BACKSTEPPING DESIGN FOR ROBUST STABILIZING CONTROL OF NONLINEAR SYSTEMS WITH TIME-DELAY

Xiaohong Jiao*† Yuanzhang Sun*
and Tielong Shen**

** Department of Electrical Engineering, Tsinghua University, Beijing, China*
*** Department of Mechanical Engineering, Sophia University, Tokyo, Japan*
† Department of Electrical Engineering, Yanshan University, Qinhuangdao, China

Abstract: Backstepping design for robust stabilizing control of nonlinear systems with time-delay is investigated in this paper. It is shown that the backstepping design for time-delay systems based on the Lyapunov-Razumikhin function is not a trivial extension of the case of nonlinear nondelay systems due to the existence of the Razumikhin condition. The key point is how to deal with the non-triangular structure form of the system after application of the Razumikhin condition. A design technique to overcome the obstacle is given so that a delay-independent state feedback stabilizing control law can be explicitly obtained by step-by-step recursion based on the construction of the Lyapunov-Razumikhin function.

Keywords: Time-delay, nonlinear systems, robust stabilization, Lyapunov-Razumikhin function, backstepping

1. INTRODUCTION

The backstepping techniques have been recognized as a powerful design method for nonlinear control systems (Isidori, 1995). This method relies on the system structure, so-called triangular structure, and with the help of this structure, the Lyapunov function whose time derivative is rendered to be negative by a feedback controller can be recursively constructed in a step-by-step way. This systematical design method of nonlinear systems is one of the notable advances in control theory in the past two decades, and very recently the attempts have been naturally made to extend the backstepping design method to nonlinear time-delay systems (Nguang, 2000; Jankovic, 2001, 2003). For instance, the backstepping design method based on Lyapunov functional has firstly presented by Nguang (2000) for nonlinear systems with time-delay, however, as commented later by Zhou, *et al.* (2002), the proposed stabilizing control law in Nguang (2000) could not be obtained constructively. In Jankovic (2001), the control Lyapunov-Razumikhin function-based stabilization method is proposed for the nonlinear time-delay systems, and the feasibility to extend the proposed method to establish a backstepping technique has been also discussed in the paper. But this method requires to check the existence of the domination function at each step of the recursive design process, and it is not given how to find the domination function. As pointed out by the author, it is a difficult task for higher dimensional systems. Therefore, developing a backstepping design technique is still an unsolved open problem in the field of the control of nonlinear systems with time-delay.

Throughout this paper a solution to this problem is provided. The class of nonlinear systems with time-delay considered in this paper is described as following form

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1) + e_1(x_{1t}) \\ \dot{x}_2 = x_3 + f_2(\tilde{x}_2) + e_2(\tilde{x}_{2t}) \\ \quad \vdots \\ \dot{x}_n = u + f_n(\tilde{x}_n) + e_n(\tilde{x}_{nt}) \end{cases} \tag{1}$$

with initial condition $x_{i0} = \phi_i(\tau)$, $\tau \in [-r, 0]$, $r > 0$ is a constant, where x_i denotes the state and $x_{it} := x_i(t+\tau)$ the delayed state. $\tilde{x}_i = [x_1 \ x_2 \cdots x_i]^T$ and $\tilde{x}_{it} = [x_{1t} \ x_{2t} \cdots x_{it}]^T$ $(i = 1, \cdots, n)$. $f_i(\cdot)$ are smooth functions with $f_i(0) = 0$ $(i = 1, 2, \cdots n)$, the related-delay functions $e_i(\cdot)$ are not necessary to be known but satisfy the linear growth condition, i.e. there exist $b_{ij} > 0 (i = 1, \cdots, n, j = 1, \cdots, i)$ such that

$$|e_i(\tilde{x}_{it})| \le \sum_{j=1}^{i} b_{ij} |x_{jt}| \tag{2}$$

Thus, the robust stabilization problem is investigated, which is to find a stabilizing controller $u = c(x)$ rendering the closed loop system globally asymptotically stable at $x = 0$ for any $e_i(\cdot)$ satisfying (2). A recursive constructing approach will be provided for the robust stabilizing controller.

