A LYAPUNOV-BASED APPROACH FOR THE CONTROL OF BIOMIMETIC ROBOTIC SYSTEMS WITH PERIODIC FORCING INPUTS

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Abstract: Bio-mimetic Robotics often deploys locomotion mechanisms (swimming, crawling, flying etc...) which rely on repetitive patterns for the actuation schemes. This directly translates into periodic forcing inputs for the dynamics of the mechanical system. Closed loop control is achieved by modulating shape-parameters (e.g. duty cycle) which directly affect the mean values of the forcing inputs. In this work, guided by an intuition inspired by linear systems theory, first a linear feedback law is derived that stabilizes a linearization of the average system, i.e. the system subject only to the average values of the forcing inputs, and then it is shown how this very feedback law can also guarantee boundedness of solutions of the original system. Boundedness is proved my means of a Lyapunov energy function easily derived in the linearized case. Unlike classical results found in literature in the areas of averaging and perturbation theory, this work instead of focussing on the existence of periodic limit cycles, simply restricts its attention on the boundedness of solutions, which directly translates into the possibility of deploying input functions which are continuous but not continuously differentiable.

Keywords: averaging, animal locomotion, non-differentiable flow, Lyapunov

1. INTRODUCTION

Recent developments in Bio-mimetic Robotics (Ayers *et al.*, 2002) led to a broad variety of bio-inspired autonomous robots mimicking locomotion of real animals. Whether swimming, crawling or flying, locomotion mechanisms are often based on repetitive, i.e. periodic, patterns (slowly) modulated by a controller via some regulatory parameters, e.g. frequency, duty cycle, etc...

As an example, consider a flapping wings Micromechanical Flying Robot (MFI) (Schenato *et al.*, 2003). Forces and torques arise from repetitive motion of wings. Periodicity of wing trajectories is modulated by the slow (compared with the wing-beat frequency) variation of certain parameters. Such periodic forces and torques represent the forcing inputs to the dynamics of a rigid body problem.

In these situations, the most intuitive approach to stabilization is considering the system as subject to an equivalent (slowly varying) average input instead of a rapidly oscillating one.

This intuition is directly imported from linear systems theory where mechanical systems display a linear low-pass filtering behavior which tends to respond mainly to the (slowly varying) average values of the inputs while rejecting its high order harmonics content.

In what follows a general class of nonlinear nonautonomous systems is considered where the time dependence is present in a parameterized family of periodic inputs. Via averaging methods, a nonlinear but autonomous system is derived whose linearized equivalent, supposed to be controllable, will provided a stabilizing feedback law. It will then proved, by means of Lyapunov energy functions, that this law can also be used to *bound* the original nonlinear nonautonomous system.

2. AVERAGING

Consider the general class of nonlinear systems represented by:

$$\dot{x} = F(x, u(d, t)) \tag{1}$$

where $x \in R^n$ represents the state variable, d is a vector of parameters, $u(d, t) \in R^m$ is a vector of forcing inputs T-periodic in t, and $F : R^n \times R^m \to R^n$ is a vector field corresponding to the dynamics of the robot. The parameter vector d is eventually modulated by a controller (in fact, in what follows, we shall always imply $d = d(x)$).

Since mechanical systems of interest are in fact affine with respect to forces, $F(x, u)$ is assumed to be *affine* in u.

This property allows one to write the average system simply as:

$$\dot{x} = \frac{1}{T}\int_0^T F(x, u(d, s))\, ds = F(x, \bar{u}(d)) \tag{2}$$

where the bar operator represents the average operation [1] over the period T and is defined as:

$$\bar{u}(d) \triangleq \frac{1}{T}\int_0^T u(d, s)\, ds \tag{3}$$

In order to allow linearization, the following two conditions are needed:

- $F(x, u)$ is *continuously differentiable* with respect both arguments.
- $\bar{u}(d)$ is *continuously differentiable.*

Note: assuming continuous differentiability of $\bar{u}(d)$ is far less restrictive than assuming continuous differentiability of $u(d, t)$, allowing thus piecewise differentiable functions such as triangular waves. This is a considerable departure from literature.

