ROBUST DECENTRALIZED H_∞ CONTROL OF MULTI-CHANNEL SYSTEMS WITH NORM-BOUNDED PARAMETRIC UNCERTAINTIES

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Abstract: This paper considers a robust decentralized H_∞ control problem for uncertain multi-channel systems. The uncertainties are assumed to be time-invariant, norm-bounded, and exist in both the system and control input matrices. Our interest is focused on dynamic output feedback. A necessary and sufficient condition for the uncertain multi-channel system to be robustly stabilizable with a specified disturbance attenuation level is derived based on the bounded real lemma, which is reduced to a feasibility problem of a nonlinear matrix inequality (NMI). A two-stage homotopy method is employed to solve the NMI iteratively. First, a decentralized controller for the nominal system with no uncertainty is computed by imposing structural constraints on the coefficient matrices of the controller gradually. Then, the decentralized controller is modified, again gradually, to cope with the uncertainties. On each stage, a variable is fixed alternately at the iterations to reduce the NMI to a linear matrix inequality (LMI). A given example shows the efficiency of this method.

Keywords: Robust decentralized control, Parametric uncertainty, Homotopy method, Nonlinear matrix inequality, LMI

1. INTRODUTION

Robust decentralized H_∞ control problems have been paid much attention. Since system models always contain uncertainties, expected performances cannot be attained if the controller is designed only for the nominal model.

It has been well known that linear-matrix-inequality (LMI)-based approaches are very powerful for centralized controller design (Boyd, *et al.*, 1994; Iwasaki and Skelton, 1994; Gahinet and Apkarian, 1994). A large number of results based on LMIs for centralized control problems have been reported in the literature. However, it is not so in the decentralized case. Decentralized H_∞ controller design problems can be formulated as feasibility problems of bilinear matrix inequalities (BMIs), but cannot be reduced to LMI problems because of the structural constraint on controllers, i.e., block-diagonal forms of coefficient matrices.

At present, there is no globally effective method to solve general BMI problems, but a number of practical techniques have been proposed. One of them is the idea of homotopy methods, whose main advantage is ability to dispense with restrictive requirement. Applications of a homotopy method to decentralized control problems have been introduced in the works of Richter and DeCarlo (1983,1984), where the method has been shown to be useful for computing decentralized state feedback in eigenvalue assignment problems. Zhai, *et al*. (2001) have solved a decentralized H_∞ control problem for multi-channel systems using a homotopy method. The problem was formulated as feasibility of a BMI. Their algorithm deforms the controller's coefficient matrices from full matrices defined by a centralized controller, to block-diagonal matrices of specified dimensions which describe a decentralized controller. Another algorithm for a decentralized H_∞ controller was proposed based on a double homotopy path method by Mehendale and

Grigoriadis (2003). Along one of the paths a full centralized structure was deformed to a block diagonal decentralized structure. Along the other path, designs were improved by solving a linear approximation to the BMI problem. Above contributions did not consider any uncertainty in the coefficient matrices.

In this paper, we consider a robust decentralized H_∞ control problem for uncertain multi-channel systems. The uncertainties are assumed to be time-invariant, norm-bounded, and exist in both the system and control input matrices. A necessary and sufficient condition for the uncertain multi-channel system to be robustly stabilizable with a specified disturbance attenuation level is derived based on the bounded real lemma, which is reduced to a feasibility problem of a nonlinear matrix inequality (NMI). A two-stage homotopy method is employed to solve the NMI iteratively. The idea of the two-stage homotopy method has been proposed by Chen, *et al.* (2004) in solving a sufficient condition for a robust decentralized H_∞ controller to exist for interconnected systems, where the dimensions of local controllers are the same as those of corresponding subsystems. First, a decentralized controller for the nominal system with no uncertainty is computed by imposing structural constraints on the coefficient matrices of the controller gradually. Then, the decentralized controller is modified, again gradually, to cope with the uncertainties. At each stage, a variable is fixed alternately at the iterations to reduce the NMI to a linear matrix inequality (LMI). A given example shows the efficiency of this method.

