SYSTEM IDENTIFICATION USING MEASUREMENTS SUBJECT TO STOCHASTIC TIME JITTER

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Abstract: When the sensor readings are perturbed by an unknown stochastic time jitter, classical system identification algorithms based on additive amplitude perturbations will give biased estimates. We here outline the maximum likelihood procedure, for the case of both time and amplitude noise, in the frequency domain, based on the measurement Discrete Time Fourier Transform (DTFT). The method directly applies to output error continuous time models, while a simple sinusoid in noise example is used to illustrate the bias removal of the proposed method.

Keywords: signal processing, sampling, stochastic properties, spectrum analysis, system identification, noise characteristics

1. INTRODUCTION

Nonuniform sampling appears in, e.g., automotive applications, radar imaging, event controlled systems and time critical applications. The non uniformity in the time stamps have different characteristics, depending on the application and in many cases the sampling is almost uniform with sampling noise corrupting the exact timing. E.g., when the sampling frequency is close to the clock frequency, when sampling requests are delayed through communication or, when imperfect mechanics are used to synchronize measurements. In these cases it is not obvious that the true measurement times are known, but instead the stochastic characteristics of the sampling jitter noise is known.

This work discusses the impact of unknown nonuniform jitter sampling on signal estimation. We describe the effect and use this knowledge to improve the maximum likelihood parameter estimation of a single sinusoid. The signal estimation problem is introduced in Section 2 and in Section 3 the effect of noise on the transform is shown. The stochastic properties of the transform is further evaluated in Section 4 and an example of parameter estimation is given in Section 5. The work is concluded and future actions are discussed in Section 6.

2. REVIEW AND PROBLEM FORMULATION

The system identification problem under consideration can be stated as estimating the parameter θ in a model structure

$$\mathcal{M} : g_\theta(t), h_\theta(t), p_\theta(\tau) \tag{1}$$

based on discrete time measurements corrupted by time and amplitude noise according to

$$\begin{aligned} y_k &= s(kT + \tau_k; \theta) + v(kT + \tau_k; \theta), \\ s(t;\theta) &= g_\theta * u(t), \\ v(t;\theta) &= h_\theta * e(t). \end{aligned} \tag{2}$$

Here u is a known input, e is noise with known characteristics, and τ_k is the sampling noise.

The sampling noise is a sequence of independent stochastic variables with probability density function (pdf) $p_\theta(\tau)$. Both the signal, noise and sampling models can be parameterized in the unknown parameter vector θ. In system identification it can be relevant to find the signal model, $g_\theta(t)$, the noise model, $h_\theta(t)$ or even the sampling model, $p_\theta(\tau)$. (Ljung 1999) treats the standard case, when $\tau_k = 0$. One example of a signal model is a continuous time output error model, with

$$s(t;\theta) = g_\theta * u(t) \tag{3a}$$
$$S(f;\theta) = G(f;\theta)U(f), \tag{3b}$$

in the time and frequency domain, respectively.

Signal estimation in the time domain, *e.g.*, the least squares (LS) solution

$$\hat{\theta}^{LS} = \arg\min_\theta \sum_{k=1}^{N} (y_k - s(kT;\theta))^2, \tag{4}$$

is less attractive for our case with time noise. For that reason, we focus on frequency domain methods where the signal model is directly expressed as a continuous time model $S(f;\theta)$. The basic idea is to derive the mean $\mu_Y(\theta)$ and covariance $R_Y(\theta)$ of the measurement DTFT $Y(f)$ in several frequencies, preferably the DFT frequencies $f = n/NT$, or a subset thereof. The central limit theorem motivates that the DTFT vector, here denoted Y, is asymptotically Gaussian ($As\mathcal{N}$) distributed,

$$Y \in As\mathcal{N}(\mu_Y(\theta), R_Y(\theta)). \tag{5}$$

The asymptotic ML estimate is given by the minimizing value in

$$\begin{aligned} l(\theta) = (Y - \mu_Y(\theta))R_Y(\theta)^{-1}(Y - \mu_Y(\theta))^T + \\ + \ln\det R_Y(\theta). \end{aligned} \tag{6}$$

Note that the covariance matrix, R_Y, will be close to diagonal if the frequency points are chosen sufficiently far apart. The explicit dependence on the underlying model structure is derived in Section 4. The minimization over $l(\theta)$ is not explicit, so a numeric search algorithm is needed. (Pintelon and Schoukens 2001) is a nice reference for maximum likelihood estimates and frequency domain identification. More estimation techniques is also found in the standard reference (Ljung 1999).

