A KALMAN FILTER-BASED STABLE DYNAMIC INVERSION FOR DISCRETE-TIME, LINEAR, TIME-VARYING SYSTEMS

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Abstract: In this paper the problem of estimating an unknown input for discrete-time, non-minimum phase, multivariable, linear time-varying systems (LTV) is considered. The initial condition of the plant may be unknown and stochastic process and measurement noise are included. The input signal is modelled as a random walk with drifts. Then it is estimated using a Kalman filter for a uniformly detectable augmented system. A necessary and sufficient condition for the detectability of the augmented system is provided. A Kalman filter-based stable dynamic inversion (SDI) for LTV systems is obtained as a consequence of our solution to the proposed problem. The inversion technique can be applied to achieve output tracking for LTV systems in the presence of non-minimum phase zeros and measurement and/or system noise. We are mainly motivated by typical need to replicate time signals in the automobile industry. Similar problems appear also in other fields as machine tool applications, aeronautic industry a.o.

Keywords: Linear time-varying system, Kalman filter, Unknown input, System inversion.

1. INTRODUCTION

Compared with state estimation, less research has been done on estimating unknown inputs for LTV systems. We refer to (Corless and Tu, 1998) for a short overview of some of the existing results. In the literature, a number of papers are devoted to the inversion of dynamical systems (Devasia *et al.*, 1996; Devasia, 1999; George *et al.*, 1999*a*; Dewilde and van der Veen, 2000). Most of these papers refrain from considering noise on the given system and/or time-varying systems.

Given a linear system Σ, our objective is to generate an input sequence $\{\hat{u}(k)\}_{k=1}^{N}$ by obtaining a suitable 'inverse system' Σ_{inv} that relates this input sequence to the given measurement sequence $\{y^d(k)\}_{k=1}^{N}$, for some given number of samples N.

The above objective was motivated by the typical need for time waveform replication in the automobile industry where the desired signal is obtained through a real-life data acquisition run, and hence corrupted by measurement and/or sensor noise. An example is the reproduction of time records (of accelerations and displacements) obtained during test drives with prototype cars. This reproduction is done on hydraulic test-rigs that enable full car endurance tests, driving comfort assessment, etc.

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for prolonged periods of time thereby saving precious resources. One high performance approach to determine the necessary inputs to the hydraulic actuators is via stable inversion of systems that naturally takes into account noise in target time histories, and is applicable to systems irrespective of the location of zeros. We note that the SDI procedure for LTI systems has successfully been applied in (Cuyper and Verhaegen, 2002; George *et al.*, 1999*a*; George *et al.*, 1999*b*) to aerospace and automobile examples to compute the desired input.

We consider in this paper discrete-time, non-minimum phase, multivariable, LTV systems. The input signal is modelled as a random walk with drifts and then it is estimated using a Kalman filter for an augmented system. Unknown initial conditions of the plant and stochastic process and measurement noise are considered. The proposed solution is based on augmenting the given uniformly detectable state-space model by a reasonable model for the input sequence and then designing a Kalman filter to provide an estimate of the input sequence from the measurements. Instead of only one model of the input sequence, we consider a set of models with a specific structure. A Kalman filter-based stable dynamic inversion (SDI) is obtained as a consequence of our solution to the proposed problem.

This paper is organized as follows: in the next section the class of systems under consideration is introduced. The model of the input and the resulting augmented system are presented in Section 3. Section 4 contains the Kalman filter-based stable dynamic inversion procedure under the assumption that the augmented system is uniformly detectable. Before the concluding section, the preservation of detectability under augmentation is discussed. More precisely, a necessary and sufficient condition for uniform detectability of the augmented system is provided.

2. PRELIMINARIES

Consider a discrete LTV system Σ of the form

$$\Sigma : \begin{cases} x(k+1) \; = \; A(k)x(k) + B(k)u(k) + G(k)w(k) \\ y(k) \; = \; C(k)x(k) + D(k)u(k) + v(k), \end{cases} \quad (1)$$

where k represents the time index normalized with the sampling period, the vectors $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ denote the system state, input and output, respectively, and $w(k) \in \mathbb{R}^n$ and $v(k) \in \mathbb{R}^p$ represent the process and measurement noise, respectively. $A(k)$, $B(k)$, $C(k)$, $D(k)$, $G(k)$ are deterministic matrices of appropriate sizes, with real entries, describing the dynamic system (plant). The matrix $D(k)$ is known as the direct transmission term, and it is typically present if the output vector represents acceleration measurements. The disturbance sequences $w(k)$ and $v(k)$ are not known but assumed to be zero-mean, identically distributed white noise sequences that are uncorrelated with the input, and with positive definite covariance matrix

