MODELING CONTINUOUS-TIME STOCHASTIC PROCESSES USING INPUT-TO-STATE FILTERS [1]

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Abstract: A novel direct approach for modeling continuous-time stochastic processes is proposed in this paper. First the observed data is passed through an input-to-state filter and the covariance of the output state is computed. The properties of the state covariance matrix is then exploited to estimate the positive real spectrum of the observed data at a set of prescribed points on the right half plane. Finally, the continuous-time parameters are obtained from the positive real spectrum estimates by solving a Nevanlinna-Pick interpolation problem. The estimated model is stable. The analytical results are illustrated using numerical simulations.

Keywords: Positive real spectrum, continuous-time, stochastic process, Nevanlinna-Pick interpolation.

1. BACKGROUND

Modeling of continuous-time stochastic processes is a fundamental research question which has received considerable interest recently. Although the signal is continuous-time, in practice one works with sampled data. One popular approach is to identify a discrete-time system from uniformly sampled data, see (Larsson, 2004) and references therein. Subsequently, the estimated discrete-time model is converted back to a corresponding continuous-time model by a nonlinear transformation. Apart from the obvious difficulty of solving nonlinear equations, this approach also suffers from three problems: (i) It does not guarantee stability of the estimated model; (ii) At fast sampling rate the associated discrete-time system poles and zeros cluster close to unity in the complex plane, leading to numerically ill-conditioned identification problem; (iii) The continuous-time parameters can be very sensitive to the sampled data. The second approach is to identify the continuous-time parameter *directly*. This approach is advantageous in many cases since one can avoid nonlinear transformations and may benefit from non-uniform sampling (Larsson, 2004). But unlike the discrete-time counterpart, the mapping from the lagged covariance estimates to the system parameters for continuous-time systems are more complicated. Hence standard discrete-time algorithms cannot be extended directly. It is also difficult to ensure stability of the estimated model.

In this paper, we propose a novel approach where we first estimate the positive real spectrum and its derivatives evaluated at some prespecified points in the right half plane. This is achieved via a linear operation on the covariance matrix of the output of an input-to-state filter. Subsequently, we present a few possibilities of estimating stable rational models from the estimates of the positive real spectrum. In this step, we need to solve a Nevanlinna-Pick interpolation problem using either the classical Nevanlinna's algorithm (Delsarte *et al.*, 1982), or a recent algorithm proposed in (Byrnes *et al.*, 2001). The algorithm proposed in (Byrnes *et al.*, 2001) proves to be more useful, because the user enjoys the freedom to *place* the spectral zeros.

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2. INPUT-TO-STATE FILTERS

The discussion in this section is applicable to any continuous-time wide sense stationary stochastic process $u(t)$. As a special application, we shall apply the results derived in this section to continuous-time ARMA and AR processes in later sections. Suppose the real-valued[1] continuous-time stochastic process $u(t)$ has an autocorrelation function

$$r_\tau := \mathcal{E}\{u(t+\tau)u(t)\}.$$

Then, the spectrum of the process is defined as

$$\phi(s) := \int_{-\infty}^{\infty} \mathrm{d}\tau\, r_\tau \mathrm{e}^{-s\tau}, \qquad s = \mathrm{i}\omega, \omega \in \mathbb{R}.$$

In this work we shall use the so-called *positive real spectrum* $f(s)$ of $u(t)$, which is defined as

$$f(s) := \int_{0}^{\infty} \mathrm{d}\tau\, r_\tau \mathrm{e}^{-s\tau}, \qquad \mathrm{Re}(s) \geq 0.$$

Consequently, it is readily verified that

$$\phi(s) = f(s) + f(-s), \qquad s = \mathrm{i}\omega, \omega \in \mathbb{R}. \quad (1)$$

