PATH PLANNING FOR NON LINEAR SYSTEMS USING TRIGONOMETRIC SPLINES.

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Abstract: This paper deals with the use of trigonometric splines (also call Schoenberg's polynomials) in order to increasing the performances of a nonlinear closed-loop system. In the first part, it is recalled that for linear systems, the couple "setting time/overshoot" can be largely improved if a path planning is used instead of a classical step input. The concept of trigonometric splines is then introduced. The remainder of this paper treats of nonlinear systems and in particular differentially flat system. After having pointed out the necessity for this kind of system to have a smooth trajectory generator, it is shown, in particular through the example of the crane, the contribution that this technique can bring. A comparison with a classical polynomials approach is performed.

Keywords: non linear system, trigonometric spline, flatness, path planning.

1. INTRODUCTION

The usual way to evaluate the dynamical behavior of a system (and specially in the case of linear system) is to analyze the response to a step. Moreover, if the system is in a closed loop control, it allows to verify that the designed control corresponds to what was expected. However, it is well-known, because of the discontinuity on its first time derivative, that this very simple trajectory is not always adapted to perform the best output.

In the case of non linear control (such as flat control law) this discontinuity can be incompatible with the control law itself because this kind of control imposes to use the successive derivatives of the input. Another problem is due to the fact that a step input is very demanding for the dynamics of the system because this special trajectory does not take into account the transient modes. Consequently, typical non linear phenomena like saturation on the control variable can appear.

The unique solution to resolve these problems consists in using a path planning generator. This generator calculates (in-line or off-line) a desired trajectory by taking into account imposed points with their associated time and eventually their successive derivatives.

Such techniques are usual in robotics because the Cartesian coordinates of the effective point of a robot cannot be directly imposed. Indeed, for a desired effective point to be reached, all the evolutions of the variable links have to be calculated. So, the number of paper dealing with this subject is extremely important, see for example (Craig, 1989).

In non linear control, some actual works propose a feedback on the trajectory planning called reference governor. The main idea of these works (Gilbert and Kolmanovsky, 2002) is to update the desired trajectory with respect to the actual position and to the final one. The paper (Mahout and Lopez, 1992) refers to work in the same spirit. For these different approaches the main idea is always the same : update the reference trajectory with respect to the effective output with the aim to take into account indirectly the dynamics of the plant to perform the best possible trajectory tracking.

In this paper, we consider path planning in the context of a non linear control. In the case of flat control design for which it is necessary to have smooth imposed trajectories, it will be shown that the use of trigonometric splines (also call Schoenberg's polynomials) can be a very interesting alternative to resolve the problem of path planning.

This paper is organized as follows. In a first part the linear case will be exposed as an example. It will be shown the influence of the input reference on the quality of the output in term of setting time and overshoot. The second part will concern the Schoenberg polynomials and the techniques associated to calculate the different coefficients of these polynomials. The third part will be about flat control and the usual way to impose smooth trajectories. In the last part the classical example of the crane will be presented where the use of trigonometric splines improves the output performances. A comparison with a classical polynomials approach is proposed

2. THE LINEAR CASE

In this section, before exposing the trigonometric splines technic, some well-known results about linear systems are recalled. This presentation shows that a path generator has a very important influence on the quality of the response of the system.