2. PROBLEM DESCRIPTION

We consider an N-channel linear time-invariant system with uncertainties, which is described by a state-space model as

$$
\begin{aligned}
\dot{x} &= (A+\delta A)x + B_1 w + \sum_{i=1}^{N} (B_{2i} + \delta B_{2i}) u_i \\
z &= C_1 x + D_{11} w + \sum_{i=1}^{N} D_{12i} u_i \qquad (1)\\
y_i &= C_{2i} x + D_{21i} w, \quad i = 1, 2, \cdots, N
\end{aligned}
$$

where $x \in R^n$ is the state, $w \in R^r$ is the disturbance input, $z \in R^p$ is the controlled output, and $u_i \in R^{m_i}$ and $y_i \in R^{q_i}$ are the control input and the measured output of channel i ($i = 1, 2, \cdots, N$), respectively. The matrices A, B_1, B_{2i}, C_1, C_{2i}, D_{11}, D_{12i}, and D_{21i} are constant and of appropriate dimensions. The matrices δA and δB_{2i} denote time-invariant uncertainties in the system and control input matrices. We suppose that the uncertainties are related as

$$
\begin{bmatrix} \delta A & \delta B_{21} & \cdots & \delta B_{2N} \end{bmatrix} = E\Delta [F_1 \ \ F_{21} \ \ \cdots \ \ F_{2N}] \qquad (2)
$$

where $E, F_1, F_{21}, \cdots, F_{2N}$ are known constant matrices and Δ is an unknown constant matrix satisfying

$$
\Delta^{\mathrm{T}} \Delta \le I. \qquad (3)
$$

We assume that there is no unstable fixed mode defined by the triplet $(C_2, A+\delta A, \ B_2 + \delta B_2)$.

We adopt a strictly proper decentralized output feedback controller described by

$$
\begin{aligned}
\dot{\hat{x}}_i &= \hat{A}_i \hat{x}_i + \hat{B}_i y_i \\
u_i &= \hat{C}_i \hat{x}_i
\end{aligned}, \quad i = 1, 2, \cdots, N \qquad (4)
$$

where $\hat{x}_i \in R^{\hat{n}_i}$ is the state of the i-th local controller and $\hat{n}_i$ is a specified dimension. The matrices $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $i = 1, 2, \cdots, N$ are constant and to be determined.

We denote the transfer function from the disturbance input w to the controlled output z of the closed-loop system obtained by applying the decentralized controller (4) to the system (1), by $T_{zw}(s)$. We say that the system (1) is robustly stabilizable with the disturbance attenuation level γ if there exists a decentralized controller (4) so that the closed-loop system is robustly stable and satisfies $\|T_{zw}\|_\infty < \gamma$ for any Δ bounded as (3), where γ is a specified positive number. The control problem of this paper is to design a decentralized controller (4) realizing such a closed-loop system.

To solve the decentralized control problem, we employ the following lemmas.

Lemma 1 (Bounded Real Lemma) (Iwasaki and Skelton, 1994; Gahinet and Apkarian, 1994). Suppose that A, B, C and D are given matrices of appropriate dimensions. Then, the following statements are equivalent:

(i) A is a stable matrix and $\|C(sI - A)^{-1}B + D\|_\infty < \gamma$.

(ii) There exists a positive definite matrix P which satisfies the LMI:

$$
\begin{bmatrix} A^{\mathrm{T}}P + PA & PB & C^{\mathrm{T}} \\ B^{\mathrm{T}}P & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{bmatrix} < 0.
$$

Lemma 2 (Petersen, 1987) Suppose that Ξ, E, and F are matrices of appropriate dimensions and Ξ is symmetric. Then,

$$
\Xi + E\Delta F + F^{\mathrm{T}}\Delta^{\mathrm{T}}E^{\mathrm{T}} < 0
$$

for all Δ satisfying $\Delta^{\mathrm{T}}\Delta \le I$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$
\Xi + \varepsilon E E^{\mathrm{T}} + \varepsilon^{-1} F^{\mathrm{T}} F < 0.
$$

3. EXISTENCE CONDITION FOR ROBUST DECENTRALIZED H_∞ CONTROLLER

To write the closed-loop system in a compact form, we define matrices in the system (1) as

$$
\begin{aligned}
B_2 &= \begin{bmatrix} B_{21} & B_{22} & \cdots & B_{2N} \end{bmatrix} \\
C_2 &= \begin{bmatrix} C_{21}^{\mathrm{T}} & C_{22}^{\mathrm{T}} & \cdots & C_{2N}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\
D_{12} &= \begin{bmatrix} D_{121} & D_{122} & \cdots & D_{12N} \end{bmatrix} \\
D_{21} &= \begin{bmatrix} D_{211}^{\mathrm{T}} & D_{212}^{\mathrm{T}} & \cdots & D_{21N}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\
\delta B_2 &= \begin{bmatrix} \delta B_{21} & \delta B_{22} & \cdots & \delta B_{2N} \end{bmatrix} \\
F_2 &= \begin{bmatrix} F_{21} & F_{22} & \cdots & F_{2N} \end{bmatrix}.
\end{aligned} \tag{5}
$$

and write the coefficient matrices of the controller (4) as

$$
\begin{aligned}
\hat{A}_D &= \mathrm{diag}\{\hat{A}_1, \hat{A}_2, \cdots, \hat{A}_N\} \\
\hat{B}_D &= \mathrm{diag}\{\hat{B}_1, \hat{B}_2, \cdots, \hat{B}_N\} \\
\hat{C}_D &= \mathrm{diag}\{\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_N\}
\end{aligned} \tag{6}
$$

and form a matrix

$$
G_D = \begin{bmatrix} \hat{A}_D & \hat{B}_D \\ \hat{C}_D & 0 \end{bmatrix}. \tag{7}
$$