$$E \begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \begin{bmatrix} v^T(j) \; w^T(j) \end{bmatrix} = \begin{bmatrix} R(k) & S^T(k) \\ S(k) & Q(k) \end{bmatrix} \Delta(k-j),$$

where Δ is the unit pulse. The initial state vector $x(0)$ is a random variable independent of $w(k)$ and $v(k)$, with the mean and covariance given by

$$E[x(0)] = \hat{x}_0,$$
$$E[(x(0) - \hat{x}_0)(x(0) - \hat{x}_0)^T] = P_0 > 0.$$

To represent the statistical information of the random variable $x(0)$ in an equation format, we introduce an auxiliary random variable $\tilde{u}(0)$ with zero mean and covariance matrix I_n. Since the covariance matrix P_0 is symmetric positive definite, it has a Cholesky factorization $P_0 = S_0 S_0^T$ with S_0 a unique lower triangular matrix with positive diagonal entries. It is easy to verify that the random variable $x(0)$ can be described through the following matrix equation: $x(0) = \hat{x}(0) + S_0\tilde{u}(0)$.

3. AN AUGMENTED SYSTEM

The SDI technique is based on augmenting the given state-space model (1) by a reasonable model for the input sequence and then designing a Kalman filter to provide an estimate of the input sequence from the measurements $\{y^d(k)\}_{k=1}^N$. For the input sequence u, we consider the following bench of models (see (Waltraud Kahle and Jensen, 1998))

$$u(k+1) = u(k) + \gamma(k)\delta(k) + G_u(k)w_u(k) \quad (2)$$

for $G_u(k)$ of appropriate sizes, some real positive sequence $\delta(k) \in \mathbb{R}^m$ and $\gamma(k) \in \mathbb{R}^{m\times m}$ diagonal matrix with all $\gamma_{ii}(k)$ components in the set $\{-1, 0, 1\}$ (the signature of the entries). The signal $w_u(k)$ is taken to be white noise with covariance Q_u, and uncorrelated with $w(k)$ and $v(k)$. Further, we incorporate $\gamma(k)\delta(k)$ into the deterministic part of the input model. Let us consider the diagonal sequence $\Delta(k) \in \mathbb{R}^{m\times m}$ such that $\Delta_{ii}(k)u(k) := \gamma_{ii}(k)\delta(k)$ if $\|u_i(k)\|_2 \geq T$, and $\Delta_{ii}(k) := \gamma_{ii}(k)\delta(k)$ if $\|u_i(k)\|_2 < T$. The multiple models (2) become

$$u(k+1) = (1 + \Delta(k))u(k) + G_u(k)w_u(k) \quad (3)$$

With the above assumptions one can add the input as a state to a new equivalent system. The resulting augmented system in compact form is as follows:

$$\Sigma_a : \begin{cases} x_a(k+1) = A_a(k)x_a(k) + G_a(k)w_a(k) \\ y(k) = C_a(k)x_a(k) + H_a(k)w_a(k) \end{cases}, \quad (4)$$

where the state and the disturbance are $x_a(k) = \begin{pmatrix} x(k)^T & u(k)^T \end{pmatrix}^T$, $w_a(k) = \begin{pmatrix} w(k)^T & v(k)^T & w_u^T(k) \end{pmatrix}^T$, and the system matrices are

$$A_a(k) := \begin{bmatrix} A(k) & B(k) \\ 0 & 1+\Delta(k) \end{bmatrix} \; G_a(k) := \begin{bmatrix} G(k) & 0 & 0 \\ 0 & 0 & G_u(k) \end{bmatrix}$$
$$C_a(k) := \begin{bmatrix} C(k) & D(k) \end{bmatrix} \;\; H_a := \begin{bmatrix} 0 & I & 0 \end{bmatrix}.$$

Remark that the augmented system Σ_a is a particular case of the most general state space formulation for detection in linear systems (Gustavson, 2000, Model (10.1)). In our augmented model, the abrupt changes appear only in the $m \times m$ lower-right block of the matrices $A_a(k)$, and $B_a(k) = D_a(k) = 0$.

