PATH-FOLLOWING FOR LINEAR SYSTEMS WITH UNSTABLE ZERO DYNAMICS

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Abstract: A path-following problem for linear systems with unstable zero dynamics is solved. While the original control variable steers the system output along the path, the path parameter θ is used as an additional control to stabilize zero dynamics with a feedback law which is nonlinear due to the path constraint. A sufficient condition for solvability of the path-following problem is given in terms of the geometric properties of the path. When this condition is satisfied, an arbitrary small $\mathcal{L}_2$ norm of path-following error can be achieved. *Copyright© 2005 IFAC.*

Keywords: Path-following, reference tracking, unstable zero dynamics

1. INTRODUCTION

A series of results, from the classical Bode integrals to recent cheap control formulae (Qiu and Davison, 1993; Seron *et al.*, 1999), show that the main obstacle to *perfect* tracking of reference signals is the instability of the zero dynamics. A possibility to avoid this obstacle is to change the input-output structure of the system by redefining the system output (Fliess *et al.*, 1998). In (Aguiar *et al.*, 2004) and in this paper, the input structure is changed by reformulating the problem as *path-following*, rather than *reference tracking*.

Several path-following problems have recently been formulated to replace the standard reference tracking problem as more suitable for certain applications (Hauser and Hindman, 1995; Skjetne *et al.*, 2002; Skjetne *et al.*, 2004; Al-Hiddabi and McClamroch, 2002). In these formulations the task is to follow a geometric path $\mathcal{Y}_d$, parameterized by a scalar θ. The path-following problem is first solved with respect to θ, leaving the choice of θ as a function of time and system state as an additional degree of freedom.

In this paper, the freedom to select θ is used to avoid the limitation imposed by unstable zero dynamics on perfect tracking. Our approach, which for the first time appears here and in the companion papers (Aguiar *et al.*, 2004), is to construct a feedback law for θ to stabilize the zero dynamics and a feedback law for the original control variable to keep the system output on the path. This decomposition allows that an arbitrary small $\mathcal{L}_2$ norm of the path-following error be achieved.

In Section 2 we define the path-following problem, while in Sections 3 and 4 we construct its solution for paths characterized by a geometric condition, Theorem 1. In Section 5 we extend Theorem 1 in several directions and consider the best achievable performance of the resulting feedback laws. We illustrate our design on an example in Section 6, and give concluding remarks in Section 7.

2. PROBLEM STATEMENT

Definition 1. A path $\mathcal{Y}_d$ is one dimensional manifold $\mathcal{Y}_d := \{y_d(\theta) = [y_{d1}(\theta), \ldots, y_{dm}(\theta)] \in I\!R^m : \theta \in I\!R^+\}$, where $y_{di} : I\!R^+ \to I\!R$ are bounded maps, $y_{di}(\cdot) \in C^{k_i}$, $k_i \geq 1$, $i = \overline{1,m}$. A repeatable path $\mathcal{V}$ is defined by $\mathcal{V} := \{x \in \mathcal{Y}_d : \ \forall \theta_1 > 0, \ \exists \theta_2 > \theta_1, \ y_d(\theta_2) = x\}$. □

[1] Supported by National Science Foundation under Grant No. ECS-9812346.

The *path-following problem* for a controllable linear time-invariant systems with uniform vector relative degree r,

$$\begin{aligned} \dot{x} &= Ax + Bu, \ x \in R^n, \\ y &= Cx, \qquad\quad u, y \in R^m, \end{aligned} \tag{1}$$

and the path $\mathcal{Y}_d$ with $k_i \geq r$, is to construct feedback laws for u and for $\theta^{(r)}$, the r^{th} derivative of θ, such that the closed-loop solutions starting in a set $\mathcal{S}_s \subseteq I\!R^n$ with nonempty interior, $\text{int}\{\mathcal{S}_s\} \neq \emptyset$, satisfy the following requirements:

R1 *Asymptotic convergence to the path:* $y(t) \to y_d(\theta(t))$ as $t \to \infty$,
R2 *Forward motion along the path:* $\dot{\theta}(t) \geq 0$ and $\lim_{t\to\infty} \theta(t) = \infty$,
R3 *Boundedness of all states.*

The requirement $\lim_{t\to\infty} \theta(t) = \infty$ is introduced to disqualify controllers which allow output of (1) to converge to a point on $\mathcal{Y}_d$. The requirement $\dot{\theta}(t) \geq 0$ prevents backward motion along $\mathcal{Y}_d$.

A key feature of the above path-following problem is the use of an additional feedback law for $\theta^{(r)}$. This design freedom is absent from the standard reference tracking problem, in which $\theta = t$. In this paper we develop two designs, for SISO systems in Section 3 and for MIMO systems in Section 4, which solve the path-following problem despite instability of the zero dynamics. These designs are applicable to a class of paths satisfying geometric conditions in Theorem 1 and Corollary 1.