It should be noted that the results presented in this paper is distinguished from those of the papers aforementioned, since the provided control law is delay-independent and the stability is guaranteed by the Lyapunov-Razumikhin function that is established recursively. As it shall be seen in the next section, recursively constructing the stabilizing control law based on the Lyapunov-Razumikhin function is not a trivial application of the existing backstepping techniques. Exactly speaking, at the i-th step of the recursive design process, the time derivative of the Lyapunov-Razumikhin function V_i, which will constitute the Lyapunov-Razumikhin function for the whole system at the final step, can not be dominated to be negative along the trajectories satisfying the Razumikhin condition, since the Razumikhin condition is presented in the whole states of the system, so that the structure of the system becomes non-triangular and the effect related to $x_j (j > i)$ can not be dominated with the auxiliary control in the i-th step, even though the original system considered has triangular structure. In this paper, a design technique is given to overcome this obstacle brought by the application of the Razumikhin condition and it is shown that if only the original system considered is of triangular form, the stabilizing control law can be always obtained recursively by the construction of the Lyapunov-Razumikhin function. A numerical example is given to demonstrate the essential idea of the presented approach.

2. MAIN RESULTS

The following lemma is a case of the Razumikhin stability theorem (Hale, 1993), i.e. a linear function ps with a constant $p > 1$ is used to replace the function $p(s)$ $(p(s) > s, \ \forall s > 0)$ in theorem. This technical lemma will serve as a basis for the explicit construction of the robust stabilizing controller.

Lemma 1 *Consider time-delay systems given by*

$$\dot{x} = f(x, x_t), \ x_0(\tau), \ \tau \in [-r, 0] \tag{3}$$

If there exist a continuous function $V(x)$ and $\mathcal{K}_\infty$ functions $\kappa_1(\cdot)$, $\kappa_2(\cdot)$ and $\kappa_3(\cdot)$ such that

$$\kappa_1(\|x\|) \le V(x) \le \kappa_2(\|x\|) \tag{4}$$

$$\dot{V}(x) \le -\kappa_3(\|x\|), \text{if} \max_{-r \le \tau \le 0} V(x_t(\tau)) < pV(x_t(0)) \tag{5}$$

then, the solution $x(t) = 0$ of the system (3) is globally asymptotically stable, where $p > 1$ is a given constant.

To demonstrate the basic idea of the recursive design, the result on the two-dimensional system of (1) is first presented, i.e. the system is

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1) + e_1(x_{1t}) \\ \dot{x}_2 = u + f_2(\tilde{x}_2) + e_2(\tilde{x}_{2t}) \end{cases} \tag{6}$$

Theorem 1. *For the system (6), a stabilizing controller is given by*

$$\begin{aligned} u = & -z_1 - f_2 + \frac{\partial \alpha_1}{\partial x_1}[x_2 + f_1] - \frac{1}{2} b_{21}^2 z_2 - \frac{1}{2} b_{11}^2 \left| \frac{\partial \alpha_1}{\partial x_1} \right|^2 z_2 \\ & - \frac{1}{2} b_{22}^2 z_2 \left[\sum_{l=1}^{2} \tilde{c}_{11}^2 (2q|z_l|) + 1 \right] - 4q^2 z_2 - \frac{1}{2} z_2 \end{aligned} \tag{7}$$

where $q > 1$ is a constant, α_1 and $\tilde{c}_{11}$ are smooth functions determined in the design procedure.

Proof. First, note that in the recursive design, Lyapunov-Razumikhin function for the whole system will be a quadratic form on z under the coordinate $z_1 = x_1$, $z_2 = x_2 - \alpha_1(x_1)$ with $\alpha_1(0) = 0$, thus, the Razumikhin condition $\max_{-r \le \tau \le 0} V(z_t(\tau)) < pV(z_t(0))$ is equivalent to $(q = \sqrt{p} > 1)$

$$\|z_t(\tau)\| < q\|z_t(0)\|, \quad \tau \in [-r, 0] \tag{8}$$

For the x_1-subsystem with x_2 viewed as a virtual control signal, a positive define function $V_1(z_1)$ is defined as

$$V_1(z_1) = \frac{1}{2} z_1^2 \quad (9)$$

then, under the condition (2), the derivative of V_1 is obtained as

$$\dot{V}_1(z_1) \le z_1\{x_2 + f_1(x_1)\} + \frac{1}{2} b_{11}^2 z_1^2 + \frac{1}{2}|x_{1t}|^2$$