We also introduce the notations

$$
\begin{bmatrix} \tilde{A} + \delta\tilde{A} & \tilde{B}_1 & \tilde{B}_2 + \delta\tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \end{bmatrix}
= \left[\begin{array}{cc|c|cc}
A + \delta A & 0_{n\times\hat{n}} & B_1 & 0_{n\times\hat{n}} & B_2 + \delta B_2 \\
0_{\hat{n}\times n} & 0_{\hat{n}\times\hat{n}} & 0_{\hat{n}\times r} & I_{\hat{n}} & 0_{\hat{n}\times m} \\
\hline
C_1 & 0_{p\times\hat{n}} & D_{11} & 0_{p\times\hat{n}} & D_{12} \\
\hline
0_{\hat{n}\times n} & I_{\hat{n}} & 0_{\hat{n}\times r} & & \\
C_2 & 0_{q\times\hat{n}} & D_{21} & &
\end{array}\right] \tag{8}
$$

where $\hat{n} = \sum_{i=1}^{N} \hat{n}_i,\ m = \sum_{i=1}^{N} m_i,\ q = \sum_{i=1}^{N} q_i,$ and

$$
\begin{aligned}
\tilde{A} &= \begin{bmatrix} A & 0_{n\times\hat{n}} \\ 0_{\hat{n}\times n} & 0_{\hat{n}\times\hat{n}} \end{bmatrix},\ \tilde{B}_2 = \begin{bmatrix} 0_{n\times\hat{n}} & B_2 \\ I_{\hat{n}} & 0_{\hat{n}\times m} \end{bmatrix}, \\
\delta\tilde{A} &= \begin{bmatrix} \delta A & 0_{n\times\hat{n}} \\ 0_{\hat{n}\times n} & 0_{\hat{n}\times\hat{n}} \end{bmatrix},\ \delta\tilde{B}_2 = \begin{bmatrix} 0_{n\times\hat{n}} & \delta B_2 \\ 0_{\hat{n}\times\hat{n}} & 0_{\hat{n}\times m} \end{bmatrix}.
\end{aligned} \tag{9}
$$

Then, the closed-loop system can be written in a compact form as

$$
\begin{aligned}
\dot{\tilde{x}} &= [\tilde{A} + \delta\tilde{A} + (\tilde{B}_2 + \delta\tilde{B}_2)G_D\tilde{C}_2]\tilde{x} + [\tilde{B}_1 + \tilde{B}_2 G_D \tilde{D}_{21}]w \\
&= [\tilde{A} + \tilde{E}\Delta\tilde{F}_1 + (\tilde{B}_2 + \tilde{E}\Delta\tilde{F}_2)G_D\tilde{C}_2]\tilde{x} + (\tilde{B}_1 + \tilde{B}_2 G_D \tilde{D}_{21})w \\
z &= (\tilde{C}_1 + \tilde{D}_{12}G_D\tilde{C}_2)\tilde{x} + (\tilde{D}_{11} + \tilde{D}_{12}G_D\tilde{D}_{21})w
\end{aligned} \tag{10}
$$

where

$$
\tilde{x} = [x^{\mathrm{T}} \quad \hat{x}^{\mathrm{T}}]^{\mathrm{T}},\ \hat{x} = [\hat{x}_1^{\mathrm{T}} \quad \hat{x}_2^{\mathrm{T}} \quad \cdots \quad \hat{x}_N^{\mathrm{T}}]^{\mathrm{T}}
$$

$$
\tilde{E} = \begin{bmatrix} E \\ 0 \end{bmatrix},\ \tilde{F}_1 = \begin{bmatrix} F_1 & 0 \end{bmatrix},\ \tilde{F}_2 = \begin{bmatrix} 0 & F_2 \end{bmatrix}.
$$

To derive (10), we have used the fact

$$
\delta\tilde{B}_2 G_D \tilde{D}_{21} = \begin{bmatrix} 0_{n\times\hat{n}} & \delta B_2 \\ 0_{\hat{n}\times\hat{n}} & 0_{\hat{n}\times m} \end{bmatrix} \begin{bmatrix} \hat{A}_D & \hat{B}_D \\ \hat{C}_D & 0 \end{bmatrix} \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} = 0. \tag{11}
$$

A necessary and sufficient condition for the existence of a robust decentralized H$_\infty$ controller is given as follows.