3. SISO DESIGN

The starting point of the design procedure is the normal form of system (1)

$$\dot{z} = A_z z + B_z y, \tag{2}$$

$$\dot{\eta} = A_c \eta + B_c u, \tag{3}$$

$$y = C_c \eta, \tag{4}$$

where (2) represents the zero dynamics driven by the output y, (3) represents m chains of r integrators and I_n is the identity matrix of dimension n

$$A_c = \begin{bmatrix} 0 & I_{m(r-1)} \\ 0 & 0 \end{bmatrix}, \quad B_c^T = \begin{bmatrix} 0 & \dots & 0 & I_m \end{bmatrix}, \quad C_c = \begin{bmatrix} I_m & 0 & \dots & 0 \end{bmatrix}.$$

The controllability of the pair (A_z, B_z) is implied by controllability of the pair (A, B). As shown in (Sannuti and Saberi, 1987), system (1) with uniform vector relative degree r can always be transformed into this form, exhibiting the transmission zeros of (1) as the eigenvalues of A_z. To focus on systems with unstable zero dynamics we let all the eigenvalues of A_z be unstable, $\text{Re}\,\lambda(A_z) > 0$.

Rewriting (2)-(4) in the error coordinates

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_r \end{bmatrix} := \eta - Y_d := \begin{bmatrix} \eta_1 - y_d(\theta) \\ \eta_2 - \dot{y}_d(\theta) \\ \vdots \\ \eta_r - y_d^{(r-1)}(\theta) \end{bmatrix}. \tag{5}$$

we note that $\dot{y}_d(\theta) = \frac{\partial y_d}{\partial \theta}\dot{\theta}$, $\ddot{y}_d(\theta) = \frac{\partial^2 y_d}{\partial \theta^2}\dot{\theta}^2 + \frac{\partial y_d}{\partial \theta}\ddot{\theta}$, etc. give rise to r new states $\theta, \dots, \theta^{(r-1)}$. The key observation is that $\theta^{(r)}$ is available as an additional control input ω in the augmented system

$$\dot{z} = A_z z + B_z(e_1 + y_d(\theta)), \tag{6}$$

$$\dot{e} = A_c e + B_c(u - y_d^{(r)}(\theta)), \tag{7}$$

$$\theta^{(r)} = \omega. \tag{8}$$

Zero dynamics (6) have two inputs, $y_d(\theta)$ and e_1. While e_1 is treated as a disturbance which is to be suppressed by u, $y_d(\theta)$ is treated as control which is to achieve boundedness of zero dynamics (6) via a feedback law for $\theta^{(r)} = \omega$. Because θ enters zero dynamics (6) through a bounded function $y_d(\cdot)$, the augmented system is *nonlinear* and our ability to design a path-following controller depends upon geometric properties of the path $\mathcal{Y}_d$.

Our approach is to decouple the task **R3** of keeping $z(t)$ bounded from the path-following task **R1**. **R3** is to be accomplished by designing a feedback law for $\theta^{(r)} = \omega$ in (8) leaving the original control variable u in (7) to be used for **R1**. Due to boundedness of $y_d(\cdot)$ and $\text{Re}\,\lambda(A_z) > 0$, the feedback law for $\theta^{(r)} = \omega$ can only achieve **R3** in a compact set.

Our design starts with the sampled-data version of zero dynamics (6). We let $\theta(t) = \hat{\theta}(t) + \widetilde{\theta}(t)$, where $\hat{\theta}(t) = \hat{\theta}_k$, $t \in [kT, (k+1)T)$, while $\widetilde{\theta}(t)$ is the intersample correction to be designed. The resulting sampled-data equivalent of the zero dynamics is

$$z_{k+1} = A_{zD} z_k + B_{zD} v_k + d_{ek} + d_{sk}, \tag{9}$$

where $T > 0$ is sampling period, $z_k := z(kT)$, $A_{zD} = e^{A_z T}$, $B_{zD} = \int_0^T e^{A_z s} ds B_z$, $v_k := y_d(\hat{\theta}_k)$ and d_{ek}, d_{sk} are disturbances due to $e_1(t)$ and $\widetilde{\theta}(t)$, respectively.

$$d_{ek} = \int_0^T e^{A_z(T-s)} B_z e_1(s + kT) ds \tag{10}$$

$$d_{sk} = \int_0^T e^{A_z(T-s)} B_z \left[y_d(\theta(s + kT)) - v_k \right] ds$$

The repeatable path $\mathcal{V}$ and the discrete-time equivalent of zero dynamics (9) are the main ingredients in stabilization of a nonlinear subsystem (6), (8), subject to **R2**, with a feedback law for $\theta^{(r)} = \omega$. The challenge here is the stabilization of linear system (9) in the presence of disturbances d_{ek}, d_{sk} and the control constraint $v_k \in \mathcal{V}$. Once this problem is solved, the construction of the path-following controller is straightforward.