When the Razumikhin condition (8) holds, $|x_{1t}| = |z_{1t}| \le \|z_t\| < q\|z\|$ holds, thus, it follows

$$\dot{V}_1(z_1) \le z_1 \left\{ x_2 + f_1(x_1) + \frac{1}{2} b_{11}^2 z_1 \right\} + \frac{1}{2} q^2 \|z\|^2 \quad (10)$$

It is clear that in the above inequality the term $\frac{1}{2}q^2\|z\|^2$ can not be cancelled with the virtual control law. But, in the virtual control law, additional function terms on z_1 must be contained in order to dominate the derivative of the Lyapunov-Razumikhin function in the final step. Thus, the virtual control law is chosen as

$$\alpha_1(\tilde{x}_1) = -f_1(x_1) - \frac{1}{2} b_{11}^2 x_1 - 4q^2 z_1 - \frac{1}{2} z_1 \quad (11)$$

such that the time derivative of V_1 satisfies

$$\dot{V}_1(z_1) \le z_1 z_2 + \frac{1}{2} q^2 \|z\|^2 - 4q^2 z_1^2 - \frac{1}{2} z_1^2 \quad (12)$$

whenever the Razumikhin condition holds.

For the system (6) a positive definite function is constructed

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 \quad (13)$$

then, under the condition (2), the following inequality is obtained

$$\begin{aligned} \dot{V}_2(z_1, z_2) \le\ & \dot{V}_1 + z_2 \left\{ u + f_2 - \frac{\partial \alpha_1}{\partial x_1}[x_2 + f_1] \right\} \\ & + |z_2|(b_{21}|x_{1t}| + b_{22}|x_{2t}|) + |z_2| \left| \frac{\partial \alpha_1}{\partial x_1} \right| b_{11}|x_{1t}| \end{aligned} \quad (14)$$

From (14) the problem how to use the Razumikhin condition in $|x_{2t}|$ arises. Let $M_2 := |z_2| b_{22} |x_{2t}|$, and note that $x_{2t} = z_{2t} + \alpha_1(x_{1t})$, where $\alpha_1(\cdot)$ has been determined in the former step, so one can find a class $\mathcal{K}$ function $c_{11}(\cdot)$ such that $|\alpha_1(x_{1t})| \le c_{11}(|x_{1t}|)$ thus, the following inequality is gotten

$$M_2 \le \frac{1}{2} b_{22}^2 z_2^2 + \frac{1}{2}|z_{2t}|^2 + |z_2| b_{22} c_{11}(|x_{1t}|) \quad (15)$$

Substituting (15) into (14) obtains

$$\begin{aligned} \dot{V}_2 \le\ & \dot{V}_1 + z_2 \left\{ u + f_2(\tilde{x}_2) - \frac{\partial \alpha_1}{\partial x_1}[x_2 + f_1] \right. \\ & \left. + \frac{1}{2}(b_{21}^2 + b_{22}^2) z_2 + \frac{1}{2} b_{11}^2 z_2 \left| \frac{\partial \alpha_1}{\partial x_1} \right|^2 \right\} \\ & + |x_{1t}|^2 + \frac{1}{2}|z_{2t}|^2 + |z_2| b_{22} c_{11}(|x_{1t}|) \end{aligned} \quad (16)$$

When the Razumikhin condition holds, $|x_{1t}| < q\|z\|$ and $|z_{2t}| < q\|z\|$ holds, thus, it follows

$$\begin{aligned} \dot{V}_2 \le\ & \dot{V}_1 + z_2 \left\{ u + f_2 - \frac{\partial \alpha_1}{\partial x_1}[x_2 + f_1] + \frac{1}{2}(b_{21}^2 + b_{22}^2) z_2 \right. \\ & \left. + \frac{1}{2} b_{11}^2 z_2 \left| \frac{\partial \alpha_1}{\partial x_1} \right|^2 \right\} + \frac{3}{2} q^2 \|z\|^2 + |z_2| b_{22} c_{11}(q\|z\|) \end{aligned} \quad (17)$$