Theorem 1. For a given constant $\gamma > 0$, the uncertain system (1) is robustly stabilizable with the disturbance attenuation level γ via a decentralized controller (4) composed of $\hat{n}_i$-dimensional local controllers, if and only if there exist a matrix G_D of (7), a positive definite matrix $\tilde{P}$, and a scalar $\varepsilon > 0$ such that

$$
\begin{aligned}
J(G_D, \tilde{P}, \varepsilon) = &\begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \phi_u & \tilde{P}\tilde{B}_1 & \tilde{C}_1^{\mathrm{T}} \\ \tilde{B}_1^{\mathrm{T}}\tilde{P} & -\gamma I & \tilde{D}_{11}^{\mathrm{T}} \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I \end{bmatrix} \\
&+ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} \end{bmatrix} \\
&+ \left\{ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} \end{bmatrix} \right\}^{\mathrm{T}} < 0
\end{aligned} \tag{12}
$$

holds, where

$$
\phi_u = \varepsilon\tilde{P}\tilde{E}\tilde{E}^{\mathrm{T}}\tilde{P} + \varepsilon^{-1}(\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2)^{\mathrm{T}}(\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2). \tag{13}
$$

Proof. From Lemma 1 for the closed-loop system (10), we see that the uncertain multi-channel system (1) is robustly stabilizable with the disturbance attenuation level γ, if and only if there exist a matrix G_D of (7) and a positive definite matrix $\tilde{P}$ such that

$$
\begin{aligned}
&\begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} & \tilde{P}\tilde{B}_1 & \tilde{C}_1^{\mathrm{T}} \\ \tilde{B}_1^{\mathrm{T}}\tilde{P} & -\gamma I & \tilde{D}_{11}^{\mathrm{T}} \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I \end{bmatrix} + \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} \end{bmatrix} \\
&+ \left\{ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} \end{bmatrix} \right\}^{\mathrm{T}} \\
&+ \begin{bmatrix} \tilde{P}\tilde{E} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta \begin{bmatrix} (\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&+ \left\{ \begin{bmatrix} \tilde{P}\tilde{E} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta \begin{bmatrix} (\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}^{\mathrm{T}} < 0
\end{aligned} \tag{14}
$$

holds. Based on Lemma 2, inequality (14) holds for any Δ satisfying (3) if and only if there exist G_D, $\tilde{P}$, and a scalar $\varepsilon > 0$ such that (12) with (13) holds. Thus, Theorem 1 has been proved.

We note that if there is no uncertainty in system matrices, Theorem 1 is reduced to the result of Zhai, *et al.* (2001).

Remark 1: In this paper, we extensively consider uncertainties in the system matrix A and the control input matrix B_2. We can also treat the dual form of (1) where uncertainties appear in the system matrix A and the measured output matrix C_2 as

$$
\begin{aligned}
\dot{x} &= (A + \delta A)x + B_1 w + \sum_{i=1}^{N} B_{2i} u_i, \\
z &= C_1 x + D_{11} w + \sum_{i=1}^{N} D_{12i} u_i, \\
y_i &= (C_{2i} + \delta C_{2i})x + D_{21i} w, \quad i = 1, 2, \cdots, N
\end{aligned} \tag{15}
$$

where

$$\begin{bmatrix} \delta A \\ \delta C_{21} \\ \vdots \\ \delta C_{2N} \end{bmatrix} = \begin{bmatrix} E_1 \\ E_{21} \\ \vdots \\ E_{2N} \end{bmatrix} \Delta F \tag{16}$$

and $E_1, E_{21}, \cdots, E_{2N}$, F are known constant matrices.

4. COMPUTATION ALGORITHM

The existence condition (12) for a robust decentralized H$_\infty$ controller is an NMI with the variables G_D, $\tilde{P}$ and ε. In order to solve this problem, we adopt the idea of the homotopy method. For this purpose, we first decompose $J(G_D, \tilde{P}, \varepsilon)$ of (12) into the nominal part $J_0(G_D, \tilde{P})$ and the perturbation part $J_u(G_D, \tilde{P}, \varepsilon)$ generated by the uncertainties, as

$$J(G_D, \tilde{P}, \varepsilon) = J_0(G_D, \tilde{P}) + J_u(G_D, \tilde{P}, \varepsilon) \tag{17}$$

where

$$\begin{aligned} J_0(G_D, \tilde{P}) &= \begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} & \tilde{P}\tilde{B}_1 & \tilde{C}_1^{\mathrm{T}} \\ \tilde{B}_1^{\mathrm{T}}\tilde{P} & -\gamma I & \tilde{D}_{11}^{\mathrm{T}} \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} \end{bmatrix} \\ &+ \left\{ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} \end{bmatrix} \right\}^{\mathrm{T}} \end{aligned} \tag{18}$$

$$J_u(G_D, \tilde{P}, \varepsilon) = \begin{bmatrix} \varepsilon\tilde{P}\tilde{E}\tilde{E}^{\mathrm{T}}\tilde{P} + \varepsilon^{-1}(\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2)^{\mathrm{T}}(\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

