NASH STRATEGY APPLIED TO ACTIVE MAGNETIC BEARING CONTROL

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Abstract: In this paper, Nash strategy is applied for controlling an active magnetic bearing. Different criteria are associated to each input of the dynamical system. The Nash controls are associated with Coupled Algebraic Riccati Equations. A state feedback from Nash strategy is designed based on a linearized system. A comparison of Nash and LQ control, applied to the non-linear system, is proposed in the case of exact and perturbed parameters knowledge.

Keywords: Active magnetic bearing, magnetic levitation, games theory, Nash strategy, multivariable system, coupled algebraic Riccati equations.

1. INTRODUCTION

Magnetic bearings are an elegant solution to avoid mechanical friction and the use of lubrification for suspending high-speed rotors. There are two kinds of magnetic bearings, a passive one with permanent magnets and an active one with electromagnets. The impossibility of controlling permanent magnets leads to the use of active magnetic bearings to stabilize the position of the rotor. In addition, for the permanent magnet system (without rotation) the Earnshaw theorem proves that stable and complete levitation cannot be achieved by using only permanents magnets (Matsumura and Yoshimoto, 1986).

The great number of applications of active magnetic bearing explains the diversity of (linear or non-linear) modeling and control approaches: backstepping (de Queiroz and Dawson, 1996), flatness (Lévine *et al.*, 1996; Ponsart, 1996), Pulse-Width-Modulating (Hall, 1990), non-linear bifurcation (Mohamed and Emad, 1993), sliding mode controller (Rundell *et al.*, 1996; Cho *et al.*, 1993) and LQ-control (Zhuravlyov, 2000). Some of them need an inboard computer like backstepping, flatness and PWM, or use only one criterion like LQ-control. Here a game theoretic approach with Nash strategy is proposed. A state feedback is designed on a tangent-linearized model and a criterion is defined for each input. Such control is then applied to the non-linear model near an equilibrium point.

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2. MAGNETIC SYSTEM MODEL

In this paper, an active magnetic bearing stabilizing problem in two directions depicted in Fig. ?? is considered. This system is composed by a planar rotor disk and two sets of stator electromagnets (the first set acts on the y-direction and the second on the x-direction).

For each direction, there is a pair of electromagnets, because an electromagnet can only attract the rotor and cannot repulse it. The rotor is positioned according to the magnetic forces F_1, F_2, F_3 and F_4 generated by the stator electromagnetic circuits. These forces are produced by the currents i_1, i_2, i_3 and i_4 in each stator coil and these currents depend on the voltages e_1, e_2, e_3 and e_4 applied to each stator. The magnetic circuits have nonlinear inductances L_1, L_2, L_3 and L_4 (because of variables air-gaps) and back-electromotive forces B_1, B_2, B_3 and B_4.

The inputs to the magnetic bearing system are the voltages e_1, e_2, e_3 and e_4. The measurable signals are the rotor y and x positions, the rotor $\dot{y}$ and $\dot{x}$ velocities and the currents i_1, i_2, i_3 and i_4. This system may be separated in a mechanical subsystem and an electrical subsystem.

Along the y-direction, the mechanical subsystem is given by

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = F_1(y,i_1) + F_2(y,i_2)$$

where m represents the mass of the rotor, while the electrical subsystem is given by

$$L_1(y,i_1)\frac{\mathrm{d}i_1}{\mathrm{d}t} + R_1 i_1 + B_1(y,i_1)\frac{\mathrm{d}y}{\mathrm{d}t} = e_1$$

$$L_2(y,i_2)\frac{\mathrm{d}i_2}{\mathrm{d}t} + R_2 i_2 + B_2(y,i_2)\frac{\mathrm{d}y}{\mathrm{d}t} = e_2$$

where R_1 and R_2 are the resistances in the first set of stator electromagnets.

Assuming that the magnetic circuit is linear (there is no magnetic saturation) and neglecting fringing and leakage, the magnetic fluxes Φ_1 and Φ_2 in the air-gaps are given by

$$\Phi_1(y,i_1) = \frac{L_0 i_1}{k-2y}$$

$$\Phi_2(y,i_2) = \frac{L_0 i_2}{k+2y}$$

where L_0 and k are positive constants depending on the system construction. This model is only admissible if the rotor is not in contact with an electromagnet, that is to say

$$-k < -g_0 < y < g_0 < k$$

where g_0 is the value of an air-gap when the rotor verifies $y = 0$. Note that $k > g_0$ because we assume that the permeability in electromagnets is finite.