Obviously, another problem arises, i.e. how to deal with the term $N_2 := |z_2| b_{22} c_{11}(q\|z\|)$. The difficulty lies in that the function $c_{11}(\cdot)$ is closely related to the virtual control law $\alpha_1(\cdot)$ designed in the former step. Thus, to eliminate this difficulty, N_2 is dealt with as follows:

$$\begin{aligned} N_2 & \le |z_2| b_{22}(c_{11}(2q|z_1|) + c_{11}(2q|z_2|)) \\ & \le \frac{1}{2} z_2^2 b_{22}^2 \sum_{l=1}^{2} \tilde{c}_{11}^2(2q|z_l|) + 2q^2 z_1^2 + 2q^2 z_2^2 \end{aligned} \quad (18)$$

where $\tilde{c}_{11}(\cdot)$ is a function satisfying the decomposition $c_{11}(s) = s\tilde{c}_{11}(s)$. Substituting (18) and (12) into (17) obtains

$$\begin{aligned} \dot{V}_2 \le\ & z_2 \left\{ u + z_1 + f_2(\tilde{x}_2) - \frac{\partial \alpha_1}{\partial x_1}[x_2 + f_1(x_1)] + \frac{1}{2} b_{21}^2 z_2 \right. \\ & \left. + \frac{1}{2} b_{22}^2 z_2 \left[\sum_{l=1}^{2} \tilde{c}_{11}^2(2q|z_l|) + 1 \right] \right\} + \frac{1}{2} b_{11}^2 z_2^2 \left| \frac{\partial \alpha_1}{\partial x_1} \right|^2 \\ & + 2q^2 z_2^2 + 2q^2 \|z\|^2 - 2q^2 z_1^2 - \frac{1}{2} z_1^2 \end{aligned}$$

whenever the Razumikhin condition holds. Therefore, a feedback law defined by (38) renders

$$\dot{V}_2(z_1, z_2) \le -\frac{1}{2} z_1^2 - \frac{1}{2} z_2^2 \quad (19)$$

along the trajectories of the whole system (6) whenever the Razumikhin condition holds. Thus, in view of lemma 1, the asymptotically stability follows from (13) and (19). □

From the design presented by theorem 1, it can be seen that the key of the recursive design is how to deal with the system without triangular structure due to the use of the Razumikhin condition and the effect of the coordinate transformation on the Razumikhin condition, so that the derivative of the Lyapunov-Razumikhin function for the whole system along the closed-loop system trajectories satisfying the Razumikhin condition is negative. Recursive application of the proposed design step described above leads to backstepping method for the system (1).

Theorem 2 *Consider the system (1) with (2). A stabilizing controller $u = c(x_1, \cdots, x_n)$, which is independent of delay, can be recursively obtained.*

Proof. As the similar design idea as theorem 1, the recursive procedure of the stabilizing control law is presented. Similarly, Lyapunov-Razumikhin function of the whole system will be a quadratic form on z under the change of coordinate

$$z_i = x_i - \alpha_{i-1}(\tilde{x}_{i-1}), \quad i = 1, \cdots, n \tag{20}$$

with $\alpha_0 = 0$, $\alpha_{i-1}(0) = 0$, then, the Razumikhin condition is equivalent to ($q>1$)

$$\|z_t(\tau)\| < q\|z_t(0)\|, \quad \tau \in [-r, 0] \tag{21}$$

First Step: For the x_1-subsystem of (1), as the same design idea as theorem 1, it follows that the virtual control law

$$\alpha_1(\tilde{x}_1) = -f_1 - \frac{1}{2}b_{11}^2 x_1 - \sum_{s=2}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}z_1 - \sum_{i=1}^{n}\frac{i(n+1-i)}{2}q^2 z_1 - \frac{1}{2}z_1 \tag{22}$$

renders $V_1(x_1)$ defined by (9) to satisfy

$$\dot{V}_1(z_1) \le z_1 z_2 + \frac{1}{2}q^2\|z\|^2 - \sum_{s=2}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{j-1}{2}n^2q^2z_1^2 - \sum_{i=1}^{n}\frac{i(n+1-i)}{2}q^2 z_1^2 - \frac{1}{2}z_1^2 \tag{23}$$

whenever the Razumikhin condition holds.