In the following we shall be concerned with estimating $f(s)$ and its derivatives at a predefined set of points $\{s_k\}_{k=1}^m$ from the observed continuous-time signal $u(t)$. The points $\{s_k\}_{k=1}^m$ are chosen such that $\mathrm{Re}(s_k) > 0$, $\forall k$. The main idea here is to use an input-to-state filter. Consider the input-to-state filter

$$\dot{z}(t) = Fz(t) + gu(t), \quad (2)$$

where F has eigenvalues at $\{-s_k\}_{k=1}^m$ and the pair (F, g) is controllable. We shall assume that the filter in (2) has a pole of order n_k at $-s_k$, while the order of the filter is n, *i.e.* F is a $n \times n$ matrix, and $\sum_{k=1}^m n_k = n$. In what follows next, we shall show that the covariance matrix of the output $z(t)$ can be used to extract the estimates of $f(s_k)$ and its derivatives. In particular, a pole of order n_k at $-s_k$ enables us to extract the derivatives of $f(s)$ up to order $n_k - 1$ evaluated at s_k. The following proposition is the first step in that direction.

Proposition 1. Assume that $f(\infty) < \infty$. Let E be the unique positive definite solution to the Lyapunov equation

$$FE + EF' + gg' = 0. \quad (3)$$

Then there exist scalar-valued functions $\{w_k\}_{k=0}^{n-1}$ of F and $f(s)$ such that

$$P := \mathcal{E}\{z(t)z'(t)\} = WE + EW', \quad (4)$$

where

$$W = \sum_{k=0}^{n-1} w_k F^k. \quad (5)$$

[1] However, the results in this section can be generalized for a complex-valued stochastic process in a fairly straightforward manner.

Proof: First note from (3) that

$$\begin{aligned}&[sI-F]^{-1}gg'[-sI-F']^{-1}\\&=[sI-F]^{-1}\{FE+EF'\}[sI+F']^{-1}\\&=[sI-F]^{-1}E-E[sI+F']^{-1}.\end{aligned}$$

Hence, using Perseval's relation we have

$$\begin{aligned}P &= \frac{-1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\phi(\mathrm{i}\omega)[\mathrm{i}\omega I-F]^{-1}gg'[\mathrm{i}\omega I+F']^{-1}\\&= WE+EW',\end{aligned}$$

where

$$\begin{aligned}W &:= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\phi(\mathrm{i}\omega)\{[\mathrm{i}\omega I-F]^{-1}.\\&= \frac{1}{2\pi\mathrm{i}}\oint_{C_R}\mathrm{d}s f(-s)[sI-F]^{-1}, \end{aligned} \quad (6)$$

where C_R is the infinite semicircular contour encircling the entire right half plane traversed in the *clockwise* direction. Note that, the second equality in (6) follows from (1); the fact that $\lim_{s\to\infty}[sI-F]^{-1} = 0$, and $f(\infty)$ is bounded. Also the contribution due to the term $[sI-F]^{-1}f(s)$ vanishes, since it is analytic in the entire right half plane. Now we know that the matrix $[sI-F]^{-1}$ commutes with F. Hence there exists polynomials $\{\alpha_k(s)\}_{k=0}^{n-1}$ such that

$$[sI-F]^{-1} = \sum_{k=0}^{n-1}\frac{\alpha_k(s)}{\Delta(s)}F^k,$$

where $\Delta(s)$ is the characteristic polynomial of F. Therefore, by setting

$$w_k = \frac{1}{2\pi\mathrm{i}}\oint_{C_L}\mathrm{d}s\frac{\alpha_k(s)f(-s)}{\Delta(s)},$$

the proposition follows. ■

Remark: The Proposition 1 gives a way to compute $\{w_k\}_{k=0}^{n-1}$ from P. Note that P is available from the observed data. The matrix E is known to the user since F and g are user defined matrices. Computation of $\{w_k\}_{k=0}^{n-1}$ from P amounts to solving a least-squares problem. Also, $\{w_k\}_{k=0}^{n-1}$ are invariant of the choice of co-ordinates of $z(t)$. To see that, consider the output state sequence $z_1(t)$ of the input-to-state filter (F_1, g_1), where

