of the SA algorithm, $\prod_\Omega$ returns the parameter back to the stable region if the update has caused some parameter to move outside of the stable region. From queuing theory, for queue stability we need $\lambda_1 \times T < \mu_1 \times T_1$ and $\lambda_2 \times T < \mu_2 \times (T - T_1)$. Thus, we have as our stable region the following

$$\frac{\lambda_1}{\mu_1} \times T < T_1 < T \times (1 - \frac{\lambda_2}{\mu_2}). \tag{19}$$

Table 2. Results for Case 1: $dE[\bar{L}_1]/d\theta$

estimator	mean	std err
SPA (RH)	-2.465	0.001
SPA (LH)	-2.465	0.001
FD (.05)	-2.475	0.024
SFM1	-147.82	0.154
SFM2	-1.713	0.002
SFM2mod	-2.188	0.006

Table 3. Results for Case 2: $dE[\bar{L}_1]/d\theta$

estimator	mean	std err
SPA (RH)	-8.3904	0.0061
SPA (LH)	-8.3835	0.0066
FD (.05)	-8.2115	0.0356
SFM1	-915.4631	1.4496
SFM2	-0.1683	0.0000
SFM2mod	-0.2412	0.0000

Table 4. Results for Case 3: $dE[\bar{L}_1]/d\theta$

estimator	mean	std err
SPA (RH)	-0.1717	0.0000
SPA (LH)	-0.1716	0.0001
FD (.05)	-0.1716	0.0003
SFM1	-11.1422	0.0004
SFM2	-0.1674	0.0000
SFM2mod	-0.1960	0.0000

Table 5. Results for Case 4: $dE[\bar{L}_1]/d\theta$

estimator	mean	std err
SPA (RH)	-20.8959	0.0424
SPA (LH)	-20.8848	0.0417
FD (.05)	-20.2334	0.1146
SFM1	-775.8161	4.0491
SFM2	-19.0584	0.0979
SFM2mod	-19.7437	0.0989

5.3.1. Gradient Estimation The results are shown in Tables 2-5. A comparison of the results shows that the simultaneous model estimator (SFM1) is large in magnitude compared to the other estimates. We take the FD estimate as the true value in Table 6 to compare the percent error of each estimator. The left and right hand estimates are extremely accurate. Both the ON/OFF (SFM2) and the modified ON/OFF (SFM2mod) models provide fair estimates of the gradient. The standard error for the four aforementioned estimators is always less than that for the FD estimates. Even though the estimates do not match the FD estimates exactly, they still look promising for the system optimization.

5.3.2. Optimization We noticed that the SFM gradient estimated were not very accurate; however, from Fig. 4 we can see that the estimates have a very important quality. These gradient estimates are 0 at the minimum of the function. This is a good sign that the SFM estimates can be used for optimization via a gradient descent algorithm such as SA. All six gradient estimation techniques cross 0 at the minimum of $\bar{L}$. Fig. 4 shows this

Table 6. Percent error of estimators with FD as true value

estimator	case 1 % error	case 2 % error
SPA (RH)	-0.72%	2.18%
SPA (LH)	-0.70%	2.10%
FD (.05)	0.00%	0.00%
SFM1	6170.05%	11048.61%
SFM2	-30.86%	-97.95%
SFM2mod	-11.87%	-97.06%

estimator	case 3 % error	case 4 % error
SPA (RH)	0.09%	3.27%
SPA (LH)	0.03%	3.22%
FD (.05)	0.00%	0.00%
SFM1	6393.84%	3734.33%
SFM2	-2.45%	-5.81%
SFM2mod	14.21%	-2.42%

estimator	avg abs error
SPA (RH)	1.57%
SPA (LH)	1.51%
FD (.05)	0.00%
SFM1	6836.71%
SFM2	34.27%
SFM2mod	31.39%

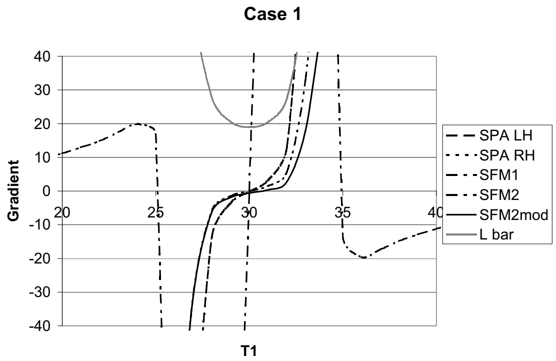

Fig. 4. Gradient Estimation Methods vs $\bar{L}$

Table 7. Near optimum range for C1

estimator	10%	5%	1%
SPA (RH)	77.0	46.4	10.9
SPA (LH)	77.8	45.6	10.9
FD (.05)	76.5	45.0	10.1
SFM1	48.9	30.2	5.7
SFM2	79.3	47.9	10.9
SFM2mod	73.6	45.0	10.0

for C1, the same property of the estimators was exhibited for C2 and C3 as well.

Because SA is an iterative algorithm, not only are we concerned with reaching the optimum, but we would like to stay near the optimum for subsequent updates. Therefore, we ran simulations and counted the number of times the average number in system was within p% (for p = 10,5,1) of the minimum average number in system based on the current T_1 from the SA algorithm.

All six gradient estimation techniques were implemented in the SA algorithm for cases C1, C2 and C3. Tables 7,8 and 9 show the number of times each estimation fell within the aforementioned optimum ranges.

Table 8. Near optimum range for C2

estimator	10%	5%	1%
SPA (RH)	92.9	89.9	60.1
SPA (LH)	92.7	90.2	61.0
FD (.05)	94.0	91.8	57.0
SFM1	3.0	2.6	1.2
SFM2	95.9	94.3	66.3
SFM2mod	95.1	91.3	10.9

Table 9. Near optimum range for C3

estimator	10%	5%	1%
SPA (RH)	43.0	21.8	4.6
SPA (LH)	43.1	22.4	4.2
FD (.05)	41.2	19.9	3.9
SFM1	43.0	21.7	4.7
SFM2	48.3	26.4	4.7
SFM2mod	42.9	21.7	4.6

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