Let there be a particular combination of parameters [2], say $d = d_0$, such that the average system has an equilibrium in $x = 0$, i.e.:

$$0 = F(0, \bar{u}_0) \qquad where \quad \bar{u}_0 \triangleq \bar{u}(d_0)$$

Consider now the average system linearized at the equilibrium:

$$\dot{x} = \left.\frac{\partial F}{\partial x}\right|_{(0,\bar{u}_0)} x + \left.\frac{\partial F}{\partial \bar{u}}\right|_{(0,\bar{u}_0)} \left.\frac{\partial \bar{u}}{\partial d}\right|_{d_0} \Delta d = Ax + B\,\Delta d \tag{4}$$

Where $\Delta d = d - d_0$.

When the linearized system is controllable [3], it is always possible to find a feedback linear law $\Delta d = -Kx$ such that the system:

$$\dot{x} = (A - BK)x$$

is exponentially stable.

Moreover, in such a case, for every positive definite Q there exist a positive definite P solution of the Lyapunov equation:

$$P(A - BK) + (A - BK)^T P = -2Q \tag{5}$$

Such a matrix can be used to define a positive definite energy function:

$$V(x) \triangleq x^T P x \tag{6}$$

whose time derivative along the trajectories of the *average* nonlinear system is given by:

$$\dot{V}(x) = x^T P F(x, \bar{u}(d_0 - Kx)) + F^T(x, \bar{u}(d_0 - Kx))Px \tag{7}$$

which can be proved to be negative definite, at least in a *bounded* domain $D \subset R^n$ around the origin, i.e.:

$$\dot{V}(x) \le -x^T(2Q)x + \mathcal{O}(\|x\|^2) \le -x^T Q x \quad \forall x \in D \tag{8}$$

Therefore, the feedback law found for the linear case also stabilizes the nonlinear average system, at least around the equilibrium.

Now, the original nonlinear system (1) does not even possess an equilibrium point. Next section will show how the feedback law previously found will actually bound system (1) around the origin.

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[1] As in classical averaging, a time integral is performed where x is considered as a frozen variable and so any function of x such as $d(x)$.

[2] Biomimetic robots are in general designed to work around a nominal set of values derived from the observation of real animals (Schenato *et al.*, 2003).

[3] This can be checked via the full rank test on $B, AB, A^2B\ldots$

3. BOUNDEDNESS VIA LYAPUNOV ENERGY FUNCTIONS

Linear feedback $d = d_0 - Kx$, applied to the original system (1), leads to:

$$\dot{x} = F(x, u(d_0 - Kx, t)) \triangleq F_c(x, t) \qquad (9)$$

Thus far, no regularity condition was needed for $u(d,t)$. In order to assure existence and uniqueness of solutions in the *sense of Caratheodory*[4] (Sastry, 1999) for the system (9), *only continuity* for $u(d,t)$ is required.

For what follows, more than continuity, a *local Lipschitz* condition shall be assumed, i.e.:

$$\exists L : \|F_c(x,t) - F_c(y,t)\| \leq L\|x-y\| \quad \forall x, y \in D, \ \forall t$$

It is possible to find families of periodic forcing inputs, e.g. triangular waves, such that $\bar{u}(d)$ is continuously differentiable while $u(d,t)$ is Lipschitz but not differentiable.

This is the main difference with most averaging theorems found in literature (see Section 10.3 in (Khalil, 1995)) which require continuity of the derivatives of vector field up to second order. In our analysis, instead, the closed-loop vector field $F_c(x,t)$ in system (9) only needs to be continuous in both x and t in order to guarantee existence and uniqueness in the sense of Caratheodory. Similarly, other groups have recently attempted to extend averaging theory to non-smooth systems (Teel *et al.*, 2003) (Iannelli *et al.*, 2003).

Clearly, there is no longer an equilibrium in $x = 0$. What is now plausible is that trajectories are attracted by, or fall into, a bounded region $D_0 \subset D$ containing the origin $x = 0$.

The idea is extending theorems such as Theorem 4.18 in (Khalil, 1995) (also reported in Appendix B) and those in (Aeyels and Peuteman, 1998), where the fact that $\dot{V}(x,t) \leq 0$ in $D - D_0$ implies that trajectories are attracted by D_0.

3.1 Boundedness

Use $V(x) = x^T P x$ as a candidate energy function and compute its time derivative along trajectories of (9):

$$\begin{aligned}\dot{V}(x_t, t) &= x_t^T P \dot{x}_t + \dot{x}_t^T P x_t \\ &= x_t^T P F_c(x_t, t) + F_c^T(x_t, t) P x_t \qquad (10)\end{aligned}$$

where, for sake of clarity, the notation x_t simply stands for $x(t)$, while $F_c(x,t)$ is defined in (9).