Let be the controlled linear system $G(s) = \frac{k}{s^2+s}$

obviously stable in closed loop. For testing in simulation this system, two kinds of input reference Y_c are considered. The first one is the step ($Y_c(t) = 1, \quad \forall t > 0$) and the second is issued to a path generator ($Y_c(t) = f(t), \quad$ such that $f(0) = 0$ et $f(T) = 1$). In these simulations, the path planning corresponds to a trigonometric spline (explained is the next part), but any other smooth function would have given similar results). In both cases, different simulations are performed with, for each one, a different value for the gain k. For each simulation, the setting time at 2% and the overshoot are measured. It can be noted that the overshoot is calculated on the error tracking $Yc(t) - Y(t)$ and not on the difference between the output and the expected final value $Yc = 1$, which is less favorable. In figure 1, the representation of the evolution of these two characteristic measurements is shown (setting time and overshoot) when k varies. These two plots present discontinuities which are due to the oscillations of the responses. Indeed when an oscillation becomes small enough to stay inside the band of 2% around the final value, therefor k increases,, the corresponding setting time is improved with value equal to the oscillation period. This abrupt variation involves the observed discontinuities. It can be observed that for a small gain ($k < 10$) the use of trigonometric spline is not interesting. Indeed, if the overshoot is globally the same, the setting time is not as good (for example at $k = 5$ the overshoot is 48% for the step and 44% for the trigonometric spline whereas the setting time increases from 7.6 seconds to 8 seconds). When k becomes bigger an important observation can be made : the response to the step input presents an overshoot of 100% and a setting time near of 7.8 seconds while the response to the trigonometric spline input converges to the point corresponding to no overshoot and a setting time near 1 second. For the response to the step input, when $k > 0.25$, there exists two conjugate complex values $\lambda_{1,2}$. So, the point of convergence can be found from the expression of the output:

$$y(t) = \frac{1}{\omega_n^2} + \frac{1}{\omega_n * \omega_p} e^{-\xi\omega_n t} \sin(\omega_p t + \phi) \qquad (1)$$

where $\omega_p = \omega_n\sqrt{(1-\xi^2)}$, $\omega_n = \|\lambda\|$, $\phi = \arg(\lambda)$ and $\xi = cos(\phi)$.

It can easily be demonstrated that, when $k \to \infty$ the maximum of the output $y(t)$ tends to 2 and the setting time is reached at T such that $e^{-\xi\omega_n T} = 0.02$, so $T = \frac{-\ln(0.02}{\Re_e(\lambda)} = 7.824$

In the case of the trigonometric spline input, an analytic expression of the output cannot be expressed. It is however possible to note that, when k increases, the setting time converges to 1 second. This value for these simulations corresponds to the final time of establishment of the trigonometric spline. It can be observed that for the overshoot, after having increased until approximately 50%, it decreases by important values of k to reach 0 finally.

In conclusion we can note that, in the linear case, without changing anything in the given proportional control law, it is possible to increase notably the performance by using a path generator.

3. THE TRIGONOMETRIC SPLINES

The term spline is usual for polynomial of a time variable τ. In the case of trigonometric splines the

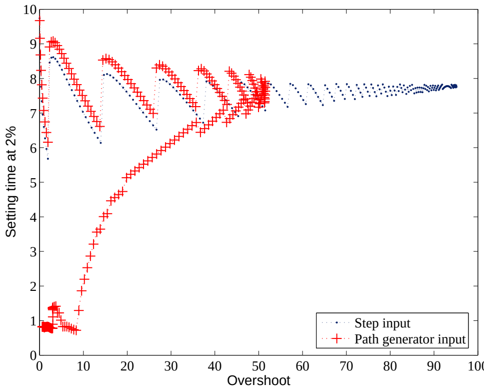

Fig. 1. The (setting time/overshoot) plane. Each point corresponds to a different value of k varying from 0.25 to 1000

time variable is not directly used but indirectly used by the way of the trigonometric functions *sin* and *cos* (Schoenberg, 1964; Simon and C., 1993).

Definition 1. A m-order trigonometric spline $y(\tau)$ satisfying the $2m$ constraints $y_i \ \ (i = 1, 2, \cdots, 2m)$ is represented uniquely by :

$$y(\tau) = \begin{array}{l} a_0 + \displaystyle\sum_{k=1}^{m-1} \left(a_k \cos(k\tau) + b_k \sin(k\tau)\right) \\ + a_m \sin\left(m\left(\tau - \gamma\right)\right) \end{array} \tag{2}$$

where γ guarantees the uniqueness of the solution and is calculated generally as :

$$\gamma = \frac{1}{2m} \sum_{k=1}^{2m} \tau_k$$

with τ_i corresponding to the values of τ where each of the $2m$ constraints are applied.

Remark 2. The trigonometric spline are 2π periodic and τ must belong to the interval $[0, 2\pi]$

Suppose n points $(y_1, y_2, \cdots, y_n)$ where y_1 defines the initial point at time τ_1 and y_n defines the final point at time τ_n. The $n-2$ other points are auxiliary points defined at time $(\tau_2, \cdots, \tau_{n-1})$.