The forces F_1 and F_2 are given by calculating virtual work

$$F_1(y,i_1) = \frac{\partial}{\partial y}\int_0^{i_1}\Phi_1(y,I)\mathrm{d}I = L_0\left(\frac{i_1}{k-2y}\right)^2$$

$$F_2(y,i_2) = \frac{\partial}{\partial y}\int_0^{i_2}\Phi_2(y,I)\mathrm{d}I = -L_0\left(\frac{i_2}{k+2y}\right)^2$$

The inductances are defined by

$$L_1(y,i_1) = \frac{\partial\Phi_1(y,i_1)}{\partial i_1} = \frac{L_0}{k-2y}$$

$$L_2(y,i_2) = \frac{\partial\Phi_2(y,i_2)}{\partial i_2} = \frac{L_0}{k+2y}$$

and the back-electromotive forces by

$$B_1(y,i_1) = \frac{\partial\Phi_1(y,i_1)}{\partial y} = \frac{2L_0 i_1}{(k-2y)^2}$$

$$B_2(y,i_2) = \frac{\partial\Phi_2(y,i_2)}{\partial y} = \frac{-2L_0 i_2}{(k+2y)^2}$$

Combining the equations above, the system in y-direction is denoted by

$$\left\{\begin{array}{lcl}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &=& \frac{L_0}{m}\left(\frac{i_1}{k-2y}\right)^2 - \frac{L_0}{m}\left(\frac{i_2}{k+2y}\right)^2\\ \frac{\mathrm{d}i_1}{\mathrm{d}t} &=& \left(\frac{k-2y}{L_0}\right)(e_1 - R_1 i_1) - 2\left(\frac{i_1}{k-2y}\right)\frac{\mathrm{d}y}{\mathrm{d}t}\\ \frac{\mathrm{d}i_2}{\mathrm{d}t} &=& \left(\frac{k+2y}{L_0}\right)(e_2 - R_2 i_2) + 2\left(\frac{i_2}{k+2y}\right)\frac{\mathrm{d}y}{\mathrm{d}t}\end{array}\right. \quad (1)$$

Similarly, along the x-direction, the mechanical subsystem is given by

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = F_3(x,i_3) + F_4(x,i_4)$$

while the electrical subsystem is given by

$$L_3(x,i_3)\frac{\mathrm{d}i_3}{\mathrm{d}t} + R_3 i_3 + B_3(x,i_3)\frac{\mathrm{d}x}{\mathrm{d}t} = e_3$$

$$L_4(x,i_4)\frac{\mathrm{d}i_4}{\mathrm{d}t} + R_4 i_4 + B_4(x,i_4)\frac{\mathrm{d}x}{\mathrm{d}t} = e_4$$

where R_3 and R_4 are the resistances in the second set of stator electromagnets.

Making the same assumptions as for the y-direction, the system in x-direction is denoted by

$$
\begin{cases}
\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = \dfrac{L_0}{m}\left(\dfrac{i_3}{k-2x}\right)^2 - \dfrac{L_0}{m}\left(\dfrac{i_4}{k+2x}\right)^2 \\
\dfrac{\mathrm{d}i_3}{\mathrm{d}t} = \left(\dfrac{k-2x}{L_0}\right)(e_3 - R_3 i_3) - 2\left(\dfrac{i_3}{k-2x}\right)\dfrac{\mathrm{d}x}{\mathrm{d}t} \\
\dfrac{\mathrm{d}i_4}{\mathrm{d}t} = \left(\dfrac{k+2x}{L_0}\right)(e_4 - R_4 i_4) + 2\left(\dfrac{i_4}{k+2x}\right)\dfrac{\mathrm{d}x}{\mathrm{d}t}
\end{cases}
\quad (2)
$$

The nominal values of the system parameters are given in Tab 1.