Example 3.1. An example is the estimation of an almost constant input. For example one would like to estimate a constant in the process white noise which occurred at unknown time instants. One can consider the following state space system with $D = G_u = 0$ and $A = B = C = G = 1$

$$\begin{cases} x(k+1) = x(k) + u(k) \; +w(k) \\ y(k) = x(k) \qquad\qquad +v(k) \end{cases}$$

This example is similar to (Gustavson, 2000, Example 10.1). There one would like to detect changes of a constant in the measurement noise instead of the process noise. The augmented system has the following matrices:

$$A_a(k) = \begin{bmatrix} 1 & 1 \\ 0 & 1+\Delta(k) \end{bmatrix}, \; C_a = [1\;0], \; G_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 3.2. One would like to estimate a function in the process white noise which has arbitrarily many changes. The unknown input will be modelled as in (3) with $G_u(k) = I$. The model in our approach is different in the sense that all varying parameters are included in the augmented system description in contrast with the approach from (Gustavson, 2000, Example 10.2) where some time-varying parameters are covariances of the noise. If we consider $G_u(k) \neq I$, then our model includes also varying covariances.

4. A KALMAN FILTER-BASED SDI

The notions of uniform detectability and uniform stabilizability for discrete-time, LTV systems and the connection with the Kalman filter are standard (Anderson and Moore, 1981): if $(C_a(k), A_a(k))$ is uniformly detectable (UD) then the optimal (Kalman filter) error covariance is bounded; furthermore, if $(A_a(k), G_a(k))$ is uniformly stabilizable (US), the Kalman filter is uniformly exponentially stable (UES). Then one can use the Kalman filter approach to estimate the state of the system Σ_a, and consequently the input $u(k)$ and the inverse Σ_{inv} of the original system Σ. Assume that the augmented system is UD. For each sequence $\Delta(k)$, one can set up a Kalman filter to estimate the state of Σ_a, which implies the estimation of the input signal for Σ:

$\hat{x}_a(k+1|k) = A_a\hat{x}_a(k|k-1) - K_a\left(C_a\hat{x}_a(k|k-1) - y(k)\right)$

where

$K_a := A_aP(k|k-1)C_a^*\left(C_aP(k|k-1)C_a^* + R\right)^{-1}$

and $P(k|k-1)$ satisfies

$P(k+1|k) = A_aP(k|k-1)A_a^*$
$-A_aP(k|k-1)C_a^*\left(C_aP(k|k-1)C_a^* + R\right)^{-1}C_aP(k|k-1)A_a^*$
$+\begin{pmatrix} G_aQG_a^* & 0 \\ 0 & Q_u \end{pmatrix}.$

More precisely, if we define

$$\Sigma_{inv} := \begin{pmatrix} 0 & I \end{pmatrix}\left(zI - (A_a - K_aC_a)\right)^{-1}K_a, \quad (5)$$

an estimation of the input is given by

$$\hat{u}(k) = \Sigma_{inv}y(k).$$

Further, we discuss how one can choose a "good" inverse of the system Σ and have a reasonable estimation of the input. We start with selection procedures for the design parameters $\Delta(k)$. For each time instant k, the mode (discrete parameter) $\Delta(k)$ of the system Σ_a akes three different values. This model incorporates the model presented in (George *et al.*, 1999*b*) for LTI systems.

Consider a given measurement sequence $\{y^d(k)\}_{k=1}^N$, for some number of samples N. A natural strategy for choosing $\Delta(k)$ is the following:

Strategy 4.1.

- For each possible sequence $\{\Delta(k)\}_{k=1}^N$ set a Kalman filter for Σ_a to find the minimum variance state estimation.
- Choose the particular sequence of $\{\Delta(k)\}_{k=1}^N$ corresponding to the smallest error.

Using the Kalman filter corresponding to the particular choice of sequence $\{\Delta(k)\}_{k=1}^N$, one obtains an inverse Σ_{inv} of the original system Σ and an estimation of the input. The number of discrete sequences $\{\Delta(k)\}_{k=1}^N$ to be considered in the first step of the above strategy is exponentially increasing with respect to N. We have to perform off-line a numerical search. For off-line analysis there are numerical approaches based on the EM algorithm or its stochastic version, the MCMC algorithm (see (Gustavson, 2000) and the references therein).

To further improve performance, one may increase the number of possible choices for $\Delta(k)$. As a draw-back, the computational load will be higher. If we restrict only to improving the transient behavior using the multiple model of the input, we consider that $\Delta(k) = 0$ for $k \geq k_1$, where $k_1 < N$. In this case the number of associated Kalman filters is reduced.