To solve the NMI (12), we propose a two-stage homotopy method. On the first stage, we consider the nominal case without uncertainty, i.e. $\delta A = 0$, $\delta B_2 = 0$. In this case, the NMI (12) is reduced to a BMI $J_0(G_D, \tilde{P}) < 0$. To solve this, we employ the technique of Zhai, *et al.* (2001) by introducing a real number λ varying from 0 to 1 as

$$H_0(G_D, \tilde{P}, \lambda) = J_0((1-\lambda)G_F + \lambda G_D, \tilde{P}) \tag{20}$$

where

$$G_F = \begin{bmatrix} A_F & B_F \\ C_F & 0 \end{bmatrix}$$

is a matrix of the same size as G_D and A_F, B_F, C_F, are coefficient matrices of an $\hat{n}$-dimensional centralized H$_\infty$ controller for the disturbance attenuation level γ. Then, a decentralized H$_\infty$ controller can be obtained by solving

$$H_0(G_D, \tilde{P}, \lambda) < 0 \tag{21}$$

using a homotopy method, where the controller's coefficient matrices are deformed from the full matrix G_F to block-diagonal matrices of specified dimensions in G_D. Suppose that a solution $(G_D, \tilde{P})$ of (21) at $\lambda = 1$ has been obtained, which we denote by $(G_{D0}, \tilde{P}_0)$.

On the second stage, we take into account uncertainties in the multi-channel system (1). In order to compute a solution of the NMI (12), we again employ a homotopy method in a different way. We introduce a real number $\tilde{\lambda} \in [0,1]$ and define the matrix function

$$H_1(G_D, \tilde{P}, \varepsilon, \tilde{\lambda}) = J_0(G_D, \tilde{P}) + \tilde{\lambda} J_u(G_D, \tilde{P}, \varepsilon). \tag{22}$$

Then,

$$H_1(G_D, \tilde{P}, \varepsilon, \tilde{\lambda}) = \begin{cases} J_0(G_D, \tilde{P}), & \tilde{\lambda} = 0 \\ J(G_D, \tilde{P}, \varepsilon), & \tilde{\lambda} = 1 \end{cases} \tag{23}$$

and the problem of finding a solution to (12) is embedded in the parametrized family of problems

$$H_1(G_D, \tilde{P}, \varepsilon, \tilde{\lambda}) < 0, \quad \tilde{\lambda} \in [0,1]. \tag{24}$$

To solve (24) from $\tilde{\lambda} = 0$ to $\tilde{\lambda} = 1$, we apply the Schur complement and consider two equivalent matrix inequalities shown as

$$\begin{aligned} &H_{11}(G_D, \tilde{P}, \varepsilon, \tilde{\lambda}) \\ &= \begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \phi_1 & \tilde{P}\tilde{B}_1 & \tilde{C}_1^{\mathrm{T}} & (\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2)^{\mathrm{T}} \\ \tilde{B}_1^{\mathrm{T}}\tilde{P} & -\gamma I & \tilde{D}_{11}^{\mathrm{T}} & 0 \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I & 0 \\ (\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2) & 0 & 0 & -\varepsilon\tilde{\lambda}^{-1} I \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \\ 0 \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} & 0 \end{bmatrix} \\ &+ \left\{ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \\ 0 \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} & 0 \end{bmatrix} \right\}^{\mathrm{T}} < 0 \end{aligned} \tag{25}$$

where $\phi_1 = \varepsilon\tilde{\lambda}\tilde{P}\tilde{E}\tilde{E}^{\mathrm{T}}\tilde{P}$, and

$$\begin{aligned} &H_{12}(G_D, \tilde{P}, \varepsilon^{-1}, \tilde{\lambda}) \\ &= \begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \phi_2 & \tilde{P}\tilde{B}_1 & \tilde{C}_1^{\mathrm{T}} & \tilde{P}\tilde{E} \\ \tilde{B}_1^{\mathrm{T}}\tilde{P} & -\gamma I & \tilde{D}_{11}^{\mathrm{T}} & 0 \\ \tilde{C}_1 & \tilde{D}_{11} & -\gamma I & 0 \\ \tilde{E}^{\mathrm{T}}\tilde{P} & 0 & 0 & -\varepsilon^{-1}\tilde{\lambda}^{-1} I \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \\ 0 \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} & 0 \end{bmatrix} \\ &+ \left\{ \begin{bmatrix} \tilde{P}\tilde{B}_2 \\ 0_{r\times(\hat{n}+m)} \\ \tilde{D}_{12} \\ 0 \end{bmatrix} G_D \begin{bmatrix} \tilde{C}_2 & \tilde{D}_{21} & 0_{(\hat{n}+q)\times p} & 0 \end{bmatrix} \right\}^{\mathrm{T}} < 0 \end{aligned} \tag{26}$$