Second Step: Since this step is a intermediate step, the virtual control law $\alpha_2(\cdot)$ is slightly different from the control law (38). For the x_2-subsystem, when the Razumikhin condition holds, the time derivative of V_2 defined as (13) is obtained as

$$\dot{V}_2 \le z_2\left\{z_3 + \alpha_2 + f_2 - \frac{\partial\alpha_1}{\partial x_1}[x_2 + f_1] + \frac{1}{2}(b_{21}^2 + b_{22}^2)z_2 + \frac{1}{2}b_{11}^2 z_2\left|\frac{\partial\alpha_1}{\partial x_1}\right|^2\right\} + \sum_{i=1}^{2}\frac{i}{2}q^2\|z\|^2 + N_2 + \dot{V}_1 \tag{24}$$

with $N_2 := |z_2|b_{22}c_{11}(q\|z\|)$, which satisfies

$$N_2 \le \frac{1}{2}b_{22}^2 z_2^2\sum_{l=1}^{2}\tilde{c}_{11}^2(nq|z_l|) + \frac{n-2}{2}b_{22}^2 z_2^2 + \frac{1}{2}\sum_{l=1}^{2}n^2q^2z_l^2 + \frac{1}{2}\sum_{l=3}^{n}c_{11}^2(nq|z_l|) \tag{25}$$

In (25), the quadratic form of z_1 in the third term can be dominated by the pre-design additional term in $\alpha_1(\cdot)$, and the fourth term will be dealt with in the later steps. These features are just the distinctive difference of backstepping design of time-delay systems from that of nonlinear non-delay systems.

Therefore, a virtual feedback law defined by

$$\alpha_2 = \frac{\partial\alpha_1}{\partial x_1}[x_2 + f_1] - \frac{1}{2}b_{22}^2 z_2\left[\sum_{l=1}^{2}\tilde{c}_{11}^2(nq|z_l|) + n - 1\right] - f_2 - \frac{1}{2}b_{11}^2\left|\frac{\partial\alpha_1}{\partial x_1}\right|^2 z_2 - \sum_{s=2}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}z_2 - \frac{1}{2}b_{21}^2 z_2 - \sum_{i=1}^{n}\frac{i(n+1-i)}{2}q^2 z_2 - \frac{1}{2}z_2 - z_1 \tag{26}$$

can render (24) with (25) and (23) to satisfy

$$\dot{V}_2 \le z_2 z_3 + \sum_{i=1}^{2}\frac{i(3-i)}{2}q^2\|z\|^2 - \sum_{l=1}^{2}\frac{1}{2}z_l^2 + \frac{1}{2}\sum_{l=3}^{n}c_{11}^2(nq|z_l|) - \sum_{i=1}^{n}\sum_{l=1}^{2}\frac{i(n+1-i)}{2}q^2 z_l^2 - \sum_{s=3}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}\sum_{l=1}^{2}z_l^2$$

whenever the Razumikhin condition holds.

Along the recursive design line in the second step, the following induction step is given.

Induction step: Suppose at the $k-1$-*th* step ($3 \le k \le n$), there are a set of virtual control laws $\alpha_i(\tilde{x}_i)$, ($i = 1, \cdots, k-1$) and a positive definite function $V_{k-1}(\tilde{z}_{k-1})$ such that

$$\dot{V}_{k-1} \le z_{k-1}z_k + \sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\sum_{l=k}^{n}\frac{k-i}{2}\eta_{(j-1)s}^2(nq|z_l|) + \sum_{i=1}^{k-1}\frac{i(k-i)}{2}q^2\|z\|^2 - \sum_{s=k}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}\sum_{l=1}^{k-1}z_l^2 - \sum_{i=1}^{n}\sum_{l=1}^{k-1}\frac{i(n+1-i)}{2}q^2 z_l^2 - \sum_{l=1}^{k-1}\frac{1}{2}z_l^2 \tag{27}$$