$$F_1 = TFT^{-1}, \qquad g_1 = T^{-1}g$$

for some nonsingular matrix T. Let the covariance matrix of $z_1(t)$ be P_1. Then it follows that $P_1 = TPT'$ and the unique positive definite solution E_1 of the Lyapunov equation $F_1E_1 + E_1F_1' + g_1g_1'$ satisfy $E_1 = TET'$. Then (4) gives

$$P_1 = W_1E_1 + E_1W_1',$$

where $W_1 = TWT^{-1}$. Now from (5) we can verify our assertion that

$$W_1 = \sum_{k=0}^{n-1} w_k F_1^k.$$

■

To proceed further, we need some notations. Let us define the sequence of $p \times p$ matrices $\{I_p^k\}_{k=0}^{p-1}$ element-wise as

$$[I_p^k]_{ij} = \delta_{j-i,k},$$

where δ_{ij} denotes the Kronecker's delta function. It is readily verified that

$$I_p^k I_p^\ell = \begin{cases} I_p^{k+\ell}, & k+\ell < p \\ 0, & \text{otherwise.} \end{cases} \qquad (7)$$

We shall also use the notation $\mathbb{C}_j^\ell = \frac{\ell!}{j!\,(\ell-j)!}$. Now we are ready for the next theorem, where we explore the way to compute $\{f(s_k)\}_{k=1}^m$ and its derivatives from $\{w_k\}_{k=0}^{n-1}$.

Theorem 1. Suppose $\{s_k\}_{k=1}^m$ are distinct, and F is chosen such that its Jordan form has the block-diagonal structure

$$\operatorname{diag}\left\{ J(-s_1,n_1) \ \cdots \ J(s_m,n_m) \right\}, \qquad (8)$$

where for each k satisfying $1 \le k \le m$, the matrix $J(-s_k,n_k)$ is an elementary $n_k \times n_k$ Jordan block

$$J(-s_k,n_k) = -s_k I_{n_k}^0 + I_{n_k}^1.$$

Define the polynomial

$$w(s) := \sum_{k=0}^{n-1} w_k s^{n-k}.$$

Denote the rth derivative of $f(s)$ evaluated at $s = s_k$ by $f^{(r)}(s_k)$. Then

$$w^{(r)}(-s_k) = (-1)^r f^{(r)}(s_k), \qquad 0 \le r < n_k$$
$$1 \le k \le m. \qquad (9)$$

Proof: Since the coefficients $\{w_k\}_{k=0}^{n-1}$ are independent of the choice of the co-ordinates of $z(t)$, we shall assume, without any loss of generality, that F is in the Jordan canonical form (8). Now

$$F^\ell = \operatorname{diag}\left\{ J^\ell(-s_1,n_1) \ \cdots \ J^\ell(-s_m,n_m) \right\},$$

and using binomial theorem and (7) gives

$$J^\ell(-s_k,n_k) = \sum_{j=0}^{\min(n_k-1,\ell)} (-s_k)^{\ell-j} \mathbb{C}_j^\ell I_{n_k}^j.$$

Therefore, from (5) we see that W also is block-diagonal: $W = \operatorname{diag}\left\{ W_1 \ \cdots \ W_m \right\}$, where W_k is a $n_k \times n_k$ block given by

$$W_k = \sum_{\ell=0}^{n-1} w_\ell \sum_{j=0}^{\min(n_k-1,\ell)} (-s_k)^{\ell-j} \mathbb{C}_j^\ell I_{n_k}^j. \qquad (10)$$