where $\phi_2 = \varepsilon^{-1}\tilde{\lambda}(\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2)^{\mathrm{T}}(\tilde{F}_1 + \tilde{F}_2 G_D \tilde{C}_2)$. We see that (25) is an LMI in G_D and ε if we fix $\tilde{P}$ and (26) is an LMI in $\tilde{P}$ and ε^{-1} if we fix G_D.

We note that the solution to (24) at $\tilde{\lambda}=0$ has been already obtained as a result of the first stage. Therefore, we choose the solution of the nominal case as the initial value $(G_{D0},\tilde{P}_0)$ in the homotopy method for the second stage. Then, we make a homotopy path to transform this initial solution at $\tilde{\lambda}=0$ to a solution at $\tilde{\lambda}=1$ as follows.

Let M be a positive integer and consider (M+1) points $\tilde{\lambda}_k = k/M$ ($k=0,1,\cdots,M$) in the interval [0,1] to generate a family of problems

$$H_1(G_D,\tilde{P},\varepsilon,\tilde{\lambda}_k)<0. \tag{27}$$

If the problem at the k-th point is feasible, we denote the obtained solution by $(G_{Dk},\tilde{P}_k)$. Then, we compute a solution $(G_{D,\,k+1},\tilde{P}_{k+1})$ of $H_{11}(G_D,\tilde{P},\varepsilon,\tilde{\lambda}_{k+1})<0$ or $H_{12}(G_D,\tilde{P},\varepsilon^{-1},\tilde{\lambda}_{k+1})<0$ by solving each as an LMI with variables being fixed as $\tilde{P}=\tilde{P}_k$ or $G_D=G_{Dk}$. If the family of problems $H_1(G_D,\tilde{P},\varepsilon,\tilde{\lambda}_k)<0$, k=1, 2, $\cdots,M$ are all feasible, a solution of the NMI (12) is obtained at $k=M$ ($\tilde{\lambda}=1$). If it is not the case, that is, both $H_{11}(G_D,\tilde{P}_k,\varepsilon,\tilde{\lambda}_{k+1})<0$ and $H_{12}(G_{Dk},\tilde{P},\varepsilon^{-1},\tilde{\lambda}_{k+1})<0$ are infeasible for some k, we consider more points in the interval $[\tilde{\lambda}_k,1]$ by increasing M, and repeat the procedure from the solution $(G_{Dk},\tilde{P}_k)$ at $\tilde{\lambda}=\tilde{\lambda}_k$.

We formulate this idea of the second stage in an algorithm for computing a robust decentralized H_∞ controller.

***Step* 1**: Initialize M to a certain positive integer, and set a certain upper bound $M_{\max}$ for M. Set k := 0. Let $\tilde{P}_k=\tilde{P}_0$ and $G_{Dk}=G_{D0}$ using the solution of the first stage.

***Step* 2**: Set $k:=k+1$ and $\tilde{\lambda}_k=k/M$. Compute a solution (G_D,ε) of $H_{11}(G_D,\tilde{P}_{k-1},\varepsilon,\tilde{\lambda}_k)<0$. If it is not feasible, go to Step 3. If it is feasible, set $G_{Dk}=G_D$, and compute a solution $(\tilde{P},\varepsilon^{-1})$ of $H_{12}(G_{Dk},\tilde{P},\varepsilon^{-1},\tilde{\lambda}_k)<0$. Then, set $\tilde{P}_k=\tilde{P}$ and go to Step 5.

***Step* 3**: Compute a solution $(\tilde{P},\varepsilon^{-1})$ of $H_{12}(G_{D,k-1},\tilde{P},\varepsilon^{-1},\tilde{\lambda}_k)<0$. If it is not feasible, go to Step 4. If it is feasible, set $\tilde{P}_k=\tilde{P}$, and compute a solution (G_D,ε) of $H_{11}(G_D,\tilde{P}_k,\varepsilon,\tilde{\lambda}_k)<0$. Then, set $G_{Dk}=G_D$, and go to Step 5.

***Step* 4**: Set M := 2M under the constraint $M\le M_{\max}$, set $\tilde{P}_{2(k-1)}=\tilde{P}_{k-1}$, $G_{D,\,2(k-1)}=G_{D,\,k-1}$, k : =2(k-1) and go to Step 2. If we cannot increase M any more, we conclude that this algorithm does not converge.