whenever the Razumikhin condition holds, where $\eta_{ij}(\cdot)$ is a class $\mathcal{K}$ function satisfying the condition

$$|\alpha_i(\tilde{x}_i)| \le \sum_{j=1}^{i-1}c_{ij}(|x_j|) = \sum_{j=1}^{i-1}\eta_{ij}(|z_j|) \tag{28}$$

with the class $\mathcal{K}$ function $c_{(j-1)s}(\cdot)$.

Thus, in the following it will be shown that for the k-*th* subsystem of (1) the time derivative of V_k also satisfies the inequality form as (27) if the positive definite function V_k is defined as

$$V_k(\tilde{z}_k) = V_{k-1}(\tilde{z}_{k-1}) + \frac{1}{2}z_k^2 \tag{29}$$

Under the condition (2), the time derivative of V_k along the trajectories of (1) satisfies

$$\dot{V}_k \le z_k\left\{x_{k+1} + f_k - \sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}[x_{i+1} + f_i]\right\} + \frac{1}{2}b_{k1}^2 z_k^2 + \frac{1}{2}z_k^2\sum_{i=1}^{k-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2 b_{i1}^2 + \frac{1}{2}k|x_{1t}|^2 + M_k + \dot{V}_{k-1} \tag{30}$$

where

$$M_k=|z_k|\sum_{j=2}^{k}b_{kj}|x_{jt}|+|z_k|\sum_{i=2}^{k-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|\sum_{j=2}^{i}b_{ij}|x_{jt}|$$

Note that $|\alpha_{j-1}(\tilde{x}_{(j-1)t})| \leq \sum_{s=1}^{j-1}\eta_{(j-1)s}(|z_{st}|)$ with class $\mathcal{K}$ functions $\eta_{(j-1)s}(\cdot)$ and (20), then,

$$\begin{aligned}M_k\leq&\frac{1}{2}z_k^2\sum_{j=2}^{k}b_{kj}^2+\frac{1}{2}\sum_{j=2}^{k}|z_{jt}|^2+\frac{1}{2}z_k^2\sum_{i=2}^{k-1}\sum_{j=2}^{i}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2b_{ij}^2\\&+\frac{1}{2}\sum_{i=2}^{k-1}\sum_{j=2}^{i}|z_{jt}|^2+|z_k|\sum_{j=2}^{k}\sum_{s=1}^{j-1}b_{kj}\eta_{(j-1)s}(|z_{st}|)\\&+|z_k|\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|b_{ij}\eta_{(j-1)s}(|z_{st}|)\end{aligned}\quad(31)$$

By substituting (31) into (30) and considering the Razumikhin condition, it follows

$$\begin{aligned}\dot{V}_k\leq&z_k\left\{x_{k+1}+f_k-\sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}(x_{i+1}+f_i)\right\}+\frac{1}{2}z_k^2\sum_{j=1}^{k}b_{kj}^2\\&+\frac{1}{2}z_k^2\sum_{i=1}^{k-1}\sum_{j=1}^{i}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2b_{ij}^2+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}q^2\|z\|^2+N_k+\dot{V}_{k-1}\end{aligned}\quad(32)$$

where

$$\begin{aligned}N_k=&|z_k|\left[\sum_{j=2}^{k}\sum_{s=1}^{j-1}b_{kj}\eta_{(j-1)s}(q\|z\|)\right.\\&\left.+\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|b_{ij}\eta_{(j-1)s}(q\|z\|)\right]\end{aligned}$$