Our design is in four steps. First, we construct a piece-wise constant feedback law $\hat{\theta}_k = p(z_k)$ for (9) to ensure boundedness of z_k despite the presence of bounded disturbances d_{ek}, d_{sk} with an estimated bound $d^\star$. In the second step, the inter-sample correction $\widetilde{\theta}(t)$ is designed so that $s(z_k, t) := p(z_k) + \widetilde{\theta}(t)$ is a C^r function as required for the implementation of $\theta^{(r)} = \omega$. This inter-sample correction acts as the disturbance d_{sk} and is to be bounded by $\frac{d^\star}{3}$. In the third step, we explicitly assign the r^{th} derivative of $s(z_k,t)$ to ω in $\theta^{(r)}=\omega$. Finally, a feedback law for u in (7) is designed such that the resulting $\|d_{ek}\|$ is also bounded by $\frac{d^\star}{3}$.

Step 1: In the SISO case both $\mathcal{Y}_d$ and $\mathcal{V}$ are convex, compact sets given by $\mathcal{Y}_d=[\underline{y}_d,\overline{y}_d]$, $\mathcal{V}=[\underline{v},\overline{v}]$, $\underline{y}_d\le\underline{v}\le\overline{v}\le\overline{y}_d$. Under the condition

$$\underline{v}<0<\overline{v} \tag{11}$$

there exist several procedures in the literature, see for example (Marruedo *et al.*, 2002) and references therein, to construct a feedback law $v_k=\sigma(z_k)$, constrained by $\sigma(z_k)\in\mathcal{V}$, which renders a neighborhood $\mathcal{S}$ of $z=0$ forward invariant for system (9). Moreover, a bound $d^\star$ can be computed to guarantee forward invariance of $\mathcal{S}$ for all disturbances d_{sk}, d_{ek} satisfying $\|d_{ek}\|+\|d_{sk}\|\le d^\star$.

Assuming that such a feedback law $v_k=\sigma(z_k)$ and the corresponding bound $d^\star$ are available, $\hat{\theta}_k=p(z_k)$ is computed from

$$p(z_k)=\min_{s>p(z_{k-1})+\theta_{min}}\{s: y_d(s)=\sigma(z_k)\}. \tag{12}$$

The constraint $\sigma(z_k)\in\mathcal{V}$ ensures that $p(z_k)$ is well defined $\forall k\ge 0$, while $p(z_k)>p(z_{k-1})+\theta_{min}$, $\theta_{min}>0$, ensures that $\{p(z_k)\}_{k=0}^\infty$ is an increasing and divergent sequence, see **R2**.

We note that condition (11) may be relaxed by allowing that $\sigma(z_k)\in\mathcal{V}$ makes forward invariant a neighborhood of any other control induced equilibrium $z=z^\star$ for system (9), see Corollary 1.

Step 2: We define $\widetilde{\theta}:[kT,(k+1)T)\to I\!R^+$, by

$$\widetilde{\theta}(t)=\begin{cases}\widetilde{\theta}_k(t), & t-kT\in[0,T^\star),\\ 0, & t-kT\in[T^\star,T),\end{cases} \tag{13}$$

where $T^\star$ is determined to guarantee $\|d_{sk}\|\le\frac{d^\star}{3}$ for any $\widetilde{\theta}_k(\cdot)\in C^r$. From (11) and (13), it can be shown that

$$T^\star\le-\frac{1}{\mu}ln(1-\frac{\mu d^\star}{3q_1}e^{-\mu T}), \tag{14}$$

implies $\|d_{sk}\|\le\frac{d^\star}{3}$, where $q_1=2c_1M\|B_z\|$, $M=\sup_{\theta\in I\!R^+}\|y_d(\theta)\|$ and $\|e^{A_z t}\|\le c_1e^{\mu t}$. The C^r increasing function $\widetilde{\theta}_k:[kT,kT+T^\star)\to I\!R^+$ provides a smooth transition from $p(z_{k-1})$ to $p(z_k)$, that is,

$$\lim_{s\to kT^+}\left[\,\widetilde{\theta}_k(s)\;\dot{\widetilde{\theta}}_k(s)\;\ldots\;\widetilde{\theta}_k^{(r)}(s)\,\right]^T=\left[\,p(z_{k-1})-p(z_k)\;0\;\ldots\;0\,\right]^T,$$

$$\lim_{s\to(kT+T^\star)^-}\left[\,\widetilde{\theta}_k(s)\;\dot{\widetilde{\theta}}_k(s)\;\ldots\;\widetilde{\theta}_k^{(r)}(s)\,\right]^T=\left[\,0\;0\;\ldots\;0\,\right]^T.$$

Thus, the feedback law $\theta=s(z_k,t)=p(z_k)+\widetilde{\theta}(t)$, $t-kT\in[0,T)$ satisfies **R2** by construction.

Step 3: To construct $\omega(t)$ in (8) we use

$$\omega=\frac{d^r}{dt^r}s(z_k,t). \tag{15}$$

because we have an explicit formula for $s(z_k,t)$ which can be differentiated symbolically.