However, from the definition of W in (6) we see

$$\begin{aligned} W_k &= \frac{1}{2\pi \mathrm{i}} \oint_{C_R} \mathrm{d}s\, f(-s)[(s+s_k)I_{n_k}^0 - I_{n_k}^1]^{-1} \\ &= \frac{-1}{2\pi \mathrm{i}} \oint_{C_R} \mathrm{d}s\, f(-s) \sum_{j=0}^{n_k} \frac{1}{(s+s_k)^{j+1}} I_{n_k}^j \\ &= \sum_{j=0}^{n_k} (1/j!)\,(-1)^j f^{(j)}(s_k) I_{n_k}^j, \qquad (11) \end{aligned}$$

where the second equality can be verified after a few steps of straightforward algebra and third equality follows from Cauchy's residue theorem. Note that W_k is a Toeplitz, upper triangular matrix, with the coefficients of $I_{n_k}^j$ appearing along the jth upper diagonal. Therefore, we can equate the coefficients of $I_{n_k}^j$ for $0 \le j \le n_k - 1$ in (10) and (11) to get (9). ■

Remark: The result in Theorem 1 can be seen as the continuous-time counterparts of some of the results in (Georgiou, 2001; Georgiou, 2002*b*; Georgiou, 2002*a*) for discrete-time processes. Compared to these results, our proofs are simpler, and our results are more general in the sense that Theorem 1 gives explicit expressions for the derivatives of $f(s)$ evaluated at s_k. An analogous proof as above can also be used to establish the corresponding results in (Georgiou, 2001; Georgiou, 2002*b*). Theorem 1 is perhaps more important to continuous-time processes than its counterpart [results in (Georgiou, 2001; Georgiou, 2002*b*)] to discrete-time processes. This is because there are numerous alternative approaches available for modeling discrete-time processes (Ljung, 1999), whereas few such results are known for the continuous-time case. ■

3. ESTIMATION OF RATIONAL MODELS

In this section we explore some possibilities of estimating rational transfer function models from the estimates of $f(s)$ and its derivatives evaluated at $\{s_k\}_{k=1}^m$. Assume that $u(t)$ has a strictly proper rational spectrum of order ν, *i.e.*,

$$\phi(s) = \frac{c(s)c(-s)}{a(s)a(-s)},$$

where

$$a(s) = s^\nu + \sum_{k=1}^{\nu} a_k s^{\nu-k}, \qquad c(s) = \sum_{k=1}^{\nu} c_k s^{\nu-k}.$$

Then the positive real spectrum $f(s)$ also admits a strictly proper rational representation as

$$f(s) = \frac{b(s)}{a(s)}, \qquad b(s) = \sum_{k=1}^{\nu} b_k s^{\nu-k}.$$

such that

$$c(s)c(-s) = a(s)b(-s) + b(s)a(-s). \qquad (12)$$

Our approach in this work is to identify the parameters $\{a_k\}_{k=1}^\nu$ and $\{b_k\}_{k=1}^\nu$ from the data. Subsequently we can evaluate the right-hand side of the equation (12). Then a spectral factorization of (12) would lead to the parameters $\{c_k\}_{k=1}^\nu$. Note that the right-hand side of (12) needs to be positive real in order to ensure the existence of a stable spectral factor. Another important issue is to ensure the stability of the estimated polynomial $a(s)$. The problem of computing $a(s)$, $b(s)$ and $c(s)$ from the *interpolation conditions* originating from $f(s)$ and its derivatives evaluated at $\{s_k\}_{k=1}^m$ is in fact a linear problem. However, when we impose the stability constraint on $a(s)$ and positivity constraint on the right-hand side of (12), we have to solve a Nevanlinna-Pick interpolation problem with a degree constraint (Delsarte *et al.*, 1982; Kimura, 1987), which is more difficult to solve. In this section, we shall give a brief account of the different possibilities for obtaining a rational model (with and without constraints) from the interpolants obtained using the method proposed in the previous section.