5. DETECTABILITY OF THE AUGMENTED SYSTEM

In this section we provide a necessary and sufficient condition for the uniform detectability of an augmented system. Before stating the main results we introduce the notations used throughout this section.

The space of "square summable" sequences is denoted $l_2^r(\mathbb{Z})$. This is a Hilbert space with the usual inner product which generates a finite energy norm. We will restrict to sequences in $l_2^r(Z)$ having finite negative support. The forward bilateral shift operator $Z : l_2^r(\mathbb{Z}) \to l_2^r(\mathbb{Z})$ on sequences $x \in l_2^r(\mathbb{Z})$ is defined by $(Zx)_i := x_{i-1}$. Z is an inner and co-inner operator on $l_2^r(\mathbb{Z})$, and $Z^* = Z^{-1}$. Then, one can express the system (1) as a system of equations in the form (Ball *et al.*, 1992)

$$\Sigma : \begin{cases} x = \mathcal{A}x + \mathcal{B}u + \mathcal{G}w \\ y = \mathbf{C}x + \mathbf{D}u + v \end{cases} . \quad (6)$$

where $\mathcal{A} := Z\mathbf{A}$, $\mathcal{B} := Z\mathbf{B}$ and $\mathcal{G} := Z\mathbf{G}$, with the block diagonal operators formed using the state-space matrix sequences denoted in boldface. We will assume that all these sequences are uniformly bounded. The transfer operator $\mathbf{T}(\lambda) : l_2^m(\mathbb{Z}) \to l_2^p(\mathbb{Z})$, $\lambda \in \mathbb{C}$, associated to the system (1)

$$T(\lambda) := \mathbf{D} + \mathbf{C}(\lambda I - \mathcal{A})^{-1}\mathcal{B}. \quad (7)$$

is well-defined (Kamen *et al.*, 1985).

A complex number λ is an *almost eigenvalue* of an operator T on l_2^r if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with, for each n, $x_n \in l_2^r$ and $\|x_n\|_2 = 1$, such that

$$\|Tx_n - \lambda x_n\|_2 \to 0 \text{ as } n \to \infty.$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an *almost eigenvector* corresponding to λ. Denote by $\sigma_a(T)$ the set of all almost eigenvalues of an operator T, known as the *approximated point spectrum* of T (Beauzamy, 1988).

A spectral test for uniform detectability of discrete-time LTV systems similar to the PBH-test is provided by the following result (Peters and Iglesias, 1999).

Theorem 1. The following statements are equivalent:

(1) The pair $(\mathbf{C}, \mathbf{A})$ is uniformly detectable.
(2) There exist a bounded operator $\mathbf{K}$ such that $\mathbf{A} + \mathbf{KC}$ is UES.
(3) There exists no almost eigenvalue of $\mathcal{A}$ with magnitude greater than or equal to 1 for which the corresponding almost eigenvector x_n satisfies $\|\mathbf{C}x_n\|_2 \to 0$ as $n \to \infty$.

For the definition of the almost eigenvalues and almost eigenvectors of an operator on a Banach spaces we refer to (Beauzamy, 1988). In case the pair $(\mathbf{C}, \mathbf{A})$ is uniformly observable, it is possible to define an operator $\mathbf{K}$ guaranteeing that $\mathbf{A}+\mathbf{KC}$ is UES in terms of the observability Gramian. In the case of uniformly detectability, a Riccati operator equation has to be solved to find a stabilizing feedback. The square root Kalman filter will provide an alternative computational attractive method for finding a stabilizing feedback.

Define the operator

$$\delta := Z\text{diag}\{I + \Delta(k)\}.$$

Assume that Theorem 1 holds. Then, we have the following test for the uniform detectability of the augmented system Σ_a.

Lemma 2. Let δ, Σ and Σ_a be as before. If the following conditions are satisfied

(i) The pair $(\mathbf{C}, \mathbf{A})$ is uniformly detectable.
(ii) $\sigma_a(\delta) \cap \sigma(\mathcal{A}) \cap \{\lambda; |\lambda| \geq 1\} = \emptyset$ (where $\emptyset$ denotes the empty set).
(iii) There exists no almost eigenvalue of $\mathcal{A}_a$ in $\sigma_a(\delta) \cap \rho(\mathcal{A}) \cap \{\lambda; |\lambda| \geq 1\}$ for which the corresponding almost eigenvector x_n satisfies
$$\|\mathbf{T}(\lambda)x_n\|_2 \to 0$$
as $n \to \infty$ ($\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$),

then the pair $(\mathbf{C}_a, \mathbf{A}_a)$ is uniformly detectable.