The order of the trigonometric spline (2) which can take into account the n above constraints must be at least equal to $m = E\left(\frac{n+1}{2}\right)$, where $E(x)$ denote the integer part of x. The determination of the $2m$ coefficients results from the resolution of a linear system of n equations.

For path planning the constraints are not always given by a set of a simple couple (time, points) but the successive time derivatives can also be associated with this set. In such case the $2m$ coefficients are issued from the extended linear system:

$$\begin{cases} y(\tau) = a_0 + \displaystyle\sum_{k=1}^{m-1} \left(a_k \cos(k\tau) + b_k \sin(k\tau)\right) \\ \qquad\qquad + a_m \sin\left(m\left(\tau - \gamma\right)\right) \\ \dot{y}(\tau) = \displaystyle\sum_{k=1}^{m=1} -k\left(a_k \sin(k\tau) + b_k \cos(k\tau)\right) \\ \qquad\qquad + m.a_m \cos\left(m\left(\tau - \gamma\right)\right) \\ \vdots \\ y^{(\beta)}(\tau) = \displaystyle\sum_{k=1}^{m-1} k^{\beta} \left(a_k \frac{\partial^{\beta} \cos(k\tau)}{\partial \tau^{\beta}} + b_k \frac{\partial^{\beta} \sin(k\tau)}{\partial \tau^{\beta}}\right) \\ \qquad\qquad + m^{\beta} a_m \dfrac{\partial^{\beta} \cos\left(m\left(\tau - \gamma\right)\right)}{\partial \tau^{\beta}} \end{cases} \tag{3}$$

Equations (3) always includes $2m$ parameters. So, if each point y_i is determined at time τ_i with β successive derivatives there are $n(\beta + 1)$ equations. In this case, a m-order trigonometric spline can interpolated $E\left(\frac{2m}{\beta+1}\right)$ points.

3.1 Time rescalling

Obviously, in a control problem, it is not realistic, if we respect remark 2, to restrict the time between 0 and 2π. The time has also to be rescaled. Suppose that a desired trajectory is defined in the time domain $[t_i, t_f]$ where t_i is the initial time and t_f the final time. It is always possible to impose the change of a variable:

$$\tau = \frac{(t - t_i) * \alpha}{(t_f - t_i)} \tag{4}$$

where α is a scalar belonging to $[0, 2\pi]$.

In the new time coordinate, the initial time is equal to zero and the final time is equal to α. The determination of the different coefficients a_0, a_k, b_k, a_m for the trigonometric spline (2) can be realised after having transformed the different time constraints with the linear transformation (4). It must be noted that as the successive derivatives $\dot{y}$,$\ddot{y}$,..,$y^{(\beta)}$ are known with respect to the time t, the determination of their values with respect to the new reference time τ introduces a scaling factor depending on the order of the derivative :

$$\begin{cases} \dot{y}(\tau) = \dot{y}(t).\left(\dfrac{\alpha}{t_f - t_i}\right) \\ \vdots \\ y^{(\beta)}(\tau) = y^{(\beta)}(t).\left(\dfrac{\alpha}{t_f - t_i}\right)^{\beta} \end{cases} \tag{5}$$

4. FLAT CONTROL

4.1 Short overview

Flat control is a quite recent control theory. It has been developed by M. Fliess, J. Lévine, Ph. Martin and P. Rouchon in 1992.(Fliess *et al.*, 1992) Consider a non linear model given by:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \tag{6}$$

with state and control vectors defined as $\mathbf{x} = \left[\, x_1 \; \cdots \; x_n \,\right]^T \in \mathbb{R}^n$ and $\mathbf{u} = \left[\, u_1 \; \cdots \; u_m \,\right]^T \in \mathbb{R}^m$. System (6) is called a *differential flat system* if, and only if, there exists a system output $\mathbf{y} \in \mathbb{R}^m$ such that:

$$\begin{cases} \mathbf{x} = A(\mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \cdots, \mathbf{y}^{(\alpha)}) \\ \mathbf{u} = B(\mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \cdots, \mathbf{y}^{(\alpha+1)}) \end{cases} \tag{7}$$

with $\mathbf{y}$ of the form:

$$\mathbf{y} = h(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, \cdots, \mathbf{u}^{(\delta)}) \tag{8}$$