Step 4: To ensure **R1**, we asymptotically stabilize subsystem (7) by setting,

$$u=y_d^{(r)}(\theta)-K_ce \tag{16}$$

where $K_c\in R^{m\times rm}$ is chosen such that $A_c-B_cK_c+\alpha I$, $\alpha>0$, is Hurwitz. We select α to satisfy

$$\alpha+\mu>\frac{3q_2}{d^\star}e^{\mu T}\|e(0)\|, \tag{17}$$

where $\|e^{(A_c-K_cB_c)t}\|\le c_2e^{-\alpha t}$ and $q_2=c_1c_2\|B_zC_c\|$, which implies that $\|d_{ek}\|\le\frac{d^\star}{3}$, $\forall k\ge 0$.

Combining (15), (16) and (17), we get that $\|d_{ek}\|+\|d_{sk}\|\le d^\star$, $\forall k\ge 0$, which guarantees boundedness of $z(t)$, $\forall t\ge 0$. A set of initial conditions for which **R1-R3** hold is given by $\mathcal{S}_s:=\mathcal{S}\times\mathcal{B}_\alpha$, where $\mathcal{B}_\alpha:=\{e\in I\!R^{rm}:\|e\|\le\frac{(\alpha+\mu)d^\star}{3q_2}e^{-\mu T}\}$. Moreover, for a sufficiently large α any compact neighborhood of $e=0$ can be contained in $\mathcal{B}_\alpha$. $\Box$

4. MIMO DESIGN

MIMO design differs only in the first two steps from SISO design due to the different properties of the repeatable path $\mathcal{V}$. In the SISO case, $\mathcal{V}=[\underline{v},\overline{v}]$ is a convex set whose interior $\mathrm{int}\{\mathcal{V}\}\neq\emptyset$ if $\underline{v}\neq\overline{v}$. In the MIMO case, $\mathcal{V}$ is not convex, and its interior is always empty, $\mathrm{int}\{\mathcal{V}\}=\emptyset$.

We briefly summarize the first two steps of the SISO design. The constructed $\theta=p(z_k)+\widetilde{\theta}(t)$, $t\in[kT,(k+1)T)$, consists of the interval $t\in[kT,kT+T^\star)$ in which θ is to provide a smooth transition from $p(z_{k-1})$ to $p(z_k)$ and the interval during which $\theta=p(z_k)$ is to stabilize the zero dynamics. The smoothing inter-sample correction $\widetilde{\theta}(t)$ is accounted for as a disturbance whose size is proportional to the length of the transition $T^\star$. To preserve the stabilizing effect of $p(z_k)$ it is beneficial to use small $T^\star$, but short transitions result in fast changes of $\theta(t)$ and large magnitudes of control signals (16), making the motion of (4) along $\mathcal{Y}_d$ jittery. These undesirable features are due to the piece-wise constant form of $\hat{\theta}=p(z_k)$ induced by (12). To decrease the control effort and obtain 'smoother' motion along $\mathcal{Y}_d$ the construction of the stabilizing part $\hat{\theta}$ is changed in the MIMO design.

Step 1: We first replace the actual input constraint $\mathcal{V}$, by its convex hull, $\mathrm{con}\,\mathcal{V}$, so that under the condition $0\in\mathrm{int}\{\mathrm{con}\,\mathcal{V}\}$ there exist several procedures for design of a feedback law $v_k=\sigma(z_k)$, $\sigma(z_k)\in\mathrm{con}\,\mathcal{V}$, which render a neighborhood $\mathcal{S}$ of $z=0$ forward invariant for system (9). Moreover, a bound $d^\star$ can be computed to guarantee forward invariance of $\mathcal{S}$ for all disturbances d_{ek}, d_{sk} satisfying $\|d_{ek}\|+\|d_{sk}\|\le d^\star$, (Marruedo *et al.*, 2002).

Then we decompose $\theta(t)$, $t\in[kT,(k+1)T)$, into $\theta(t)=\hat{\theta}(t)+\widetilde{\theta}(t)$, where $\hat{\theta}(t)$ is a piece-wise linear, *continuous*, nondecreasing function of time

$$\hat{\theta}(t)=d_{ki}t+n_{ki},\; t\in\left[\tau_{ki},\tau_{k(i+1)}\right), \tag{18}$$

$$\lim_{t\to\tau_{ki}^-}\hat{\theta}(t)=\lim_{t\to\tau_{ki}^+}\hat{\theta}(t),\;\; i=0,N_k, \tag{19}$$

with $\tau_{k0} = kT$, $\tau_{kN_k} = (k+1)T$ and N_k, $\mathcal{P}_{kN_k} \in \mathbb{P}^c_{kN_k} := \{[d, n, \tau] : d \in \mathbb{D}_{kN_k}, n \in I\!R^{N_k}, \tau \in \mathbb{T}_{kN_k}\}$,