PROOF. According to Theorem 1, we have to prove that for all almost eigenvectors $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{A}_a$ we have that

$$\mathbf{C}_a x_n \to 0 \Rightarrow x_n \to 0 \text{ (as } n \to \infty), \quad (8)$$

where the convergence is in the sense of the norm on l_2^r. Partition x_n as $\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix}$, corresponding to the partition of $\mathcal{A}_a$. Then,

$$\begin{bmatrix} \mathcal{A} - \lambda I & \mathcal{B} \\ 0 & \delta - \lambda I \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} \to 0 \Rightarrow \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} \to 0 \quad (9)$$

for $\lambda \in \sigma_a(A_a) \cap \{\lambda; |\lambda| \geq 1\}$ the corresponding almost eigenvalue for x_n. From the lower part of (9) it follows that $x_n^2 \to 0$ or $\lambda \in \sigma_a(\delta) \cap \{\lambda; |\lambda| \geq 1\}$ and $\xi_n := x_n^2/\|x_n^2\|$ is a corresponding almost eigenvector (remark that here we shall consider a coercive subsequence of x_n^2 for which we use the same notation).

Case I: If $x_n^2 \to 0$, from the upper part of (9), we have that $(\mathcal{A}-\lambda I)x_n^1 \to 0$ (as $n \to \infty$). If $x_n^1 \to 0$ then (8) is satisfied. When x_n^1 does not converge to zero, one can construct $x_n^1/\|x_n^1\|$ an almost eigenvector of $\mathcal{A}$ corresponding to λ (a coercive subsequence of x_n^1 should be considered). Since $(\mathbf{C}, \mathbf{A})$ is UD (see assumption (i)), it follows that $\mathbf{C}x_n^1 \to 0 \Rightarrow x_n^1 \to$ (as $n \to \infty$). Consequently, in this case (8) is satisfied.

Case II: Assume now that $\lambda \in \sigma_a(\delta) \cap \{\lambda; |\lambda| \geq 1\}$ and ξ_n is a corresponding almost eigenvector. Since (ii) is satisfied, we have that $\lambda \in \sigma_a(\delta) \cap \rho(\mathcal{A}) \cap \{\lambda; |\lambda| \geq 1\}$. The top row in (9) reads

$$\begin{bmatrix} \mathcal{A} - \lambda I & \mathcal{B} \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} \to 0.$$

This is equivalent to

$$x_n^1 - (\lambda I - \mathcal{A})^{-1} B x_n^2 \to 0. \tag{10}$$

for all almost eigenvectors $(x_n)_{n\in\mathbb{N}}$ corresponding to λ.

$$\begin{aligned} \mathbf{C}_a x_n &= \begin{bmatrix} \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \mathbf{C}x_n^1 + \mathbf{D}x_n^2 \\ &= \mathbf{C}(x_n^1 - (\lambda I - \mathcal{A})^{-1} B x_n^2) + \mathbf{T}(\lambda) x_n^2 \end{aligned}$$

Using (10) and assumption (iii), we obtain that (8) holds, and the proof is completed.

For LTI systems the above lemma would imply the following result.

Lemma 3. Let $\delta = I$, Σ and Σ_a be as before but time invariant. If the following conditions are satisfied

- (i) The pair $(\mathbf{C}, \mathbf{A})$ is detectable,
- (ii) $1 \notin \sigma(\mathcal{A})$,
- (iii) The system Σ has no transmission zeros at $\lambda = 1$, $(T(1) \neq 0)$,

then the pair $(\mathbf{C}_a, \mathbf{A}_a)$ is detectable.

Define the operator

$$N(\lambda) := \begin{bmatrix} \mathcal{A} - \lambda I & \mathcal{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}.$$

The following refinement of the result from Lemma 2 can be proved.

Lemma 4. Let δ, Σ and Σ_a be as before and assume that the pair $(\mathbf{C}, \mathbf{A})$ is uniformly detectable. Then, the following two conditions are equivalent.