***Step* 5**: If $k<M$, go to Step 2. If $k=M$, the obtained $(G_{DM},\tilde{P}_M,\varepsilon)$ is a solution of the NMI (12).

Remark 2. At each of Steps 2 and 3, we suggest solving two LMIs obtained by fixing one of the variables in (25) or (26). It is theoretically not necessary to deal with the second one, but according to authors' experiences, it improves the convergence of the algorithm.

Remark 3. In Step 4, we may simply set M : = 2M, k=0, and go back to Step 2. This means that we compute a different homotopy path from the beginning.

5. AN EXAMPLE

We present an example to demonstrate the efficiency of the two-stage homotopy algorithm. We deal with a two-channel system, where the coefficient matrices of the nominal system are

$$A=\begin{bmatrix} 1.0 & -1.0 & -2.2 & -1.0 & 2.0 & 2.0 & 0.2 & -2.0 \\ 2.1 & -5.1 & -1.2 & 0 & 1.1 & 1.0 & 0.1 & -0.7 \\ 2.1 & -1.0 & -3.2 & -0.9 & 2.0 & 2.0 & 0.2 & -2.0 \\ 8.3 & -10.4 & -7.4 & -1.0 & 7.4 & 7.0 & 0.1 & -6.5 \\ 2.2 & -4.0 & -1.3 & 0 & 0.2 & 1.1 & 0.1 & 0.2 \\ -2.2 & 7.8 & 3.2 & 0.3 & -7.2 & -2.3 & -0.9 & 1.3 \\ 2.4 & 5.1 & -0.2 & -0.9 & -4.0 & 2.0 & -2.8 & -2.0 \\ -1.2 & 6.0 & 2.2 & 0.2 & -6.2 & -0.2 & -1.0 & 0.2 \end{bmatrix}$$

$$B_1=\begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ B_{21}=\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix},\ B_{22}=\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$C_1=\begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$C_{21}=\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

$$C_{22}=\begin{bmatrix} -2 & 1 & 3 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}$$

$$D_{11}=0_{4\times4},\ D_{121}=\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}},\ D_{122}=\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$D_{211}=\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},\ D_{212}=\begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}$$

and the uncertainties are defined by

$$E_1=\begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.1 & 0.1 & 0.4 & 0.7 & 0.1 \end{bmatrix}^{\mathrm{T}}$$

$$F_1=\begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.2 & 0.4 & 0.1 & 0.6 & 0.1 \end{bmatrix}$$

$$F_2=\begin{bmatrix} 0.3 & 0.2 \end{bmatrix}.$$

We set the disturbance attenuation level to be achieved as 2.9.

On the first stage, we consider the case of no uncertainty. We obtain the initial value for the homotopy method of the first stage by solving a centralized H$_\infty$ control problem for the nominal system as

$$G_F = \left[\begin{array}{cccccccc|cc} -2.95 & -2.42 & -1.68 & -1.34 & 2.11 & -2.46 & 1.62 & -1.77 & -0.22 & 0.73 \\ -1.03 & -2.46 & 0.14 & 1.14 & -0.10 & -1.76 & -2.28 & 1.30 & 0.05 & 0.19 \\ -1.04 & -1.47 & -3.02 & -0.35 & 0.45 & -1.74 & -2.15 & -0.38 & -0.38 & 0.85 \\ 1.13 & 1.38 & 2.04 & -2.80 & -1.53 & 4.21 & 0.15 & -0.65 & 0.21 & -1.27 \\ 0.63 & 2.46 & -0.88 & 2.57 & -3.75 & 2.25 & -4.16 & 2.77 & 2.60 & -0.34 \\ -0.49 & -0.25 & -0.88 & 3.08 & 1.98 & -7.27 & 1.60 & 2.52 & 1.05 & 1.79 \\ 0.39 & 0.80 & 3.21 & -0.04 & -1.56 & 3.04 & -3.12 & -1.19 & 0.74 & -1.13 \\ -0.10 & 0.60 & 1.36 & -1.57 & -2.32 & 2.07 & -0.83 & -3.15 & -1.70 & -0.95 \\ \hline -1.60 & -0.14 & 0.33 & 0.30 & 0.63 & 0.23 & -0.6 & -2.35 & 0 & 0 \\ -0.50 & -0.76 & 0.70 & 0.29 & 0.65 & -1.28 & 0.08 & -0.16 & 0 & 0 \end{array}\right]$$