For the well studied special case of amplitude noise only, we have

$$\mu_Y(f;\theta) = S * \Gamma_N^p(f) \tag{7}$$
$$R_Y(f;\theta) = \Phi_{vv}(f;\theta), \tag{8}$$

where $\Phi_{vv}(f;\theta)$ is the noise spectra and * denotes convolution. The frequency window $\Gamma_N^p(f)$ includes

- *Leakage*: local behavior around f depending on the number of samples.
- *Aliasing*: $\Gamma_N^p(f) = \Gamma_N^p(f + k/T)$ is a periodic function summing up all signal part frequencies being multiples of the sampling frequency $1/T$ according to Poisson's summation formula. That is, the continuous time OE model (3b) is summed up using Poisson's summation formula implicitly in this formulation.

These well-known facts have to be modified when time noise is introduced. Aliasing still occurs, but higher frequencies are damped and distorted, and the leakage effect becomes frequency dependent.

This work will concentrate on describing the properties of the transform, when the DTFT of the sequence y_k is produced, given the sampling pdf, $p(\tau)$, and additive white Gaussian amplitude noise, v, i.e., the model structure in Eq. (1) is reduced to

$$\mathcal{M} : g_\theta(t), h_\theta(t) = \delta(t), p_\theta(\tau) = p(\tau).$$

These properties will be used to perform the ML estimate for θ, in $g_\theta(t)$.

Nonuniform sampling, both deterministic and stochastic, is described in (Bilinskis and Mikelsons 1992) and (Marvasti 2001). The Dirichlet transform is discussed as the accurate way of performing frequency transform of nonuniform time samples in (Bilinskis and Mikelsons 1992). The common factor for these works is the complete knowledge of the time stamps, t_k, which is the main difference from this work.

System identification in the frequency domain has also been treated before, for uniform sampling in the time domain, both theoretical aspects (Gillberg and Ljung 2005) and identification in an automotive application (Gillberg and Gustafsson 2005). These publications address the problem assuming that the sampling times are known, but are also aiming at frequency domain identification based on nonuniform time domain sampling.

3. FREQUENCY TRANSFORM WITH NOISE

As stated we constrain the investigation to additive white Gaussian noise v and a completely known sampling noise pdf. To clearly show the impact of the two noise types in Eq. (2), we describe the transform for the combined sampling model

$$y_k = z_k + v_k \tag{9a}$$
$$z_k = s(kT + \tau_k) \tag{9b}$$
$$s(t) = \int S(f)e^{i2\pi ft}\,\mathrm{d}f \tag{9c}$$

where both noises are independent identically distributed. The measurement noise, v_k, is white and Gaussian with $E[v_k v_l] = \sigma^2\delta(k-l)$, and the sampling noise, τ_k, is limited to the interval $[-T/2, T/2]$ and $E[\tau_k] = 0$.

This section describes the effect of the noises on the Discrete Time Fourier Transform (DTFT). The stochastics of the transform will differ for the two noise models. The DTFT of the sequence y_k is

$$\begin{aligned} Y(f) &= \sum_{k=0}^{N-1} y_k e^{-i2\pi fkT} \\ &= \sum_{k=0}^{N-1} (z_k + v_k) e^{-i2\pi fkT} \\ &= Z(f) + \underbrace{\sum_{k=0}^{N-1} v_k e^{-i2\pi fkT}}_{\hat{V}(f)}, \end{aligned} \tag{10}$$

where, using Eq. (9c), the sampling noise part becomes

$$\begin{aligned} Z(f) &= \sum_{k=0}^{N} z_k e^{-i2\pi fkT} \\ &= \sum_{k=0}^{N-1} \int S(\varphi) e^{i2\pi\varphi(kT+\tau_k)} \, \mathrm{d}\varphi e^{-i2\pi fkT} \\ &= \int S(\varphi) W(f,\varphi) \, \mathrm{d}\varphi, \\ W(f,\varphi) &= \sum_{k=0}^{N-1} e^{i2\pi(\varphi-f)kT} e^{i2\pi\varphi\tau_k}. \end{aligned} \tag{11}$$

Note that this continuous time frequency domain approach is perhaps the only way to explicitly separate the signal and time errors. The assumption of Gaussian noise together with the fact that the sampling noise are independent identically distributed random variables gives that the DTFT of the sequence y_k in Eq. (9a) is distributed as

$$Y \in As\mathcal{N}(E[Z], \mathrm{Cov}(Z) + \mathrm{Cov}(\hat{V})), \tag{12}$$

according to the central limit theorem.