The $\mathbf{y}$ output is called a *flat output* of system (6), being necessary to have the same number of outputs than inputs (that is $y \in \mathbb{R}^m$). It has been shown that having found a flat output, the control is defined by

$$\mathbf{u}(t) = B(\mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \cdots, \mathbf{v}) \tag{9}$$

linearizes the system (6) between the flat output y and a auxiliary input v in the Brunovsky form :

$$y_j^{(\alpha_j+1)} = v_j \quad j = 1, \ldots, \dim(u) \tag{10}$$

Path planning and trajectory tracking can also be implemented, since for a desired flat output $\mathbf{y}_d(t)$ the asymptotic tracking of this trajectory can be achieved by a classical feedback on $\mathbf{v}$

$$v_j = y_{d,j}^{(\alpha_j+1)} + \sum_{k=1}^{\alpha_j} \sigma_j^k \left(y_j^{(k)} - y_{d,j}^{(k)} \right) \quad j = 1, \ldots, \dim(u) \tag{11}$$

Where σ_j^k denote the coefficients of a Hurwtiz polynomial which ensures the desired tracking dynamics.

4.2 The problem of trajectory tracking

Equation (11) shows that flat control needs, for each flat output, to use a well-defined trajectory $\left(y_d, \dot{y}_d, \ddot{y}_d, \cdots, y_d^{(\alpha+1)}\right)$. Indeed, if the trajectory is not completely defined (i.e the function is not smooth enough and the successive derivatives are null) then the most interesting characteristic of the flat control, which consists in taking into account the dynamics of the plant, is not correctly exploited. Generally (Levine, 2004) polynomial splines are used to describe a desired path planning. If only a point-to-point path is used to define

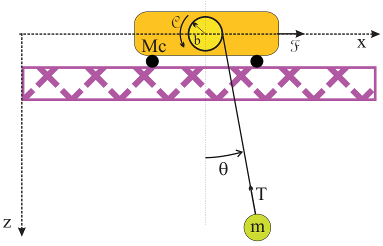

Fig. 2. The crane

the path a well-defined trajectory must verify the $(\alpha + 2)$ initial conditions at time t_i:

$$\overline{Y}_i = \left(y_d(t_i), \dot{y}_d(t_i), \ddot{y}_d(t_i), \cdots, y_d^{(\alpha+1)}(t_i) \right)$$

and the $(\alpha + 2)$ final conditions at time t_f :

$$\overline{Y}_f = \left(y_d(t_f), \dot{y}_d(t_f), \ddot{y}_d(t_f), \cdots, y_d^{(\alpha+1)}(t_f) \right)$$

A simple and usual solution consists in constructing the $(2\alpha+3)$ degree polynomial spline satisfying these $(2\alpha + 4)$ conditions :

$$y(t) = \sum_{k=0}^{2\alpha+3} c_k \tau^k(t) \tag{12}$$

with $\tau(t) = \frac{t - t_i}{t_f - t_i}$

Obviously it is possible to impose intermediate points. In this case the degree of the polynomial spline, which corresponds to this trajectory, will increase.

5. CONTRIBUTION OF THIS APPROACH THROUGH THE EXAMPLE OF THE CRANE

This section presents through the example of the crane, one of the most famous examples in flat control (Fliess *et al.*, 1993). Three interesting points in the use of trigonometric spline: the efficiency, the infinite derivability and the optimality of the calculated path.

5.1 Flatness and control structure

The dynamical equations of the crane (Fig. 2) can be decomposed into two parts :

- The trolley position (D) and the rope length(R) are controlled by a PID controller.
- The (x, z) coordinates of the load m which are controlled by a nonlinear flat controller.