By using the property of the class $\mathcal{K}$ function, Young's Inequality and the function decomposition,

$$\begin{aligned}N_k\leq&\frac{1}{2}z_k^2\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2b_{ij}^2\left[\sum_{l=1}^{k}\tilde{\eta}_{(j-1)s}^2(nq|z_l|)\right.\\&\left.+n-k\right]+\frac{1}{2}\sum_{i=2}^{k}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\sum_{l=k+1}^{n}\eta_{(j-1)s}^2(nq|z_l|)\\&+\frac{1}{2}z_k^2\sum_{j=2}^{k}\sum_{s=1}^{j-1}b_{kj}^2\left[\sum_{l=1}^{k}\tilde{\eta}_{(j-1)s}^2(nq|z_l|)+n-k\right]\\&+\sum_{i=2}^{k}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}\sum_{l=1}^{k}z_l^2\end{aligned}\quad(33)$$

in which, $\eta_{(j-1)s}(a)=a\tilde{\eta}_{(j-1)s}(a)$. Thus, a virtual feedback law defined by

$$\begin{aligned}\alpha_k=&-z_{k-1}-f_k+\sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}[x_{i+1}+f_i]-\frac{1}{2}z_k\sum_{j=1}^{k}b_{kj}^2\\&-\frac{1}{2}z_k\sum_{j=2}^{k}\sum_{s=1}^{j-1}b_{kj}^2\left[\sum_{l=1}^{k}\tilde{\eta}_{(j-1)s}^2(nq|z_l|)+n-k\right]\\&-\frac{1}{2}z_k\sum_{i=1}^{k-1}\sum_{j=1}^{i}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2b_{ij}^2-\sum_{s=k}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}z_k\\&-\frac{1}{2}z_k-\frac{1}{2}z_k\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2b_{ij}^2\cdot\\&\left[\sum_{l=1}^{k}\tilde{\eta}_{(j-1)s}^2(nq|z_l|)+n-k\right]-\sum_{i=1}^{n}\frac{i(n+1-i)}{2}q^2z_k\\&-z_k\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\frac{(k-i)n^2q^2}{2}\tilde{\eta}_{(j-1)s}^2(nq|z_k|)\end{aligned}$$

renders (32) with (33) and (27) to satisfy

$$\begin{aligned}\dot{V}_k\leq&z_kz_{k+1}+\sum_{i=2}^{k}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\sum_{l=k+1}^{n}\frac{k+1-i}{2}\eta_{(j-1)s}^2(nq|z_l|)\\&+\sum_{i=1}^{k}\frac{i(k+1-i)}{2}q^2\|z\|^2-\sum_{i=1}^{n}\sum_{l=1}^{k}\frac{i(n+1-i)}{2}q^2z_l^2\\&-\sum_{s=k+1}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}\sum_{l=1}^{k}z_l^2-\sum_{l=1}^{k}\frac{1}{2}z_l^2\end{aligned}$$

whenever the Razumikhin condition holds.

Obviously, this recursive procedure will terminate at the *n-th* step where

$$V(\tilde{z}_n)=V(\tilde{z}_{n-1})+\frac{1}{2}z_n^2\quad(34)$$

with $\alpha_i(\tilde{x}_i)$ $(i=1,2,\cdots,n-1)$ and

$$\begin{aligned}\alpha_n=&\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}[x_{i+1}+f_i]-\frac{1}{2}z_n\sum_{i=1}^{n-1}\sum_{j=1}^{i}\left|\frac{\partial\alpha_{n-1}}{\partial x_i}\right|^2b_{ij}^2\\&-\frac{1}{2}z_n\sum_{j=2}^{n}\sum_{s=1}^{j-1}\sum_{l=1}^{n}b_{nj}^2\tilde{\eta}_{(j-1)s}^2(nq|z_l|)-\frac{1}{2}z_n\sum_{j=1}^{n}b_{nj}^2-f_n\\&-\frac{1}{2}z_n\sum_{i=2}^{n-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\sum_{l=1}^{n}\left|\frac{\partial\alpha_{n-1}}{\partial x_i}\right|^2b_{ij}^2\tilde{\eta}_{(j-1)s}^2(nq|z_l|)\\&-z_n\sum_{i=2}^{n-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\frac{(n-i)n^2q^2}{2}\tilde{\eta}_{(j-1)s}^2(nq|z_n|)-\frac{1}{2}z_n\\&-\sum_{i=2}^{n}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}z_n-\sum_{i=1}^{n}\frac{i(n+1-i)}{2}q^2z_n-z_{n-1}\end{aligned}$$

is such that

$$\dot{V}\leq z_n(u-\alpha_n(\tilde{x}_n))-\frac{1}{2}\sum_{l=1}^{n}z_l^2\quad(35)$$

along the trajectories of the whole system (1) whenever the Razumikhin condition holds.