4. FIRST MOMENTS WITH SAMPLE NOISE

The moments of the transform has an explicit dependence on the signal transform $S(f;\theta)$. In the following derivation the dependence on the parameter θ will be implicit. The moments of the transform in Eq. (5) is given as

$$\begin{aligned} \mu_Y(f) = E[Y(f)] &= E[Z(f)] \\ &= \int S(\varphi) E[W(f,\varphi)] \, \mathrm{d}\varphi, \\ &= \int S(\varphi) \mu_W(f,\varphi) \, \mathrm{d}\varphi \end{aligned} \tag{13}$$

and

$$R_Y(f,\varphi) = R_Z(f,\varphi) + R_{\hat{V}}(f,\varphi),$$

with

$$\begin{aligned} R_Z(f,\varphi) &= \mathrm{Cov}(Z(f), Z(\varphi)) \\ &= \iint S(\eta) \, \mathrm{Cov}(W(f,\eta), W(\varphi,\zeta)) S(\zeta)^* \, \mathrm{d}\eta \, \mathrm{d}\zeta, \\ &= \iint S(\eta) R_W(f,\eta,\varphi,\zeta) S(\zeta)^* \, \mathrm{d}\eta \, \mathrm{d}\zeta. \end{aligned} \tag{14}$$

The addition to the covariance from the measurement noise is simply

$$\begin{aligned} R_{\hat{V}}(f,\varphi) &= \mathrm{Cov}(\hat{V}(f), \hat{V}(\varphi))), \\ &= \sigma^2 \sum_{k=0}^{N-1} e^{-i2\pi(f-\varphi)kT}. \end{aligned}$$

The following Lemmas show the statistics for the stochastic window, W, needed in Eqs. (13) and (14).

Lemma 1. (Mean value). The mean value, μ_W, with respect to τ_k, of the stochastic window, $W(f,\varphi)$, defined in Eq. (11), is

$$\mu_W(f,\varphi) = \gamma^g(-\varphi) \Gamma_N^p(f-\varphi),$$

where $\gamma^g(f)$ is the characteristic function of τ and $\Gamma_N^p(f)$ is the Dirichlet function.

Proof:

$$\begin{aligned} \mu_W(f,\varphi) &= E[W(f,\varphi)], \\ &= \sum_{k=0}^{N-1} e^{i2\pi(\varphi-f)kT} E[e^{i2\pi\varphi\tau_k}] \\ &= \gamma^g(-\varphi) \underbrace{\frac{1-e^{i2\pi(\varphi-f)NT}}{1-e^{i2\pi(\varphi-f)T}}}_{\Gamma_N^p(f-\varphi)}, \end{aligned}$$

where we recognize the characteristic function and the Dirichlet function. ∎

The term $\gamma^g(f) = E[e^{-i2\pi f\tau_k}]$, is the characteristic function for τ_k and models damping corresponding to the sampling noise. The Dirichlet function, $\Gamma_N^p(f)$, becomes the periodic window arising from the actual sampling,

$$\Gamma_N^p(f) = e^{-i\pi f(N-1)T} \frac{\sin(\pi fNT)}{\sin(\pi fT)}.$$

Thus, when there is no sampling noise, the expected value is the usual periodic window, Γ_N^p, since, in that case $\gamma^g(f) = 1$. Also, in that case, $E[Y(f)] = \int S(\varphi)\Gamma_N^p(f-\varphi)\,\mathrm{d}\varphi$, which is equal to the DTFT of $s(kT)$. Note also that since $\gamma^g(f)$ is the characteristic function (CF) for the distribution of τ_k, the exact value can be found in textbooks on probability theory for standard distributions.