Parameter	Nominal Value
m	2 kg
k	0.0020125 m
L_0	0.0003 H.m
R_1	1 Ω
R_2	1 Ω
R_3	1 Ω
R_4	1 Ω

Table 1. System parameters

2.1 Linearized model

The two sets (1) and (2) of physical equations are independant, even if they have the same structure. The state vectors are respectively

$$
z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \\ i_1 \\ i_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_5 \\ z_6 \\ z_7 \\ z_8 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ i_3 \\ i_4 \end{pmatrix}
$$

for the y-direction and x-direction. The inputs are voltages e_1, e_2, e_3 and e_4. In the following, only the system on the y-direction is considered. The non-linear (affine in the inputs) model becomes from (1):

$$
\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \left(\dfrac{k-2z_1}{L_0}\right) & 0 \\ 0 & \left(\dfrac{k+2z_1}{L_0}\right) \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}
+ \begin{pmatrix} z_2 \\ \dfrac{L_0}{m}\left(\dfrac{z_3}{k-2z_1}\right)^2 - \dfrac{L_0}{m}\left(\dfrac{z_4}{k+2z_1}\right)^2 \\ -\left(\dfrac{k-2z_1}{L_0}\right) R_1 z_3 - 2\left(\dfrac{z_3}{k-2z_1}\right) z_2 \\ -\left(\dfrac{k-2z_1}{L_0}\right) R_2 z_4 - 2\left(\dfrac{z_4}{k-2z_1}\right) z_2 \end{pmatrix}
\quad (3)
$$

It was shown in (Ponsart, 1996) that this system was controllable. In this paper we consider only the linearized model near an equilibrium point.

The equilibrium points for the system (3) are[2]

[2] There are two other solutions mathematically possible for z_{10}, but they are not physically possible.

$$
\begin{cases}
z_{10} = \dfrac{k}{2}\dfrac{R_1 e_{20} - R_2 e_{10}}{R_1 e_{20} + R_2 e_{10}} \\
z_{20} = 0 \\
z_{30} = \dfrac{e_{10}}{R_1} \\
z_{40} = \dfrac{e_{20}}{R_1}
\end{cases}
$$

Stabilizing the rotor near the position $y = 0$ imposes the equilibrium point

$$
\begin{cases}
z_{10} = z_{20} = 0 \\
z_{30} = z_{40} = I_0
\end{cases}
$$

With the notations $z_i = z_{i0} + \tilde{z}_i$ and $e_j = e_{j0} + \tilde{e}_j$ ($1 \le i \le 4$ and $1 \le j \le 2$), the tangent linearization of (3) is

$$
\frac{\mathrm{d}\tilde{z}}{\mathrm{d}t} = A\tilde{z} + B_1\tilde{e}_1 + B_2\tilde{e}_2 \quad (4)
$$

where:

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{8I_0^2L_0}{mk^3} & 0 & \dfrac{2I_0^2L_0}{mk^2} & -\dfrac{2I_0^2L_0}{mk^2} \\ 0 & -\dfrac{2I_0}{k} & -\dfrac{kR_1}{L_0} & 0 \\ 0 & \dfrac{2I_0}{k} & 0 & -\dfrac{kR_2}{L_0} \end{bmatrix}
$$

and

$$
B_1 = \begin{pmatrix} 0 \\ 0 \\ \dfrac{k}{L_0} \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{k}{L_0} \end{pmatrix}, \quad \tilde{z} = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \\ \tilde{z}_4 \end{pmatrix}
$$

The system (4) is controllable (the pairs (A, B_1) and (A, B_2) are controllable) if and only if the current $I_0 \neq 0$. It's the premagnetization current.

3. A GAME THEORETIC APPROACH

3.1 Expressions of criteria

Each input $\tilde{e}_1$ and $\tilde{e}_2$ must optimize its own objective, which is translated in a problem of minimization of a criterion (respectively J_1 and J_2). Here the criteria J_1 and J_2 can be writen as

$$
\begin{cases}
J_1(\tilde{e}_1, \tilde{e}_2) = \dfrac{1}{2}\displaystyle\int_{t_0}^{+\infty}\left(\tilde{z}^T Q_1 \tilde{z} + \dfrac{\tilde{e}_1^2}{R_1}\right)\mathrm{d}t \\
J_2(\tilde{e}_1, \tilde{e}_2) = \dfrac{1}{2}\displaystyle\int_{t_0}^{+\infty}\left(\tilde{z}^T Q_2 \tilde{z} + \dfrac{\tilde{e}_2^2}{R_2}\right)\mathrm{d}t
\end{cases}
\quad (5)
$$

where

$$
Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & L_1(z_{10}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_2(z_{10}) \end{bmatrix}$$