The corresponding equations are :

$$\Sigma_1 : \begin{cases} m\ddot{x} = -T\sin(\theta) \\ m\ddot{z} = -T\cos(\theta) + mg \\ x = R\sin(\theta) + D + b\cos(\theta) \\ z = R\cos(\theta) - b\sin(\theta) \end{cases} \tag{13}$$

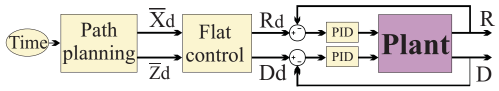

Fig. 3. Structure of the controller

$$\Sigma_2 : \begin{cases} M_c\ddot{D} = \mathcal{F} + T\sin(\theta) - f_1\dot{D} \\ J\dfrac{\ddot{R}}{b} = \mathcal{C} + bT - f_2\dfrac{\dot{R}}{b} \end{cases} \quad (14)$$

All the variables of these equations are represented on the figure 2 except for the friction on the trolley f_1 and the friction on the roll f_2.

The subsystem Σ_1 (and consequently the whole system) is flat with the flat output (x, z) because all the state variables verify the formulation (7) :

$$\begin{cases} R = \dfrac{1}{g-\ddot{z}}\left[z\sqrt{\ddot{x}^2 + (\ddot{z}-g)^2} - b\ddot{x}\right] & (a) \\ D = x - \dfrac{1}{g-\ddot{z}}\left[b\sqrt{\ddot{x}^2 + (\ddot{z}-g)^2} - z\ddot{x}\right] & (b) \\ \theta = \arctan\left(\dfrac{\ddot{x}}{\ddot{z}-g}\right) & (c) \\ T = m\sqrt{\ddot{x}^2 + (\ddot{z}-g)^2} & (d) \end{cases} \quad (15)$$

The subsystem Σ_2 is controlled by the way of two PID which adapt the force $\mathcal{F}$ and the torque $\mathcal{C}$ with respect to the difference between the trolley position D (resp. the rope length R) and the desired position D_d (resp. the desired length R_d). Therefor, the flat control has to determine the necessary trolley position reference D_d and rope length R_d reference, including the dynamic of the plant, by using the equations (15a) and (15b). Thus it needs to use the output of a path generator for each component of the flat output $\overline{X_d} = [x_d, \dot{x_d}, \ddot{x_d}]^T$ and $\overline{Z_d} = [z_d, \dot{z_d}, \ddot{z_d}]$, (the path trajectories have to be at least twice derivable). Globally, the structure of the controller can be represented by the figure 3.

5.2 Interest of the method trough three simulation results

In this section, three comparisons of simulation results are presented. The comparisons are related to the "path generator", which is either a standard polynomial spline (equation 12) or a trigonometric spline (equation 2). The degree of the polynomial spline depends on the simulation but is always the same for both trajectories .

5.2.1. Efficiency The first simulation corresponds to a nominal case for a point-to-point trajectory. We impose the initial condition $\overline{X}_i = (0, 0, 0), \overline{Z}_i = (-10, 0, 0)$ and the final condition $\overline{X}_f = (20, 0, 0), \overline{Z}_f = (-11, 0, 0)$ that leads to 5th-order polynomials .

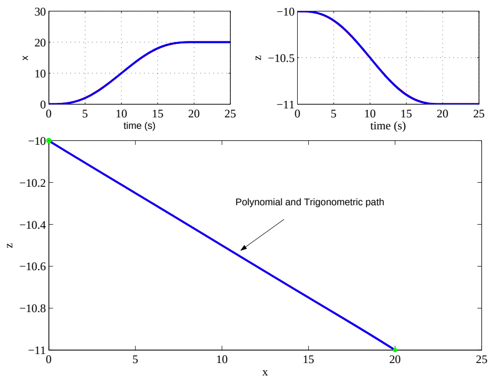

Fig. 4. Simulation n°1 : nominal case

Figure 4 shows that the two kinds of trajectory are globally equivalent. Indeed, the trajectories are sufficiently derivable (with respect to the order of the flat control (7) and satisfy the initial and final conditions. We can conclude that the trigonometric spline are well-adapted to perform a good trajectory tracking. Moreover, it can be noted that the determination of parameters for both cases are equivalent and consists in the resolution of a linear system.