In (Souders *et al.* 1990), the Fourier transform of $E[s(t+\tau_k)]$ is calculated to show the bias introduced by the jitter sampling. The effect is shown to be a linear filter on $S(f)$, and since $E[DFT(y_k)] = DFT(E[y_k])$ the result in the

Lemma above shows the different effect the jitter sampling has on the DFT compared to on the Fourier transform. Basically, $\mathcal{F}(E[s(t+\tau)]) = S(f)\gamma^g(f)$ in the case of even noise distributions.

Lemma 2. (Covariance). The covariance, R_W, of W is given as

$$\begin{aligned} R_W(f,\eta,\varphi,\zeta) + \mu_W(f,\eta)\mu_W(\varphi,\zeta)^* \\ &= E[W(f,\eta)W^*(\varphi,\zeta)], \\ &= \Lambda_\tau^T(\eta,\zeta)\Lambda_N(f-\eta,\varphi-\zeta). \end{aligned}$$

The two factors correspond to parts depending on the sampling noise, Λ_τ, and on the finite sampling, Λ_N, and they are

$$\Lambda_\tau(f,\varphi) = \begin{pmatrix} \gamma^g(-f)\gamma^g(\varphi) \\ \gamma^g(\varphi - f) \\ \gamma^g(-f)\gamma^g(\varphi) \end{pmatrix}$$

and

$$\Lambda_N(f,\varphi) = \begin{pmatrix} \Sigma_N^p(f,-\varphi) \\ \Gamma_N^p(f-\varphi) \\ \Sigma_N^p(-\varphi,f) \end{pmatrix},$$

respectively. The functions were defined in Lemma 1, except for

$$\begin{aligned} \Sigma_N^p(f,\varphi) &= \sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\gamma^p(f)^k\gamma^p(\varphi)^l \\ &= \begin{cases} \frac{1}{(1-\gamma^p(\varphi))}\left[\Gamma_N^p(f) - \Gamma_N^p(f+\varphi)\right], & \gamma^p(\varphi)\neq 1, \\ \rho^2\frac{\rho^N(-1+N(\rho^{-1}-\rho^{-2}))+1}{(1-\rho)^2}, & \begin{cases}\gamma^p(f)=\gamma^p(\varphi), \\ \rho=\gamma^p(f)\neq 1\end{cases} \\ \frac{N(N-1)}{2}, & \gamma^p(f)=\gamma^p(\varphi)=1, \end{cases} \end{aligned}$$

with $\gamma^p(f) = e^{-i2\pi fT}$.[1]

Proof: The second moment of the stochastic window, W, is by definition

$$\begin{aligned} &E[W(f,\eta)W^*(\varphi,\zeta)] \\ &= \frac{1}{N^2}\sum_{k,l} e^{i2\pi(\eta-f)kT}E[e^{i2\pi(\eta\tau_k-\zeta\tau_l)}]e^{-i2\pi(\zeta-\varphi)lT} \\ &= \sum_k\sum_{l<k} + \sum_k\sum_{l=k} + \sum_k\sum_{l>k} \\ &= V_1 + V_2 + V_3. \end{aligned}$$

It will be helpful to note that

$$\begin{aligned} \sum_{k=0}^{N-1}a^k\sum_{l=0}^{k-1}b^l &= \sum_{k=0}^{N-1}a^k\begin{cases}\frac{1-b^k}{1-b}, & b\neq 1 \\ k, & b=1\end{cases} \\ &= \begin{cases} \frac{1}{1-b}\left(\frac{1-a^N}{1-a} - \frac{1-(ab)^N}{1-ab}\right), & a,b,ab\neq 1 \\ \frac{1}{1-b}\left(\frac{1-a^N}{1-a} - N\right), & b=\frac{1}{a}\neq 1 \\ \frac{1}{1-b}\left(N - \frac{1-b^N}{1-b}\right), & b\neq a = 1 \\ a^2\frac{a^N(-1+N(a^{-1}-a^{-2}))+1}{(1-a)^2}, & a\neq b=1, \\ \frac{N(N-1)}{2}, & a=b=1 \end{cases} \end{aligned}$$

which immediately gives $\Sigma_N^p(f,\varphi)$ as in the Lemma. The first term of the second moment above is