The criteria (5) includes the kinetic energy of the rotor $\frac{1}{2}m\dot{y}^2$, the electromagnetic energy in the air gap $\frac{1}{2}L_j(z_{10})\tilde{i}_j^2$ and the Joules effect energy $\tilde{e}_j^2/R_j$ near the equilibrium point $(1 \leq j \leq 2)$. For $z_{10} = 0$, we have the relations

$$L_1(z_{10}) = L_2(z_{10}) = \frac{L_0}{k}$$

Note that the pairs $\left(\sqrt{Q_1}, A\right)$ and $\left(\sqrt{Q_2}, A\right)$ are observable. The aim here is to minimize the control energy.

3.2 Nash Strategy

The problem formulation is symmetric in the inputs. The voltages $\tilde{e}_1$ and $\tilde{e}_2$ have the same hierarchical level. To find a compromise or an equilibrium, the Nash strategy is chosen. The pair $(\tilde{e}_1^*, \tilde{e}_2^*)$ corresponds to a Nash equilibrium (Başar and Olsder, 1982; Ho, 1970; Ho *et al.*, 1965) if the relations for each admissible voltages $(\tilde{e}_1, \tilde{e}_2)$ are verified:

$$\begin{cases} J_1\left(\tilde{e}_1^*, \tilde{e}_2^*\right) \leq J_1\left(\tilde{e}_1, \tilde{e}_2^*\right) \\ J_2\left(\tilde{e}_1^*, \tilde{e}_2^*\right) \leq J_2\left(\tilde{e}_1^*, \tilde{e}_2\right) \end{cases} \tag{6}$$

At Nash equilibrium $(\tilde{e}_1^*, \tilde{e}_2^*)$, the controller j increases his criteria J_j, if he decides not to use $\tilde{e}_j^*$, without cooperation with the other controller.

The necessary conditions for a Nash equilibrium for criteria J_1 and J_2 (5) with the dynamical constraint (4) are

$$\begin{cases} \dfrac{\partial H_j}{\partial \tilde{e}_j} = 0 \\ \dot{\psi}_j = -\dfrac{\partial H_j}{\partial \tilde{z}} \end{cases}, \quad 1 \leq j \leq 2 \tag{7}$$

where ψ_j is the costate vector associed with the dynamical constraint (4), and H_j the Hamiltonian

$$H_j = \frac{1}{2}\left(\tilde{z}^T Q_j \tilde{z} + \frac{\tilde{e}_j^2}{R_j}\right) + \psi_j^T\left(A\tilde{z} + B_1\tilde{e}_1 + B_2\tilde{e}_2\right)$$

For this system, (7) leads us to

$$\begin{cases} \tilde{e}_1^* = -R_1 B_1^T \psi_1 \\ \tilde{e}_2^* = -R_2 B_2^T \psi_2 \end{cases}$$

and

$$\begin{cases} \dot{\tilde{z}} = A\tilde{z} - S_1\psi_1 - S_2\psi_2 \\ \dot{\psi}_1 = -Q_1\tilde{z} - A^T\psi_1 \\ \dot{\psi}_2 = -Q_2\tilde{z} - A^T\psi_2 \end{cases}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \tilde{z} \\ \psi_1 \\ \psi_2 \end{pmatrix} = N \begin{pmatrix} \tilde{z} \\ \psi_1 \\ \psi_2 \end{pmatrix} \tag{8}$$

with

$$N = \begin{bmatrix} A & -S_1 & -S_2 \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{bmatrix} \tag{9}$$

where $S_j = R_j B_j B_j^T$ $(1 \leq j \leq 2)$.