3.1 *Method 1: linear interpolation*

The simplest method for estimating $\{a_k\}_{k=1}^{\nu}$ and $\{b_k\}_{k=1}^{\nu}$ from the interpolants is to solve a linear system of equations. Here to keep the description simple, we shall assume that the eigenvalues of F are distinct and we have estimates $\hat{f}_k$ of $f(s_k)$ for $1 \leq k \leq n$. The generalization for interpolation conditions involving derivatives of $f(s)$ is straightforward. For this approach we need $n \geq 2\nu$. Recall that, we are required to solve

$$\hat{f}_k a(s_k) = b(s_k), \qquad 1 \leq k \leq n. \tag{13}$$

Let us define

$$V = \begin{bmatrix} s_1^{\nu-1} & \cdots & s_1 & 1 \\ s_2^{\nu-1} & \cdots & s_2 & 1 \\ \vdots & \ddots & \vdots & \\ s_n^{\nu-1} & \cdots & s_n & 1 \end{bmatrix}, \qquad v = \begin{bmatrix} a_1^{\nu} \\ a_2^{\nu} \\ \vdots \\ a_n^{\nu} \end{bmatrix},$$

and $D = \mathrm{diag}\{\ \hat{f}_1 \ \cdots \ \hat{f}_n\ \}$. Then (13) gives

$$Dv = -DV\mathbf{a} + V\mathbf{b}, \tag{14}$$

where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_{\nu} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_{\nu} \end{bmatrix}.$$

Hence the estimates of **b** and **a** are obtained by solving (14). If $n > 2\nu$, we need to solve (14) in a least-squares sense. We point out that if the estimates $\{\hat{f}_k\}_{k=1}^{n}$ are consistent, then estimates of **a** and **b** obtained by solving (14) are also consistent. However, if $\{\hat{f}_k\}_{k=1}^{n}$ are not accurate enough due to small-sample estimation errors, it is not possible to guarantee the stability of the estimated $a(s)$ polynomial. Also, one cannot ensure the positive realness condition in (12).

3.2 *Method 2: Nevanlinna-Pick interpolation*

The classical way of solving the Nevanlinna-Pick interpolation problem is to employ the Nevanlinna's algorithm for solving interpolation in a rational Schur function [2] (Delsarte *et al.*, 1982). The popular way of mapping the Nevanlinna-Pick interpolation problem in our context to the Schur function interpolation problem is to use a bilinear transform

$$z = \frac{1+s}{1-s} \Leftrightarrow s = \frac{z-1}{z+1}. \tag{15}$$

For all z satisfying $|z| > 1$ in (15), it is readily verified that $\mathrm{Re}(s) > 0$. The idea here is to convert the *continuous-time* problem to an equivalent *discrete-time* problem of finding a rational function $h(z)$ of degree ν analytic outside the unit disc such that $h(z_k) = \hat{f}_k$, for $1 \leq k \leq \nu+1$, where $z_k = (1+s_k)/(1-s_k)$. Note that in this framework we need $n = \nu + 1$. Once we know $h(z)$, we can map it back to $f(s)$ using

$$f(s) = h\left\{\frac{1+s}{1-s}\right\}. \tag{16}$$

Next, the *equivalent* discrete-time Nevanlinna-Pick problem is converted to a Schur function interpolation problem by mapping the positive real interpolants $\{\hat{f}_k\}_{k=1}^{\nu+1}$ into the interior of unit disc: $x_k = (1-\hat{f}_k)/(1+\hat{f}_k)$, where we seek for a rational Schur function $\sigma(z)$ such that $\sigma(z_k^{-1}) = x_k$. The Schur function $\sigma(z)$ can then be used to get $h(z)$ as

$$h(z) = \frac{1-\sigma(z^{-1})}{1+\sigma(z^{-1})}. \tag{17}$$

A solution to the above problem exists if and only if the Nevanlinna-Pick Hermitian matrix

$$N = \left[\frac{\bar{\hat{f}}_i + \hat{f}_j}{1-\bar{z}_i^{-1} z_j^{-1}} : i,j = 1,2,\ldots,\nu\right]$$

is non-negative definite, while the solution is unique if and only if N is rank deficient. Note that $\bar{x}$ denotes the conjugate of x. Nevanlinna's algorithm for computing a solution $\sigma_1(z)$ is described as follows.