$$\begin{aligned} &V_1(f,\varphi,\eta,\zeta) \\ &= \gamma^g(-\eta)\gamma^g(\zeta)\sum_{k=0}^{N-1}e^{i2\pi(\eta-f)kT}\sum_{l=0}^{k-1}e^{-i2\pi(\zeta-\varphi)lT} \\ &= \gamma^g(-\eta)\gamma^g(\zeta)\Sigma_N^p(f-\eta,\zeta-\varphi), \end{aligned}$$

since $\gamma^g(f) = E[e^{-i2\pi f\tau_k}]$ was the CF of τ_k. The second term is the sum over $l=k$,

$$\begin{aligned} V_2(f,\varphi,\eta,\zeta) &= \sum_{k=0}^{N-1}e^{i2\pi(\eta-f-\zeta+\varphi)kT}\gamma^g(\zeta-\eta) \\ &= \gamma^g(\zeta-\eta)\Gamma_N^p(f-\eta-\varphi+\zeta) \end{aligned}$$

and, finally, the third term is

$$\begin{aligned} &V_3(f,\varphi,\eta,\zeta) \\ &= \gamma^g(-\eta)\gamma^g(\zeta)\sum_{l=0}^{N-1}e^{-i2\pi(\zeta-\varphi)lT}\sum_{k=0}^{l-1}e^{i2\pi(\eta-f)kT} \\ &= \gamma^g(-\eta)\gamma^g(\zeta)\Sigma_N^p(\zeta-\varphi,f-\eta). \end{aligned}$$

Identification of terms gives the result of the Lemma. ■

5. EXAMPLE

This section describes an example of parameter estimation for the signal

$$s(t;\,\theta) = a_0\sin(2\pi f_0 t) \tag{15a}$$

$$\theta = \begin{pmatrix} a_0 & f_0 \end{pmatrix}^T. \tag{15b}$$

The transform of $s(t;\,\theta)$ is then

$$S(f;\,\theta) = \frac{a_0}{2i}\left(\delta(f-f_0) - \delta(f+f_0)\right). \tag{15c}$$

From the expression of the transform statistics in Lemmas 1 and 2, it is straightforward to use Eqs. (13) and (14) to calculate the mean value and covariance in the distribution in Eq. (12).

To find the best parameter vector θ the maximum likelihood estimator will be used. The DTFT is calculated for the frequencies $f = n/NT$, $n = 0,..,N-1$, i.e., we get the DFT, and will be stacked in the vector Y^N. The likelihood function for both sampling cases is then

$$\begin{aligned} l(\theta) = (Y^N - \mu(\theta))^*C(\theta)^{-1}(Y^N - \mu(\theta)) + \\ + \ln\det(C(\theta)), \end{aligned} \tag{16}$$

where $\mu(\theta)$ is the mean vector with the kth element being

$$(\mu(\theta))_k = E[Y(\frac{k}{NT})|\theta],$$

and $C(\theta)$ is the covariance matrix with element n,m being

$$(C(\theta))_{n,m} = \mathrm{Cov}(Y(\frac{n}{NT}),Y(\frac{m}{NT})\big|\,\theta).$$

[1] Comparing the notation, $\gamma^p(f)$ is the periodic (p) counterpart to the more general (g) characteristic function, $\gamma^g(f)$. Superscript p is used to indicate that a variable is constructed based on $\gamma^p(f)$ and vice versa for g.

Then,

$$\hat{\theta} = \arg\min_{\theta} l(\theta) \tag{17}$$

is the ML-estimation of the amplitude and frequency of the sinusoid.

The sampling of the signal, $s(t)$, is done with jitter sampling noise, $z_k = s(kT + \tau_k)$, and we will compare two ways of estimating the parameters. First we model the noise as additive and use the common frequency domain ML method to estimate the parameters. This crude approximation is compared to using the fact that the noise is jitter on the measurement times to produce the ML estimate.