Solutions of (8) are looked for, in particular forms, if assuming the existence of K_1 and K_2 defined by $\psi_1 = K_1\tilde{z}$ and $\psi_2 = K_2\tilde{z}$. In the case of criteria with infinite horizon (5), K_1 and K_2 verify the Coupled Algebraic Riccati Equations:

$$\begin{matrix} A^T K_1 + K_1 A + Q_1 - K_1 S_1 K_1 - K_1 S_2 K_2 = 0 \\ A^T K_2 + K_2 A + Q_2 - K_2 S_1 K_1 - K_2 S_2 K_2 = 0 \end{matrix} \tag{10}$$

With these commands, the closed loop becomes

$$\frac{\mathrm{d}\tilde{z}_c}{\mathrm{d}t} = \left(A - S_1 K_1 - S_2 K_2\right)\tilde{z}_c \tag{11}$$

The eigenvalues of $(A - S_1K_1 - S_2K_2)$ are included in the set of eigenvalues of N (8). There can exist different solutions (K_1, K_2) for the equations (9). All these solutions can be determined by the invariants spaces of N (Abou-Kandil *et al.*, 2003). When it's possible the dichotomical solution (the solution which leads to the fastest dynamic $A - S_1K_1 - S_2K_2$) is selected. For the linearized system (4) the controls are:

$$\begin{cases} \tilde{e}_1^* = -R_1 B_1^T K_1 \tilde{z}_c \\ \tilde{e}_2^* = -R_2 B_2^T K_2 \tilde{z}_c \end{cases} \tag{12}$$

where $\tilde{z}_c$ is given by (10). Next the voltages $\tilde{e}_1^*$ and $\tilde{e}_2^*$ (11) are applied to the non-linear system (3). The next section presents some numerical simulations.

4. NUMERICAL RESULTS

First of all, to compare the Nash control, LQ control is taken as reference control on the linearized system

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}t} = A\tilde{z} + \begin{bmatrix} B_1 & B_2 \end{bmatrix}\begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} \tag{13}$$

and the criterion's weighting matrices are:

$$Q = Q_1 + Q_2, \quad R = \begin{bmatrix} \dfrac{1}{R_1} & 0 \\ 0 & \dfrac{1}{R_2} \end{bmatrix} \tag{14}$$

H denotes the Hamiltonian matrix associated with the LQ control resolution:

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \quad (15)$$

The fastest admissible dynamic (10) -called the dichotomical solution (Jungers and Abou-Kandil, 2004)- is selected to determine K_1 and K_2 (9) and naturally the stable solution of LQ control (Fig 1).

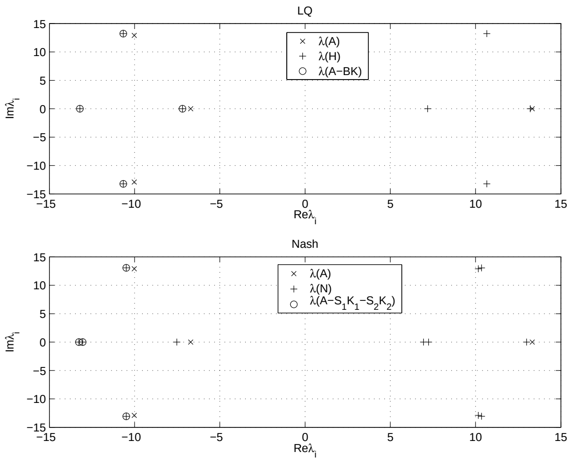

Fig. 1. Distribution of selected eigenvalues for LQ and Nash control

The simulation shows the regulation of the position of the rotor's center, with an initial position perturbation set to $y(0) = 0.45g_0$. First, the simulation was performed assuming exact knowledge of the parameter values of the system (1).

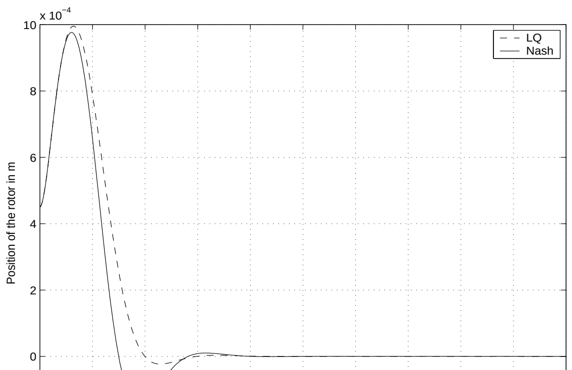

Fig. 2. Position of rotor for LQ and Nash regulation with exact parameter knowledge and an initial position error $y(0) = 0.45g_0$.