[4] Discontinuous inputs (PWM) can be included if solutions in the *sense of Filippov* (Sastry, 1999) are considered.

Checking that $\dot{V}(x,t) \leq 0$ in a whole region surrounding the equilibrium *for all* t could often fail, yet being true *most of the time*.

In the case of systems forced by T-periodic inputs, this idea actually leads to a useful test.

Instead of checking the sign of $\dot{V}(x,t)$, consider:

$$\begin{aligned}\Delta V_T(x_t, t) &\triangleq \int_t^{t+T} \dot{V}(x_s, s)\, ds \\ &= V(x_{t+T}, t+T) - V(x_t, t) \qquad (11)\end{aligned}$$

which after substituting (10) becomes:

$$\Delta V_T(x_t, t) = \int_t^{t+T} [x_s^T P F_c(x_s, s) + F_c^T(x_s, s) P x_s]\, ds \qquad (12)$$

Clearly, if $\Delta V_T(x_t, t) \leq 0$ simply means that $V(x_{t+T}, t+T) \leq V(x_t, t)$. Therefore after a period T, the trajectory shall stay on a lower (or at most equivalent) energetic level, where energy levels of $V(x)$ are ellipsoids and their energy decreases down to zero as x approaches the origin.

The purpose now is estimating the sign of (12) without actually knowing x_s, solution of the original system (9), for $s \in [t, \ t+T]$.

To this end, consider for the moment only the first integrand of (12):

$$\begin{aligned}x_s^T P F_c(x_s, s) = x_t^T P F_c(x_t, s) + [x_s - x_t]^T P F_c(x_s, s) + \\ + x_t^T P [F_c(x_s, s) - F_c(x_t, s)]\end{aligned}$$

The first addendum, once integrated over the $[t, \ t+T]$ time interval, is nothing but the first term of the right side of Eq.(7) multiplied by T , i.e.:

$$\begin{aligned}\int_t^{t+T} x_t^T P F_c(x_t, s)\, ds &= \int_t^{t+T} x_t^T P F(x_t, u(d_0 - Kx_t, s))\, ds \\ &= T\, x_t^T P F(x_t, \bar{u}(d_0 - Kx_t))\end{aligned}$$

Therefore, (12) can be rewritten as:

$$\begin{aligned}\Delta V_T(x_t, t) &= \Delta V_a + \Delta V_b + \Delta V_c \\ \Delta V_a &\triangleq T\,[x_t^T P F(x_t, \bar{u}(d_0 - Kx_t)) + \\ &\quad + F^T(x_t, \bar{u}(d_0 - Kx_t)) P x_t] \\ \Delta V_b &\triangleq \int_t^{t+T} 2[x_s - x_t]^T P F_c(x_s, s)\, ds \\ \Delta V_c &\triangleq \int_t^{t+T} 2[F_c(x_s, s) - F_c(x_t, s)] P x_t\, ds\end{aligned}$$

As long as the trajectory is confined in D, the following inequalities hold:

$$\begin{array}{l}\forall x_t \in D, \\ \forall x_s \in D \\ \forall t \in \mathbb{R}, \\ \forall s \in [t, \ t+T]\end{array} \Rightarrow \left\{ \begin{array}{rl} \|x_t\| & \leq r_D \\ \|F_c(x_s, s)\| & \leq \|F_{max}\| \\ \|x_s - x_t\| & \leq T\,\|F_{max}\| \\ \|F_c(x_s, s) - F_c(x_t, s)\| & \leq L\|x_s - x_t\| \\ & \leq L\,T\,\|F_{max}\| \end{array} \right. \qquad (13)$$

where L is the previously defined Lipschitz constant and:

$$\begin{aligned}r_D &\triangleq \max_{x \in D} \|x\| \\ \|F_{max}\| &\triangleq \max_{t \in [0,T], \ x \in D} \|F_c(x,t)\|\end{aligned}$$

Now use Eq.(7) together with inequalities (8) and (13) to get:

$$\begin{aligned}\Delta V_a &\leq -T\,x_t^T Q x_t \\ \Delta V_b &\leq 2T^2 r_D L\|P\|\|F_{max}\| \\ \Delta V_c &\leq 2\|P\|\|F_{max}\|^2\end{aligned}$$