(1) Compute the Fenyves array of complex numbers

$$x_{i,k} = \frac{(1-\bar{z}_{k-1}^{-1} z_i^{-1})(x_{i,k-1} - x_{k-1,k-1})}{(z_i^{-1} - z_{k-1}^{-1})(1-\bar{x}_{k-1,k-1} x_{i,k-1})}$$

for $1 \leq i < \nu+1, i \leq k < \nu+1$, where $x_{i,1} = x_i$ for $1 \leq i \leq \nu+1$.

(2) Initialize [3] $\sigma_{\nu+1}(z) = x_{\nu+1,\nu+1}$. From the *diagonal* entries $x_{k,k}$ of Fenyves array compute recursively the general solution $\sigma(z)$ as

$$\sigma_k(z) = \frac{x_{k,k}(1-z\bar{z}_k^{-1}) + (z-z_k^{-1})\sigma_{k+1}(z)}{1-z\bar{z}_k^{-1} + \bar{x}_{k,k}(z-z_k^{-1})\sigma_{k+1}(z)},$$

Taking $\sigma(z) = \sigma_1(z)$ we can use the mappings in (16) and (17) to reconstruct the corresponding *continuous-time* function $f(s)$. The function $f(s)$ derived in this way satisfies the positivity and stability constraints. However, it can be shown that the associated spectrum

[2] A rational function analytic inside the unit disc having modulus not larger than unity is called a Schur function.

[3] We emphasize that without the degree constraint, one is free to choose any Schur function for $\sigma_{\nu+1}(z)$ such that $\sigma_{\nu+1}(z_{\nu+1}) = x_{\nu+1,\nu+1}$. This provides a way of seeing the non-unique nature of the solution.

has zeros in the locations $\{-s_k\}_{k=1}^{\nu}$. Recall that we need $f(\infty) = 0$. Now note from (16) and (17) that

$$f(\infty) = 0 \Leftrightarrow h(-1) = 0 \Leftrightarrow \sigma(-1) = 1.$$

One way to incorporate this constraint would be to evaluate ν interpolants from the data and include the final interpolation condition in the form

$$x_{\nu+1} = 1,\ z_{\nu+1} = -1.$$

However, it can shown in this case that Nevanlinna's algorithm gives a degenerate spectrum which is identically zero. This limits the application of Nevanlinna's algorithm where the underlying spectrum is strictly proper. Another main disadvantage of Nevanlinna's algorithm is that it assigns spectral zeros at $\{-s_k\}_{k=1}^{\nu}$. This can be inconvenient for practical purposes. The reason is that the values of s_k are typically chosen near the frequency points of interest, and that forcing spectral zeros at $-s_k$ would seriously distort the spectral information at these frequency points. This problem will be illustrated in the next section.

One alternative is Byrnes-Georgiou-Lindquist algorithm. The idea here is to use use so-called *spectral zero assignability theorem* as stated below.

Theorem 2. Let a set of points $\{z_k\}_{k=1}^{\nu}$ and $\{\hat{f}_k\}_{k=1}^{\nu}$ be given so that $|z_k| > 0$ and $\mathrm{Re}(\hat{f}_k) > 0$, and the matrix N is non-negative definite. Then for every monic polynomial $c_d(z)$ of order ν, there exists a unique order ν monic Schur stable polynomial $a_d(z)$ and an associated order ν monic polynomial $b_d(z)$ such that

$$\lambda c_d(z)c_d(z^{-1}) = a_d(z)b_d(z^{-1}) + b_d(z)a_d(z^{-1}),$$
$$b_d(z_k)/a_d(z_k) = \hat{f}_k,\ 1 \le k \le \nu,$$

where the gain factor λ is determined by another prespecified interpolation condition [$f_0 = b_d(\infty)/a_d(\infty)$ for example].