5.2.2. Infinity derivability The second point concerns the infinity derivability of the trigonometric spline. Indeed from (2) it can be remarked that whatever the degree of the considered spline is, its derivatives are continuous and not strictly null. This property is not true in the case of polynomials spline where the derivatives become null when the order of the derivative is bigger to the order of the polynomial. The proposed simulation illustrates this advantage : only initial and final points are imposed to determine the trajectory and leads to first order polynomials (for example $z_d(t) = 0.033t + 10$ or $z_d(\tau) = 12 + 2\sin(\tau - \gamma)$).

The simulations (figure 5) show that the trajectory generated by the polynomial spline involves oscillations due to the fact that the second derivative is null and does not respect the order of the flat model. Conversely, the trigonometric spline performs trajectory correctly tracked by the system, without any visible oscillation during the travel. At the terminal point, however, it can be observed that an important oscillation exist. This oscillation appears because the final speed reference is not null when the final point is reached.

5.2.3. Optimality This third simulation is based on the first one but an obstacle has been added between the initial and the final point. To perform the avoidance, two supplementary points have been taken into account to design the trajectory. These two points correspond to the following coordinates $\overline{X}_A = (5, 1, 0.1), \overline{Z}_A = (-5, 1, 0.15)$ and

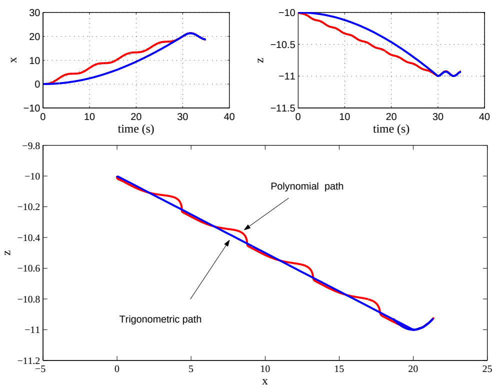

Fig. 5. Simulation n°2 : first order polynomials

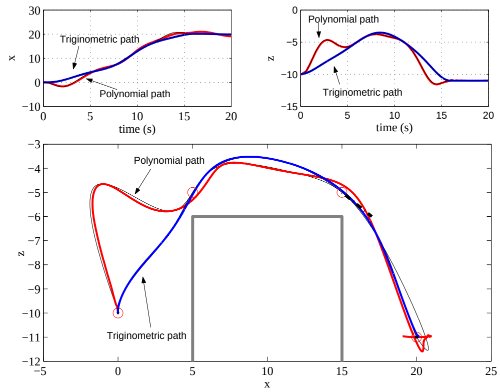

Fig. 6. Simulation n°3 : obstacle avoiding

$\overline{X}_B = (15, 2, 0.1)$, $\overline{Z}_B = (-5, -1, -1)$. The point A must be reached at time $t = 6s$ and the point B at time $t = 11s$. Different approaches can be considered for such a path planning. The first one consists in calculating 3 trajectories (respectively $[X_i A]$, $[A, B]$ and finally $[B, X_f]$) and this case can be reduced to the previous simulation (repeated three times). The second approach (used in this section) consists in taking into account all the constraints in the same path. Consequently the trajectory has to satisfy 12 constraints (4 positions, 4 speeds and 4 accelerations), that leads to a 11th order spline polynomial and trigonometric splines. It can be observed that a very important difference between the two paths. Indeed, the figure (6) shows that the path described by the spline polynomial respects the constraints but is far from optimum. This phenomenon is known in robotics under the name of "wandering". Conversely the path described by the trigonometric spline is perfectly adapted to the desired trajectory. In the case of robot motion (Simon and C., 1991; Simon and C., 1992), there is some elements of proof that show that the proposed path is optimal.

6. CONCLUSION

In this paper was presented the use of trigonometric splines for the path planning. In a first part it has been shown the interest of using such smooth trajectory in the case of linear system. In a second part, for non linear systems, it has been shown the effectiveness of such an approach. The main property exploited is that the function describing the trigonometric spline is smooth (its derivatives are not strictly null). This property implies that the use of trigonometric splines is particularly well adapted for the generation of trajectory in the control of flat systems. Another appreciable property of the trigonometric splines is that the phenomenon of "wandering" disappears and gives some optimality for the calculated path. An actual extension of this work consists in performing an on-line version of this trajectory generation which will be able to take into account the actual position (and the successive derivatives) of the system to re-calculate the parameters of the trigonometric spline.

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