Notice in the figure (Fig 2) that LQ and Nash control lead to almost the same response. Moreover the position stays in the air-gap ($-g_0 \leq y(t) \leq g_0$), so these two solutions are admissible for this initial position error. LQ and Nash controls seem to lead to equivalent solutions. We propose now to study these LQ and Nash controls on the system with perturbed parameters. The next simulation consider a pertubation of $+20\%$ on parameters of (1), except for g_0.

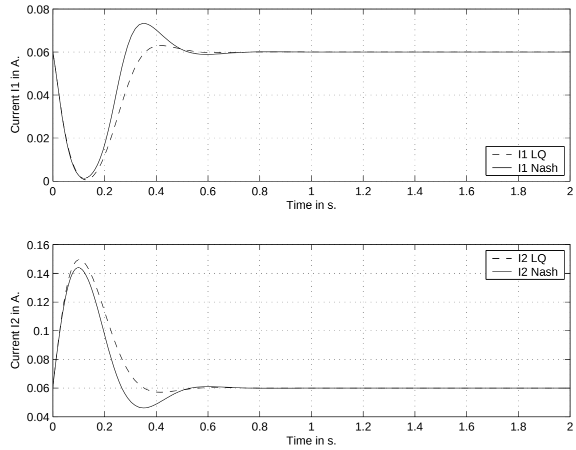

Fig. 3. Currents I_1 and I_2 for LQ and Nash regulation with exact parameter knowledge and an initial position error $y(0) = 0.45g_0$.

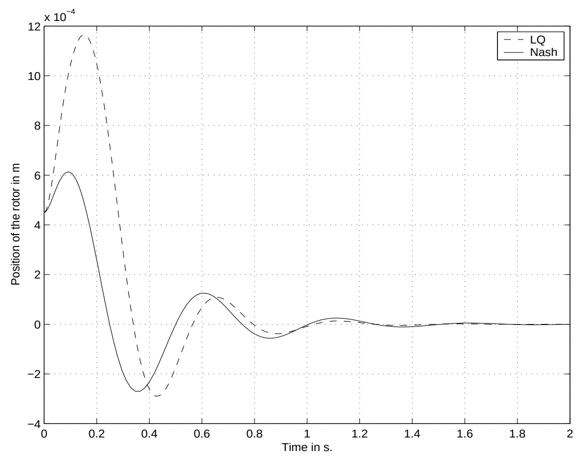

Fig. 4. Position of rotor for LQ and Nash regulation with perturbed parameter knowledge and an initial position error $y(0) = 0.45g_0$.

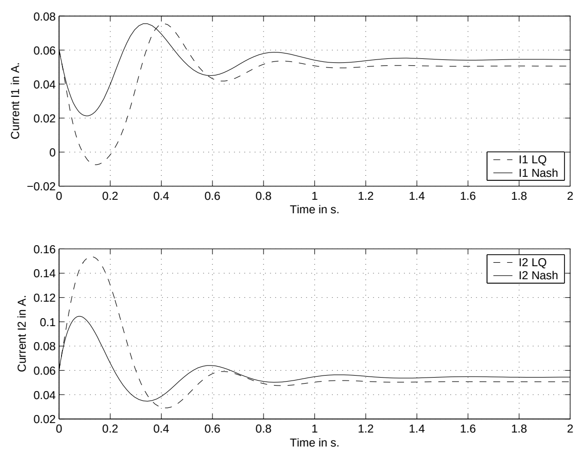

Fig. 5. Currents I_1 and I_2 for LQ and Nash regulation with perturbed parameter knowledge and an initial position error $y(0) = 0.45g_0$.

Notice in (Fig 4) that with this initial error of position and this perturbation of 20% of the parameters, the LQ control is not physically admissible, because the position $y(t)$ is not always in the air-gap $[-g_0; g_0]$. Inversely the Nash control is physically admissible, furthermore with smaller currents (Fig 5) than the LQ control. In this case, for the considered perturbation of parameters, the Nash control is more robust than the LQ one. Nevertheless in our actual knowledge, there are no general results on the robustness of Nash control.

Finally we present a map of initial error of position and velocity corresponding of an acceptable trajectory for Nash control, assuming that the currents I_1 and I_2 are equilibrium currents I_0 (Fig 6).