Proof: See (Byrnes *et al.*, 2001) and references therein.

It is shown in (Byrnes *et al.*, 2001) that the problem of solving $a_d(z)$ and $b_d(z)$ can be formulated as a convex optimization problem. Moreover, an iterative algorithm is given in (Byrnes *et al.*, 2001). We propose to solve for $h(z)$ using Byrnes-Georgiou-Lindquist algorithm and then use (16) get $f(s)$. For every presumed locations of spectral zeros, Theorem 2 guarantees a stable denominator polynomial. If we are interested in an AR process, we can set spectral zeros at $s = -\infty$ (equivalently, $z = -1$). For ARMA processes, we need to estimate the spectral zeros using a suitable method, *e,g,* the method in Section 3.1.

4. CASE STUDIES

In this section we present the estimation results obtained using the proposed algorithms in numerical simulation experiments. In our first example, we consider a continuous-time AR process with

$$a(s) = s^3 + 0.3s^2 + 9s + 0.9, \qquad c(s) = 4. \qquad (18)$$

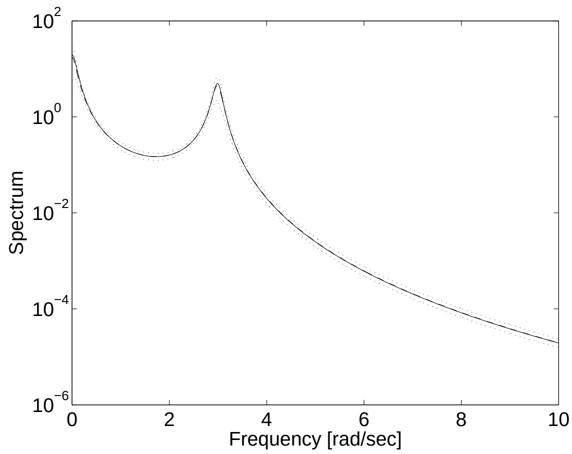

Fig. 1. Comparison of the mean of estimated spectrum (dashed line) and the true spectrum (solid line). The mean $\pm$ standard deviation of the estimated spectrum is shown in dotted lines.

The stochastic process is sampled at 20 Hz. The sampled version of the stochastic process is simulated such that the second order statistics of the sampled process resembles that of the original continuous-time process, see (Söderström, 2002; Larsson, 2004) for details. The output of input-to-state filter is computed using the discretization technique known as state variable filter (SVF), see (Garnier and Young, 2004) and references therein. In the discretization of the input-to-state filters, the input signal is assumed to vary linearly in between the sampling instants [commonly referred to as the first order hold]. The estimation is carried out using Byrnes-Georgiou-Lindquist algorithm by prespecifying all the spectral zeros at the infinity. The interpolation points are chosen as $\{1, 1+3\mathrm{i}, 1-3\mathrm{i}, 1.5\}$. The observation time of the precess is taken as 5 minuits. This means that 6000 samples are used for estimation. In Figure 1 we compare the mean and standard deviation of the estimated spectrum with the true spectrum. The numerical mean and standard deviations are based on 100 Monte-Carlo simulations [4].

In the next example we consider an ARMA model with $a(s)$ same as in (18), and

$$c(s) = s^2 + 0.5s + 6.$$

At the first stage a preliminary estimate of the parameters are obtained using the least-squares based approach in (14). Then the spectral zeros so obtained are used in Byrnes-Georgiou-Lindquist's algorithm to determine the associated stable denominator and gain factor. The interpolation points for the least-squares based approach in (14) are chosen as $\{1, 1+3\mathrm{i}, 1-$

[4] Note that due to estimation errors in the interpolants $\hat{f}_k$ the Nevanlinna-Pick matrix N may fail to be non-negative definite. However, such situation is not encountered in this example.