Type 1 Approximate the signal samples as corrupted by measurement noise but not sample noise, $\tau_k = 0$ in Eq. (9b):

$$E[Y(\frac{k}{NT})] = \frac{a_0}{2i}\big(\Gamma_N^p(\frac{k}{NT} - f_0) + \\ - \Gamma_N^p(\frac{k}{NT} + f_0)\big),$$

$$\mathrm{Cov}(Y(\frac{n}{NT}), Y(\frac{m}{NT})) = \frac{\sigma^2}{N}\Gamma_N^p(\frac{n-m}{NT}), \\ = \frac{\sigma^2}{N}\sum_{r=-\infty}^{\infty}\delta(\frac{n-m}{N} - r)$$

Type 2 Let the signal samples be corrupted by sample noise but not measurement noise, $v_k = 0$ in Eq. (9a):

$$E[Y(f)] = \frac{a_0}{2i}\big(\gamma^g(-f_0)\Gamma_N^p(f - f_0) + \\ - \gamma^g(f_0)\Gamma_N^p(f + f_0)\big).$$

The covariance is given from Lemma 2 and Eqs. (15c) and (14).

Plots of the likelihood function, Eq. (16), averaged over 50 Monte Carlo realizations of τ_k, using the sampling noise model (Type 2), are given in Figure 1. The global minimum is clearly shown, but it is obvious that the likelihood function also has local minima.

To be able to show the performance of the estimates from Eq. (17), 16 Monte Carlo runs were made for 17 different frequencies and 19 different amplitudes. In Figure 2, the resulting mean errors and the standard deviations are shown. The estimation of the frequency is fairly similar for the two methods, (Type 1 and 2), while in the estimation of the amplitude, the model with measurement noise (Type 2) gives a large bias. This is easily explained with the results from Lemma 1, where it was shown that the difference in mean value, between Type 1 and 2 was an amplitude scaling with the CF $\gamma^g(f)$. For both cases the standard deviation increases for larger values of the true parameters, f^0 and a^0. Though, the standard deviation is significantly smaller for $\hat{a}$ when using the Type 2 model.

To further explain the increase in the standard deviation we study the sampling noise model (Type 2) further. In Figure 3, the standard deviation is depicted for $\hat{f}$ and $\hat{a}$ as a function of both the true frequency and the true amplitude. These plots show, that it is harder to estimate a higher frequency independent of the amplitude, and this property also affects the ability to estimate the amplitude, when a^0 is large. It is always easy to estimate zero amplitude regardless of its frequency.

6. CONCLUSIONS AND FUTURE WORK

We have performed a preliminary investigation of system identification in the frequency domain for nonuniformly sampled signals, where the actual sample times are perturbed from the nominal uniform values by an unknown realizations of a stochastic process. A continuous time model frequency domain maximum-likelihood approach was taken. It was shown that for estimation of amplitude and frequency in a single sinusoid, our method gives unbiased estimates and a smaller variance compared to a crude amplitude error approximation of the time errors. In future work, we will evaluate our method on output error models, and also consider parameterized sampling noise pdf's.

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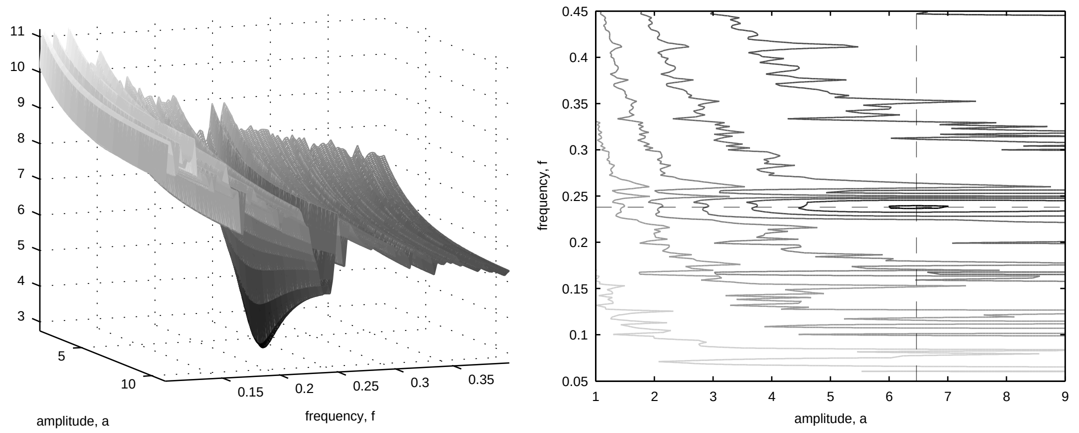

Fig. 1. The maximum likelihood function, $l(\theta) = l(a, f)$, Eq. (16), for sampling noise, Type 2. The true parameters are marked with dashed lines in the contour plot to the right. The global minimum is clearly visible in the mesh-plot to the left. 50 Monte Carlo runs have been used to remove some of the variations.