$\mathbb{D}_{kN_k} := \{x \in I\!R^{N_k} : x_i \geq 0\}$,
$\mathbb{T}_{kN_k} := \{x \in I\!R^{N_k - 1} : x_1 \geq kT,$
$x_{N_k-1} \leq (k+1)T,\ \ x_i \geq x_{i-1},\ i = 2, N_k - 1\}$,

are unknown parameters. At $t = kT$, N_k and $\mathcal{P}_{kN_k}$ are computed such that $\hat{\theta}(t)$ satisfies (19), while requiring that the solution of (6) with $\theta = \hat{\theta}(t)$ and the solution of (9) with stabilizing feedback law $v_k = \sigma(z_k)$, starting from the same initial condition $z(kT) = z_k$, are sufficiently close at $t = (k+1)T$. In other words, N_K and $\mathcal{P}_{kN_k} \in \mathbb{P}^c_{kN_k}$ are computed such that (19) holds and

$$E^c(\mathcal{P}_{kN_k}) := \| \int_0^T e^{A_z(T-s)} B_z y_d(\hat{\theta}(s + kT))ds \\ - B_{zD}\sigma(z_k)\| \leq \frac{d^\star}{3}. \tag{20}$$

Lemma 1. For any $d^\star > 0$ and any $\sigma \in \mathrm{con}\,\mathcal{V}$, there exist parameters N_k and $\mathcal{P}_{kN_k} \in \mathbb{P}^c_{kN_k}$ such that $E^c(\mathcal{P}_{kN_k}) \leq \frac{d^\star}{3}$. ■

The existence of N_k and $\mathcal{P}_{kN_k} \in \mathbb{P}^c_{kN_k}$ satisfying (19), (20) is established in Lemma 1 by explicitly computing them. Its proof is omitted due to the space limitation. However, the computed slopes of $\hat{\theta}$, d_{ki}, may be unnecessarily steep, leading to the closed-loop behavior similar to the one obtained with discontinuous $\hat{\theta}$, see (12). This is why Lemma 1 only serves as an existence result, while the actual computation is performed as follows.

Continuous Spanning Algorithm

S1: Set $N_k = 1$ and select $\omega_0 > 0$.

S2: Let $N_k := N_k + 1$ and compute

$$\min_{\mathcal{P}_{kN_k} \in \mathbb{P}^c_{kN_k}} J^c_{kN_k}(\mathcal{P}_{kN_k}) := \min_{\mathcal{P}_{kN_k} \in \mathbb{P}^c_{kN_k}} (\hat{\theta}((k+1)T) - \theta(kT) - \omega_0 T)^2$$

subject to $E^c(\mathcal{P}_{kN_k}) \leq \frac{d^\star}{3}$, $\hat{\theta}((k+1)T) - \theta(kT) \geq \theta_{min}$ and (19).

S3: If the above minimization has a feasible solution, set $\mathcal{P}^\star_{kN_k} := \arg\min J^c_k(\mathcal{P}_{kN_k})$, $N^\star_k := N_k$. Otherwise, go to *S2*.

Lemma 1 ensures that the minimization in *S2* has a feasible solution for a finite N_k. The computed parameters $\mathcal{P}^\star_{kN^\star_k}$ define the feedback law for $\hat{\theta}$ by

$$\hat{\theta} = p(z_k, t) = d^\star_{ki} t + n^\star_{ki},\ t \in [\tau^\star_{k(i-1)}, \tau^\star_{ki}),$$

where $i = 1, N^\star_k$, $\tau^\star_{k0} = kT$ and $\tau^\star_{kN^\star_{k+1}} = (k+1)T$.

Since $\omega_0 > 0$ represents the desired velocity of (4) along $\mathcal{Y}_d$, the minimization of $J^c_{kN_k}$ is to maintain the desired average velocity during each sample interval. Numerical difficulties may arise in *S2*, but in many cases it is possible to provide a good initial guess of parameters N_k and $\mathcal{P}_{kN_k}$ based on $\sigma(z_k)$ and $\theta(kT)$ to significantly reduce the computational effort.

Step 2: The smoothing correction $\widetilde{\theta}$ is defined in the same spirit as in the SISO design

$$\widetilde{\theta} = \begin{cases} \widetilde{\theta}_{ki}(t),\ t \in [\underline{\tau}^\star_{ki}, \overline{\tau}^\star_{ki}),\ \ i = 0, N^\star_k, \\ 0, \qquad \forall i = 0, N^\star_k,\ \ t \notin [\underline{\tau}^\star_{ki}, \overline{\tau}^\star_{ki}), \end{cases}$$

where $T^\star$ satisfies (14) and $\underline{\tau}^\star_{k0} = kT$, $\underline{\tau}^\star_{ki} = \tau^\star_{ki} - \frac{T^\star}{2(N^\star_k+1)}$, $i = 1, N^\star_k - 1$, $\underline{\tau}^\star_{kN_k} = T - \frac{T^\star}{(N^\star_k+1)}$, $\overline{\tau}^\star_{ki} = \underline{\tau}^\star_{ki} + \frac{T^\star}{(N^\star_k+1)}$, $i = 0, N^\star_k$. C^0 functions $\widetilde{\theta}_{ki}(\cdot)$ are chosen to make the resulting $\theta(t) = p(z_k, t) + \widetilde{\theta}(t)$ nondecreasing and C^r. Since the length of the interval on which $\widetilde{\theta}(t) \neq 0$, $t \in [kT, (k+1)T)$, is $T^\star$, it follows that $\|d_{sk}\| \leq \frac{d^\star}{3}$. □

The designs in Sections 3 and 4 provide a constructive proof for the following Theorem.