Therefore if $x_t \in D, \forall t \in \mathbb{R}$, then Equation (11) can be bounded as follows:

$$\Delta V_T(x_t, t) \leq T(-x_t^T Q x_t + T\,b) \qquad (14)$$

where the positive scalar b is defined as:

$$b \triangleq 2r_D L\|P\|\|F_{max}\| + 2\|P\|\|F_{max}\|^2 \qquad (15)$$

In order to prove boundedness of solutions, define Ω and Λ set families and their boundaries as follow:

$$\begin{aligned}\Omega_\lambda &\triangleq \left\{x \in R^n : \|x^T Q x\| \leq \lambda\right\} \\ \Lambda_\lambda &\triangleq \left\{x \in R^n : \|x^T P x\| \leq \lambda\right\} \\ \partial\Omega_\lambda &\triangleq \left\{x \in R^n : \|x^T Q x\| = \lambda\right\} \\ \partial\Lambda_\lambda &\triangleq \left\{x \in R^n : \|x^T P x\| = \lambda\right\}\end{aligned} \qquad (16)$$

Furthermore, define $B_r = \{x \in R^n \mid \|x\| \leq r\}$ and $\partial B_r = \{x \in R^n \mid \|x\| = r\}$. It is now possible to state and prove the boundedness property.

Lemma 1. For every positive $r > 0$, it is possible to find $c > 0$ and $T_0 > 0$ such that $\forall T \leq T_0$ every trajectory, solution of (9) and starting in Λ_c at time t_0, is confined in B_r for $t \geq t_0$.

Proof: The set D contains the origin at its interior, therefore $\exists r_0 > 0$ such that $B_{r_0} \subset D$. Consider $r_1 = \min\{r, r_0\}$, clearly $B_{r_1} \subset D$. The Λ sets are concentric ellipsoids and therefore it is always possible to find $c > 0$ small enough such that $\|x\| < r_1\ \forall x \in \Lambda_c$.

Define $dist(\Lambda_c, \partial B_{r_1})$ as the distance between the set Λ_c and ∂B_{r_1}, it is nonzero due to the previous choice of c. By the third inequality of (13), for every $T < T_1 = dist(\Lambda_c, \partial B_{r_1})/\|F_{max}\|$, any solution of (9) such that $x_t \in \Lambda_c$ at some time t will be confined in B_{r_1} for a whole period T, i.e. $x_s \in B_{r_1}\ \forall s \in [t,\ t+T]$. Therefore, since $B_{r_1} \subset D$, for trajectories such that $x_t \in \Lambda_c$ at some time t, inequality (14) holds valid.

The Ω sets are also concentric ellipsoids, therefore a $T_2 > 0$ small enough can always be found such that $\Omega_{Tb} \subset \Lambda_c\ \forall T \leq T_2$, where b is defined in (15). For every point x which is not in the interior of Ω_{Tb}, $\|x^T Q x\| \geq Tb$ holds true and in particular, given the validity of (14) for points in Λ_c, the following holds true:

$$\Delta V_T(x_t, t) \leq 0 \quad \forall x_t \in \partial\Lambda_c$$

By defining $T_0 = \min\{T_1, T_2\}$ and by recalling definition (11), this simply means that for every $T \leq T_0$ $V(x_{t+T}, t+T) \leq V(x_t, t)$ $\forall x_t \in \partial\Lambda_c$, i.e. whenever $x_t \in \partial\Lambda_c$ then $x_{t+T} \in \Lambda_c$ and, by construction of Λ_c, it can never leave B_{r_1} for all time in $[t,\ t+T]$. This proves the Lemma since $B_{r1} \subset B_r$ and therefore a trajectory starting in Λ_c is allowed to pass its boundary but shall always make return in Λ_c within a period of time and never leave B_r. ■