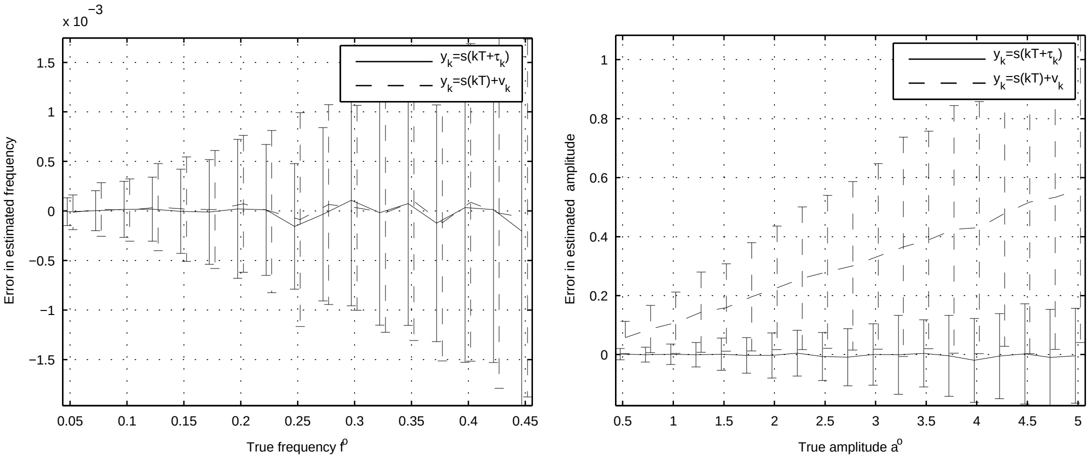

Fig. 2. Mean errors and standard deviations in estimated parameters using the two different signal models, Type 1 and 2. The graphs are slightly shifted horizontally to give better visibility. The mean and standard deviation are calculated over all Monte Carlo runs as well as over all amplitudes, a^0, or all frequencies, f^0, respectively.

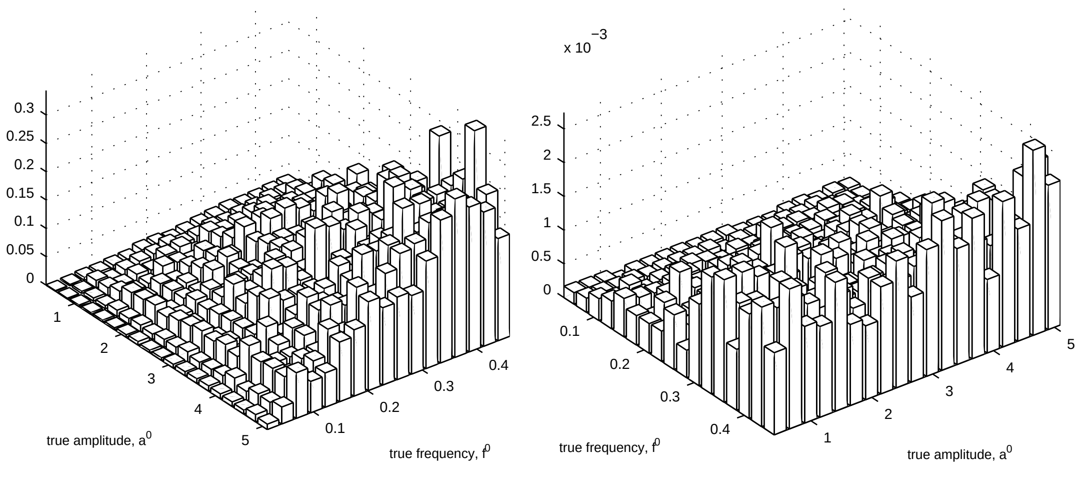

Fig. 3. The standard deviation for $\hat{a}$ (left) and $\hat{f}$ (right), using the model with sampling noise, Type 2, as a function of both true amplitude and true frequency.