Theorem 1. The path-following problem for system (1) and path $\mathcal{Y}_d$ is solvable if $0 \in \mathrm{int}\{\mathrm{con}\,\mathcal{V}\}$. ■

5. EXTENSIONS

In this section we first relax the geometric condition on the path $\mathcal{Y}_d$ and replace the assumption that (1) be a square system with uniform vector relative degree r by assumption that (1) be right invertible. We then modify the continuous spanning algorithm to decrease computational effort and examine the performance of the path-following controllers.

5.1 Relaxed Condition on the Path $\mathcal{Y}_d$

To keep solutions $z(t)$ of (6) bounded it is sufficient that a feedback law for $\theta^{(r)} = \omega$ makes forward invariant a neighborhood $\mathcal{S}$ of any control induced equilibrium $z = z^\star$. With $\mathcal{V}_{B_z} := \{B_z v : v \in \mathrm{int}\{\mathrm{con}\,\mathcal{V}\}\}$ and $\mathcal{R}(A_z) := \{A_z z : z \in I\!R^{n-rm}\}$ we have the following result.

Corollary 1. The path-following problem for system (1) and a path $\mathcal{Y}_d$ is solvable if $\mathcal{R}(A_z) \cap \mathcal{U}_{B_z} \neq \emptyset$. ■

The proof is omitted due to the space limitation.

5.2 Class of Linear Systems

System (1) was assumed to be square and with uniform vector relative degree to ensure that it can be transformed into the form (2)-(4), where the zero dynamics are driven by the output *only*, and not by any of its derivatives. Now, we consider a *right invertible* system

$$\begin{aligned} \dot{x} &= \overline{A}x + \overline{B}u,\ x \in R^n,\ \ u \in I\!R^p \\ y &= \overline{C}x, \qquad\quad y \in I\!R^m,\ \ p \geq m. \end{aligned} \tag{21}$$

From Proposition 4 in (Saberi *et al.*, 1990), we conclude that there exists a precompensator

$$\begin{aligned} \dot{q} &= A_{pc}q + B_{pc}\overline{u},\ q \in R^{\overline{n}}, \\ u &= C_{pc}q, \qquad\qquad \overline{u} \in I\!R^m, \end{aligned} \tag{22}$$

such that cascaded system (22), (21) with input $\overline{u}$ and output y is square and has a uniform vector relative degree. Applying Theorem 1 to this cascade we obtain the following Corollary.

Corollary 2. The path-following problem for a right invertible system (21) and a path $\mathcal{Y}_d$ is solvable if $0 \in \mathrm{int}\{\mathrm{con}\,\mathcal{V}\}$. ■

5.3 Decreasing the Computational Effort

Now we focus on the computation of $\hat{\theta}$. Parameters N_k and $\mathcal{P}_{kN_k}$ are computed to make the overall effect of $y_d(\hat{\theta}(t))$, $t \in [kT,(k+1)T)$, on (6) sufficiently similar to the effect of a stabilizing feedback law $v_k = \sigma(z_k)$ on sampled-data version of zero dynamics (9), see (20). However, to guarantee stability of zero dynamics (6), it is enough to compute N_k and $\mathcal{P}_{kN_k}$ such that the effect of $y_d(\hat{\theta}(t))$, $t \in [kT,(k+1)T)$,

$$\mathcal{M}(\mathcal{P}_{kN_k}) = \int_0^T e^{A_z(T-s)} B_z y_d(\hat{\theta}(s+kT))ds$$

makes increments of a Lyapunov function $V(z_k)$ negative along the closed-loop solutions of (6),

$$V(z_{k+1}) - V(z_k) \le -c\|z_k\|^2, \;\; c > 0. \qquad (23)$$

We make the following assumption.

Assumption 1. Under the feedback law $v_k = \sigma(z_k)$, constrained by $\sigma(z_k) \in \operatorname{con}\mathcal{V}$, $0 \in \operatorname{int}\{\operatorname{con}\mathcal{V}\}$, the difference of Lyapunov function $V = z^T P z$, $P = P^T > 0$ along solutions of $z_{k+1} = A_{zD} z_k + B_{zD}\sigma(z_k)$ satisfies (23) for all $z_k \in \Omega_{\overline{\rho}} := \{z \in I\!R^{n-rm} : V(z) \le \overline{\rho}\}$. □

Zero dynamics (6) will be stabilized by $\hat{\theta}(t)$ if

$$\begin{aligned} & z_k^T(A_{zD}^T P A_{zD} + cI - P)z_k + 2\mathcal{M}(\mathcal{P}_{kN_k})^T P A_{zD} z_k \\ & + \mathcal{M}(\mathcal{P}_{kN_k})^T P \mathcal{M}(\mathcal{P}_{kN_k}) \le 0 \end{aligned} \qquad (24)$$

holds for all $k \ge 0$. We use (24) to replace the constraint (20) in *S2* of the continuous spanning algorithm. The existence of parameters N_k and $\mathcal{P}_{kN_k}$ for which $\mathcal{M}(\mathcal{P}_{kN_k})$ satisfies (24) follows directly from Lemma 1. In our experience, this modification significantly reduces the computational effort of the continuous spanning algorithm because $S_1^c := \{\mathcal{P}_{kN_k} \in \mathbb{P}^c_{kN_k} : E^c(\mathcal{P}_{kN_k}) \le \frac{d^\star}{3}\}$ is contained in $S_2^c := \{\mathcal{P}_{kN_k} \in \mathbb{P}^c_{kN_k} : \mathcal{M}(\mathcal{P}_{kN_k})$ satisfies (24)$\}$, $S_1^c \subseteq S_2^c$.