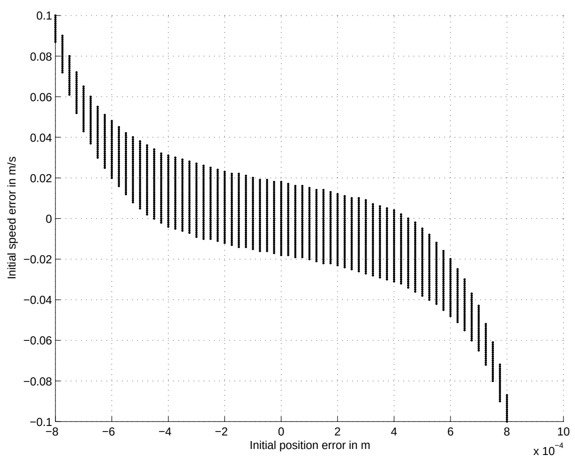

Fig. 6. Map of initial error of position and velocity for an acceptable trajectory for Nash control for non perturbed system

According to this figure, when there is a big initial position error, the control is admissible only with an opposite velocity error.

5. CONCLUSION

In this paper, a game theoretic approach using Nash strategy is proposed to control an active magnetic bearing. Multiple inputs on the system are considered, with separate criterion for each input. A state feedback, designed by the resolution of Coupled Algebraic Riccati Equations, associated with Nash strategy is compared with a LQ control. For exact parameters knowledge, the simulation results seem equivalent, but for perturbed parameters, the control derived from Nash strategy is better than LQ control. In future work, the trade off between different solutions of Coupled Algebraic Riccati Equations and the robustness will be considered.

REFERENCES

Abou-Kandil, H., G. Freiling, V. Ionescu and G. Jank (2003). *Matrix Riccati Equations in Control and Systems Theory*. Birkhäuser.

Başar, T. and G. J. Olsder (1982). *Dynamic Noncooperative Game Theory*. Academic Press.

Cho, D., Y. Kato and D. Spilman (1993). Sliding mode and classical controllers in magnetic levitation systems. *IEEE Control Systems* **13**(1), 42–48.

de Queiroz, M. S. and D. M. Dawson (1996). Nonlinear control of active magnetic bearings: a backstepping approach. *IEEE Transactions on Control Systems Technology* **4**(5), 545–552.

Hall, S. A. (1990). Pulse-width-modulating control of a nonlinear electromagnetic actuator. *IEEE Transactions on Industrial Electronics* **37**(3), 241–249.

Ho, Y. C. (1970). Survey paper: Differential games, dynamic optimization and generalized control theory. *Journal of Optimization Theory and Applications* **6**(3), 179–209.

Ho, Y. C., A. E. Bryson and S. Baron (1965). Differential games and optimal pursuit-evasion strategies. *IEEE Transactions on Automatic Control* **AC-10**(4), 385–389.

Jungers, M. and H. Abou-Kandil (2004). Robustesse de différentes commandes issues de la stratégie de Nash. In: *CIFA'04*. Douz Tunisie.

Lévine, J., J. Lottin and J.-C. Ponsart (1996). A nonlinear approach to the control of magnetic bearings. *IEEE Transactions on Control Systems Technology* **4**(5), 524–544.

Matsumura, F. and T. Yoshimoto (1986). System modeling and control design of a horizontal-shaft magnetic-bearing system. *IEEE Transactions on Magnetics* **Mag 22**(3), 196–203.

Mohamed, A. M. and F. P. Emad (1993). Nonlinear oscillations in magnetic bearing systems. *IEEE Transactions on Automatic Control* **38**(8), 1242–1245.

Ponsart, J.-C. (1996). Asservissements numériques de paliers magnétiques. Application aux pompes à vide. PhD thesis. Université de Savoie, France.

Rundell, A. E., S. V. Drakunov and R. A. DeCarlo (1996). A sliding mode observer and controller for stabilization of rotational motion of a vertical shaft magnetic bearing. *IEEE Transactions on Control Systems Technology* **4**(5), 598–608.

Zhuravlyov, Y. N. (2000). On lq-control of magnetic bearing. *IEEE Transactions on Control Systems Technology* **8**(2), 344–350.