This centralized H$_\infty$ controller achieves the disturbance attenuation level 2.23 for the nominal system. Then, for the nominal system, we design a decentralized H$_\infty$ controller composed of two local controllers (4) whose dimensions are $\hat{n}_1 = 2$ and $\hat{n}_2 = 3$. We obtain the coefficient matrices of the decentralized H$_\infty$ controller by using the homotopy method of Zhai, *et al.* (2001) with M=128 as

$$\hat{A}_1 = \begin{bmatrix} 2.18 & 3.72 \\ -5.58 & -6.46 \end{bmatrix}, \ \hat{B}_1 = \begin{bmatrix} 0.94 \\ -0.05 \end{bmatrix}$$

$$\hat{C}_1 = \begin{bmatrix} 7.69 & 9.53 \end{bmatrix}$$

$$\hat{A}_2 = \begin{bmatrix} -15.97 & -19.38 & -5.25 \\ -3.62 & -14.60 & 9.08 \\ 0.91 & -0.75 & 0.65 \end{bmatrix}, \ \hat{B}_2 = \begin{bmatrix} -0.11 \\ 0.02 \\ -0.05 \end{bmatrix}$$

$$\hat{C}_2 = \begin{bmatrix} 3.59 & 8.30 & -1.51 \end{bmatrix}.$$

The disturbance attenuation level achieved by this controller for the nominal system is 2.33.

Next, by taking the above decentralized controller as the initial value for the homotopy method of the second stage, a robust decentralized H$_\infty$ controller is computed. With M=64 in the proposed algorithm, we obtain the coefficient matrices

$$\hat{A}_1 = \begin{bmatrix} -0.61 & 1.27 \\ -20.74 & -23.00 \end{bmatrix}, \ \hat{B}_1 = \begin{bmatrix} 1.16 \\ 0.42 \end{bmatrix}$$

$$\hat{C}_1 = \begin{bmatrix} 24.44 & 28.67 \end{bmatrix}$$

$$\hat{A}_2 = \begin{bmatrix} -30.56 & -18.75 & 19.36 \\ -19.32 & -45.29 & 11.44 \\ -20.54 & 11.85 & -47.83 \end{bmatrix}, \ \hat{B}_2 = \begin{bmatrix} -0.07 \\ -0.23 \\ -0.54 \end{bmatrix}$$

$$\hat{C}_2 = \begin{bmatrix} 6.18 & 2.08 & 8.77 \end{bmatrix}.$$

The disturbance attenuation level achieved by this controller is 2.58.

6. CONCLUSION

This paper has considered a robust decentralized H$_\infty$ control problem for uncertain multi-channel systems. The uncertainties are assumed to be time-invariant, norm-bounded, and exist in both the system and control input matrices. A necessary and sufficient condition for the uncertain multi-channel system to be robustly stabilizable with a specified disturbance attenuation level has been derived based on the bounded real lemma. A two-stage design method based on the idea of homotopy has been employed, where a decentralized controller for the nominal system with no uncertainty is computed first by imposing structural constraints on the coefficient matrices gradually. Then, the decentralized controller is modified, again gradually, to cope with the uncertainties.

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REFERENCES

Boyd, S., R. E. El. Ghaoui, R. E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM studies in Applied Mathematics. Philadelphia PA: SIAM.

Chen, N., M. Ikeda, W. Gui (2004). Robust decentralized H$_\infty$ control of interconnected systems: A design method using homotopy. *Preprints of The 10th IFAC/IFORS/IMACS/IFIP Symposium on Large Scale Systems: Theory and Applications*, Osaka, Japan, pp.330-336.

Gahinet, P., and P. Apkarian (1994). A linear matrix inequality approach to H$_\infty$ control. *International Journal of Robust and Nonlinear Control*, **4**, 421-448.

Iwasaki, T., and R. E. Skelton (1994). All controllers for the general H$_\infty$ control problem: LMI existence conditions and state space formulas. *Automatica*, **30**, 1307-1317.

Mehendale, C.S., and K.M. Grigoriadis (2003). A homotopy method for decentralized control design. *Proceedings of the American Control Conference*. Denver, Colorado, pp. 5023-5027.

Petersen I.R. (1987). A stabilization algorithm for a class of uncertain linear systems. *Systems & Control Letters*, **8**, 351-357.

Richter, S., and R. DeCarlo (1983). Continuation methods: Theory and applications. *IEEE Transactions on Automatic Control*, **28**, 660-665.

Richter, S., and R. DeCarlo (1984). A homotopy method for eigenvalue assignment using decentralized state feedback. *IEEE Transactions on Automatic Control*, **29**, 148-158.

Zhai, G., M. Ikeda, and Y. Fujisaki (2001). Decentralized H$_\infty$ controller design: A matrix inequality approach using a homotopy method. *Automatica*, **37**, 565-573.