5.4 $\mathcal{L}_2$ Performance

It is of theoretical interest to determine the best performance

$$\mathcal{I} := \int_0^\infty \|y(t) - y_d(\theta(t))\|^2 dt, \qquad (25)$$

achievable with a path-following controller. As shown in (Qiu and Davison, 1993; Seron *et al.*, 1999) for tracking of constant references, this performance is limited by the effort of $y(t)$ required to stabilize the unstable zero dynamics. In contrast, if the path-following problem is solvable such limitation is overcome by a path-following controller.

Corollary 3. Let the path-following problem be solvable for system (1) and a path $\mathcal{Y}_d$. Then for any $\epsilon > 0$, there exists a path-following controller which achieves $\mathcal{I} \le \epsilon$. ■

Corollary 3 is a consequence of two facts. First, for any $\epsilon > 0$ and any initial condition $e(0)$ there exist $\alpha > 0$ in (16) for which the solution of (7), (16) satisfies $\mathcal{I} \le \epsilon$. Moreover, such choice of (16) is beneficial for the stability of the zero dynamics, because faster convergence of $e(t)$ reduces the disturbing effect of d_{ek} on (9).

6. EXAMPLE

The system consisting of a mass $\mathcal{X}$ moving in a plane, and carrying a mass $\mathcal{Z}$ is modelled by

$$\ddot{x} = 2H(\dot{z} - \dot{x}) + f, \qquad x, f \in I\!R^2 \qquad (26)$$

$$\ddot{z} = 2H(\dot{x} - \dot{z}) + G(z - x), \;\; z \in I\!R^2 \qquad (27)$$

where $H = \operatorname{diag}(h_1, h_2) > 0$, $G = \operatorname{diag}(g_1, g_2) > 0$. Variables $x = [x_1, x_2]^T$ and $z = [z_1, z_2]^T$ denote the positions of the masses. The bottom mass $\mathcal{X}$ is controlled by forces $f = [f_1, f_2]^T$, while the top mass $\mathcal{Z}$ is under the action of friction force $H(\dot{x} - \dot{z})$. Because the bottom mass $\mathcal{X}$ is convex, the top mass $\mathcal{Z}$ is acted upon by the gravity force $G(z-x)$ which drives it away from the equilibrium $z = x$. This leads to the unstable zero dynamics with respect to the output $y = x$.

Our objective is to move the mass $\mathcal{X}$, that is $y = x$, along a circular path $\mathcal{Y}_d$ of radius R,

$$y_d(\theta) = \left[\, R\sin\theta \;\; R\cos\theta \,\right]^T. \qquad (28)$$

Because $y_d(\theta)$ is periodic, we have $\mathcal{Y}_d = \mathcal{V} = \{u \in I\!R^2 : \|u\| = R\}$. For output $y = x$, system (26), (27) has uniform vector relative degree $r = 2$, and its normal form is

$$\ddot{y} = \overline{u}, \qquad y \in I\!R^2, \qquad (29)$$

$$\dot{\zeta}_s = A_s\zeta_s + B_s y, \quad \zeta_s \in I\!R^2, \qquad (30)$$

$$\dot{\zeta} = A\zeta + By, \qquad \zeta \in I\!R^2, \qquad (31)$$

where $A_s = \operatorname{diag}(\lambda_{s1}, \lambda_{2s})$, $B_s = \operatorname{diag}(b_{s1}, b_{s2})$, $A = \operatorname{diag}(\lambda_1, \lambda_2)$ and $B = \operatorname{diag}(b_1, b_2)$. Subsystem (30) is the stable, and subsystem (31) is the unstable part of the zero dynamics, that is, λ_{s1}, λ_{s2} are the stable and λ_1, λ_2 are the unstable transmission zeros of system (26), (27). For boundedness of $\zeta_s(t)$ it sufficient that $y(t)$ be bounded, thus we will ignore subsystem (30).