- (1) The pair $(\mathbf{C}_a, \mathbf{A}_a)$ is uniformly detectable.
- (2) There exists no almost eigenvalue of $\mathcal{A}_a$ in $\sigma_a(\delta) \cap \{\lambda; |\lambda| \geq 1\}$ for which the corresponding almost eigenvectors x_n satisfy $\|N(\lambda)x_n\|_2 \to 0$ as $n \to \infty$.

For LTI systems, the above lemma becomes.

Lemma 5. Let $\delta = I$, Σ and Σ_a be time-invariant, and assume that the pair $(\mathbf{C}, \mathbf{A})$ is detectable. Then, the following two conditions are equivalent.

- (1) The pair $(\mathbf{C}_a, \mathbf{A}_a)$ is detectable.
- (2) The system Σ has no left-invariant zeroes at $\lambda = 1$, i.e., $\mathrm{rank}(N(1)) = r + m$ (where r and m are the dimensions of the state and the input, respectively).

The approximated point spectrum is invariant under rotations. This implies that we need only to consider the almost eigenvalues which are real and have modulus greater then one.

It can be proved that if conditions (i) and (ii) from Lemma 2 are satisfied then condition (2) of Lemma 4 implies condition (iii) in Lemma 2. For LTI systems this corresponds to the fact that the set of invariant zeros contains the complete set of transmission zeroes and some, but not necessarily all, of the decoupling zeros. Similar results providing the preservation of the uniform observability can be proved using the same techniques. Lemma 5 is an extension of a similar result stated in (George *et al.*, 1999*b*) which holds only if the original system has no poles ar $\lambda = 1$.

If the LTI system is controllable and observable, then the invariant zeros are the same with the transmission zeros. It would be interesting to see what will be the equivalent version for LTV systems. We leave this as an open question.

Answers for many questions regarding the spectrum and the approximate point spectrum for weighted shifts operators are provided in (Ben-Artzi and Gohberg, 1991; Shields, 1974). To give an idea of the analysis performed in the above mentioned papers, we conclude this section with some comments on the SISO LTV model of the input. We pay attention to the homogeneous system $u(k+1) = \alpha(k)u(k)$ where $\alpha(k) := 1 + \Delta(k)$. By construction, $\alpha(k)$ is invertible for all $k \in \mathbb{N}$, and $\sup\{|\alpha(k)|, |\alpha(k)^{-1}|\} < \infty$. We consider the following two real numbers

$$\kappa_+ = \lim_{l\to\infty} \sup_{k\in\mathbb{N}} \| \prod_{i=0}^{l-1} \alpha(k+i) \|^{1/l}$$

$$\kappa_- = \lim_{l\to\infty} \inf_{k\in\mathbb{N}} \mu(\| \prod_{i=0}^{l-1} \alpha(k+i) \|)^{1/l},$$

where $\mu(T)$ is the smallest singular value of T. Then the spectrum of the one-dimensional weighted shift δ is given by the annulus

$$\sigma(\delta) = \{\lambda \in \mathbb{C}; \kappa_- \leq |\lambda| \leq \kappa_+\}. \tag{11}$$

For particular cases of the sequence $\{\alpha(k)\}_{k\in\mathbb{N}}$ one can compute κ_- and κ_+, see for example (Halmos, 1967, Problem 77).

6. CONCLUSIONS AND FUTURE WORKS

6.1 Conclusions

In this note, we considered the problem of estimating an unknown input for discrete-time, non-minimum phase, multivariable, LTV systems. The input signal is modelled as a random walk with drifts. From the original systems and the models of the input an augmented system is formed. If the augmented systems is uniformly detectable, the an estimation of the unknown input is obtained choosing from a family of Kalman filters the one which gives the minimal covariance error. A necessary and sufficient condition for uniform detectability of the augmented system is provided. One consequence of this approach, the transient behavior in estimating unknown inputs will be improved even for LTI systems. A Kalman filter-based stable dynamic inversion for LTV systems is also obtained as a consequence of our procedure. A Matlab toolbox for SDI for LTI systems has successfully been used in applications, see for example (Cuyper and Verhaegen, 2002).

6.2 Future Works

A class of LTV systems which we will consider further will be linear, discrete-time, time-varying systems for which the state-space becomes LTI for $k \to \infty$. For this systems one can perform an inner-outer factorization to extract the system zeros which are in, or very closed to, the additional spectrum introduced by the augmentation procedure. Simulations for particular examples will be also performed by the authors. It is desirable to extend the inversion Matlab toolbox for particular classes of LTV systems.

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