We can summarize our results in the following theorem

Theorem 1. Consider the following system:

$$\dot{x} = F(x, u)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $F(x,u)$ is *continuously differentiable* with respect to both its arguments and it is *affine* in u. Let the function $u = u(d,t)$ be *locally Lipschitz* with respect to the parameter vector $d \in \mathbb{R}^p$ and T-periodic with respect to t. Let the averaged input $\bar{u}$ defined as:

$$\bar{u} \triangleq \bar{u}(d) = \int_0^T u(d,s)ds$$

and assume that $\bar{u}(d)$ is *continuously differentiable* with respect to d. Suppose there exist a feedback $d = d(x)$, where $d(x)$ is *continuously differentiable* with respect to x, such that the origin is an *exponentially stable* equilibrium point of $\bar{F}_c \triangleq F(x, \bar{u}(d(x)))$. For each $r > 0$, there exists a domain $D_r \subset B_r = \{x : ||x|| \leq r\}$, and $T_0 > 0$, such that if $x_0 \in D_r$ and $T < T_0$ then

$$x(t) \in B_r$$

for all $t \geq t_0$, where $x(t)$ is the solution of the closed loop system

$$\dot{x} = F(x, u(d(x), t)), \qquad x(t_0) \in D_r.$$

4. CONCLUSIONS AND FUTURE WORK

In this work, a class of nonlinear nonautonomous systems is considered which is of interest in Biomimetic Robotics. Such systems are time dependent in the sense that time periodic inputs are used as forcing inputs for a mechanical system.

A simple control law is derived from the linearization of the time-averaged equivalent system. Such a control law is then fed-back into the original system and boundedness of solution is analyzed in relation to the time period of the forcing inputs.

Differently from classical results in averaging and perturbation theory, which focus on the existence of limit cycles, only local Lipschitz continuity for the forcing inputs is needed, instead of continuous differentiability. The approach is based on Lyapunov energy functions.

As part of future work, the authors intend extending their approach based on Lyapunov energy functions to a larger class of systems, where forcing inputs are discontinuous, e.g. Pulse Width Modulated (PWM) systems. The reason is that even if the original system is discontinuous, the average system can still be smooth enough to be linearized and therefore a Lyapunov function can be easily derived. In order to use this Lyapunov as a candidate one for the original nonlinear system, only conditions for the existence of piecewise Lipschitz solutions of the original system are needed. We are currently investigating the application of these ideas to the flapping wing control of micromechanical flying insect (Yan *et al.*, 2001) and to the development of biomimetic robots for minimally-invasive biomedical surgery.

Appendix A

Because of the linearization:

$$F(x, \bar{u}(d_0 - Kx)) = (A - BK)x + G(x)$$

where the function $G(x)$ satisfies:

$$\frac{\|G(x)\|}{\|x\|} \to 0 \quad as \quad \|x\| \to 0$$

Therefore, for any $\gamma > 0$, there exists $r > 0$ such that

$$\|G(x)\| < \gamma\|x\|, \quad \forall\|x\| < r$$

Hence,

$$\begin{aligned}\dot{V}(x) &= x^T PF(x, \bar{u}(d_0 - Kx)) + F^T(x, \bar{u}(d_0 - Kx))Px \\ &= x^T P[(A - BK)x + G(x)] + [(A - BK)x + G(x)]^T Px \\ &= x^T[P(A - BK) + (A - BK)^T P]x + x^T PG(x) + \\ &\quad +G^T(x)Px \\ &= -x^T(2Q)x + x^T PG(x) + G^T(x)Px \\ &< -x^T(2Q)x + 2\gamma\|P\|\|x\|^2\end{aligned}$$

Appendix B

Here is the theorem referred to by previous sections:

Theorem 4.18 (Khalil, 1995): *Let $D \subset R^n$ be a domain that contains the origin and $V : [0, \infty] \times D \to R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|) \qquad \text{(B-1)}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x), \quad \forall\|x\| \ge \mu > 0 \qquad \text{(B-2)}$$

$\forall t \ge 0$ and $\forall x \in D$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$\mu < \alpha_2^{-1}(\alpha_1(r)) \qquad \text{(B-3)}$$

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \le \alpha_2^{-1}(\alpha_1(r))$, there is $T \ge 0$ (dependent on $x(t_0)$ and μ) such that the solution of $\dot{x} = f(t, x)$ satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall t_0 \le t \le t_0 + T \qquad \text{(B-4)}$$

$$\|x(t)\| \le \alpha_2^{-1}(\alpha_1(r)), \quad \forall t \ge t_0 + T \qquad \text{(B-5)}$$

Moreover, if $D = R^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then (B-4) and (B-5) hold for any initial state $x(t_0)$, with no restriction on how large μ is.

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