With the error coordinates $e := y - y_d(\theta)$ and $\widetilde{u} := \overline{u} - \frac{\partial y_d}{\partial\theta}\ddot{\theta} - \frac{\partial^2 y_d}{\partial\theta^2}\dot{\theta}^2$, we get

$$\begin{aligned} \dot{\zeta} &= A\zeta + B(y_d(\theta) + e_1), \\ \ddot{e} &= \widetilde{u}, \qquad\qquad \ddot{\theta} = \omega. \end{aligned} \qquad (32)$$

The sampled-data zero dynamics in (32) are

$$\zeta_{k+1} = A_D\zeta_k + B_D v_k + d_{ek} + d_{sk} \qquad (33)$$

where $A_D = \operatorname{diag}(\lambda_{D1}, \lambda_{D2})$, $B_D = \int_0^T e^{At}dtB = \operatorname{diag}(b_{D1}, b_{D2})$ and d_{ek}, d_{sk} are disturbances given by (10). It can be shown that the feedback law

$$\begin{aligned} \sigma(\zeta_k) &= \begin{cases} \overline{v}_k, & \|\overline{v}_k\| \le R \\ \dfrac{\overline{v}_k}{\|\overline{v}_k\|}R, & \|\overline{v}_k\| > R \end{cases} \\ \overline{v}_k &:= -B_D^{-1}\operatorname{diag}(\sqrt{\lambda_{D1}^2 - 1}, \sqrt{\lambda_{D2}^2 - 1})\zeta_k \end{aligned} \qquad (34)$$

renders the set $\mathcal{S} := \{\zeta \in I\!R^2 : \|B_D^{-1}\zeta\|^2 \le \frac{R^2}{\lambda_M^2 - 1}\}$ forward invariant for system (33) with input constraints $v_k \in \operatorname{con}\mathcal{V} = \{v \in I\!R^2 : \|v\| \le R\}$, as long as $\|d_{ek}\| + \|d_{sk}\| \le d^\star$,

$$d^\star = R\frac{b_{D1}b_{D2}}{\sqrt{b_{D1}^2+b_{D2}^2}}\frac{(\lambda_m-\sqrt{\lambda_m^2-1})\sqrt{\lambda_m^2-1}}{\lambda_M^2-1},$$

where $\lambda_M = \max\{\lambda_{D1},\lambda_{D2}\}$, $\lambda_m = \min\{\lambda_{D1},\lambda_{D2}\}$. The rest of the design exactly follows the procedure in Section 4.

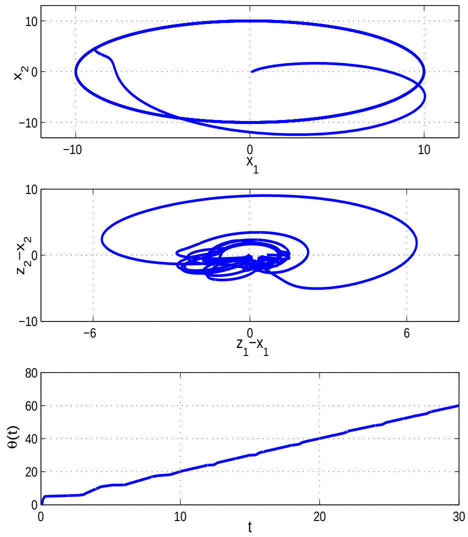

Fig. 1. Position of the lower mass, $x_1(t)$ versus $x_2(t)$ (top), difference in positions of the upper and the lower mass, $z_1(t) - x_1(t)$ versus $z_2(t) - x_2(t)$ (middle), resulting $\theta(t)$ (bottom).

A typical behavior of closed-loop system (26), (27) and (16), using the *continuous spanning* algorithm coupled with (34), is shown on Figs. 1 and 2 for initial conditions $x(0) = [0,0]^T$, $\dot{x}(0) = [0,0]^T$, $z(0) = [0,0]^T$, $\dot{z}(0) = [0,0]^T$, and $H = \mathrm{diag}(8,5)$, $G = \mathrm{diag}(3,2)$, $K_c = 15\left[\, I_2 \;\; I_2 \,\right]$, $T = 3$, $R = 10$.

We set the desired average velocity of the bottom mass $\mathcal{X}$ along $\mathcal{Y}_d$ to $\omega_0 = 2$. It is our experience that the constrained minimization in *S2* is feasible for $N_k^\star \le 3$, and that its solution $\mathcal{P}^\star_{kN_k^\star}$ satisfies $J^c_{kN_k^\star}(\mathcal{P}^\star_{kN_k^\star}) = 0$. Thus, the designed feedback law achieves that $\mathcal{X}$ follows the circular path (28) with strictly forward motion and the desired average velocity $\omega_0 > 0$, while keeping the difference in positions of $\mathcal{X}$ and $\mathcal{Z}$ bounded.

7. CONCLUSION

To the best of our knowledge, the idea to use the path parameter θ for stabilization of the unstable zero dynamics appears for the first time here and in the companion paper (Aguiar *et al.*, 2004). Taken together, the two papers demonstrate that the idea is feasible for linear systems with unstable zero dynamics and the paths characterized by the condition in this paper. Because the burden of keeping the zero dynamics bounded is transferred to θ, controlled via its r^{th} derivative, the control u can be designed to achieve arbitrary small $\mathcal{L}_2$ norm of the path-following error. The θ part of the path-following controller is nonlinear because of the path constraint and its complexity is due to the non-convex nature of the constraint. The proposed procedure solves this problem for a discrete-time version of the zero dynamics, with an additional inter-sample correction.

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