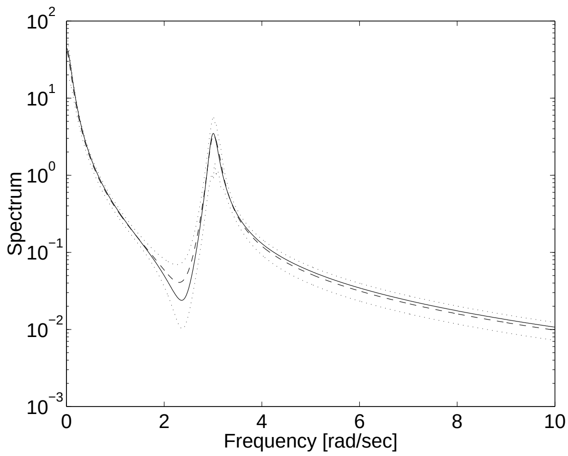

Fig. 2. Comparison of the mean of estimated spectrum (dashed line) and the true spectrum (solid line). The mean $\pm$ standard deviation of the estimated spectrum is shown in dotted lines.

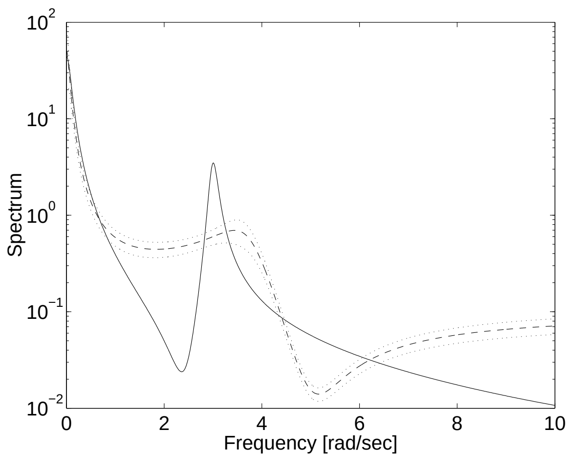

Fig. 3. Comparison of the mean of estimated spectrum (dashed line) and the true spectrum (solid line)using Nevanlinna's algorithm. The mean $\pm$ standard deviation of the estimated spectrum is shown in dotted lines.

$3i, 1+2i, 1-2i, 1.5\}$, among which the first four are used in Byrnes-Georgiou-Lindquist algorithm. The number of data samples, sampling frequency and observation time is the same as the previous example. A comparison between the estimated spectrum and the true spectrum is depicted in Figure 2. In about 25% cases the least-squares based failed to give a positive real $f(s)$. Finally, in Figure 3 we show the spectrum estimation results obtained using Nevanlinna's algorithm. The interpolation points are chosen as $\{1, 0.5+5i, 0.5-5i, 0.5\}$. As mentioned earlier, we can see the estimated spectral zeros are located at the interpolation points, which seriously distorts the spectrum.

5. CONCLUSIONS

In this paper we have proposed a novel direct approach for modeling continuous-time stochastic processes. The main idea is to use an input-to-state filters to sample the positive real spectrum in some prescribed points in the right half plane. Subsequently, a suitable prescribed method can be used to interpolate these data to obtain the required model. The Byrnes-Georgiou-Lindquist's algorithm can be used to obtain stable AR models with a high degree of accuracy. For ARMA models, we estimate the spectral zeros using a least-squares method, and subsequently compute the poles and the gain factor using Byrnes-Georgiou-Lindquist's algorithm. We caution that this method of estimating spectral zeros is ad-hoc. Thus, the problem of estimating both the numerator and denominator polynomials from the interpolation data satisfying the stability and positivity constraint is an open research problem.

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