Hence, by choosing the feedback control law $u=c(x_1,\cdots,x_n)=\alpha_n(\tilde{x}_n)$, it finally follows

$$\dot{V}\leq-\frac{1}{2}\|z\|^2\quad(36)$$

whenever the Razumikhin condition holds. Thus, in view of lemma 1, the asymptotically stability follows from (34) and (36). □

3. NUMERICAL EXAMPLE

To illustrate the proposed recursive method, a two-dimensional system is as example simulated

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1) + e_1(x_1, x_{1t}) \\ \dot{x}_2 = u + f_2(x_1, x_2) + e_2(x_1, x_{1t}, x_2, x_{2t}) \end{cases} \tag{37}$$

where $f_1(x_1) = x_1^2 + 2x_1$, $f_2(x_1, x_2) = x_1x_2 + x_1^2 + x_2^2$. Unknown functions $e_1(\cdot)$ and $e_2(\cdot)$ satisfy

$$|e_1(x_1, x_{1t})| \leq |x_{1t}|, \quad |e_2(x, x_t)| \leq |x_{1t}| + |x_{2t}|$$

By applying theorem 1 to the system, a robust stabilizing controller is obtained

$$u = -x_1 - f_2 + \frac{\partial \alpha_1}{\partial x_1}(x_2 + f_1) - \frac{1}{2} z_2 \left(\frac{\partial \alpha_1}{\partial x_1}\right)^2 - 4.04 z_2 (z_1^2 + z_2^2) - 56.6 z_2 \tag{38}$$

where $\alpha_1(x_1) = -x_1^2 - 7.04x_1$, $z_2 = x_2 - \alpha_1(x_1)$.

In simulation, the initial conditions are chosen as

$$\phi_1(\tau) = 0.1\tau^2, \quad \phi_2(\tau) = -0.8 \sin(\tau + \frac{\pi}{2})$$

When $\tau \in [-0.2, 0]$ and the uncertainties are described by $e_1(x_1, x_{1t}) = x_{1t} \sin x_1$, $e_2(x, x_t) = x_{1t} \cos x_1 + x_{2t} \sin x_1x_2$, the response of the closed loop system (37) with (38) is shown in Figure 1. Figure 2 is the case of $\tau \in [-7, 0]$ and $e_1(x_1, x_{1t}) = x_{1t} \cos 0.6x_1$, $e_2(x, x_t) = x_{1t} \sin 0.5x_{2t} + x_{2t} \cos(x_1 + x_2)$. The simulation results demonstrate that the system with the delay-related uncertainty can be stabilized by the robust feedback controller constructed recursively.

4. CONCLUSIONS

The robust stabilization problem for general nonlinear time-delay systems with triangular structure is investigated. A Lyapunov-Razumikhin function based version of similar backstepping approach is developed. The provided control law is independent of the state-delayed, so that the value of the delay is allowed to be unknown.

REFERENCES

Hale, J.K.and S.M. Lunel(1993). Introduction to functional differential equations, *Springer, New York.*

Isidori,A. (1995). Nonlinear control systems, *Springer, New York.*

Jankovic, M. (2001). Control Lyapunov Razumikhin function and robust stabilization of time-delay systems, *IEEE Trans. Automat. Contr.,* **47**(7), pp.1048-1060.

Jankovic, M. (2003). Control of nonlinear systems with time delay, *Proc. of the 42th IEEE CDC*, pp.4545-4550.

Nguang, S.K. (2000). Robust stabilization of a class of time-delay nonlinear systems, *IEEE Trans.Automat. Contr.* **45**(4), pp.756-762.

Zhou, S., G.Feng and S.K.Nguang (2002). Comments on "Robust stabilization of a class of time-delay nonlinear systems", *IEEE Trans. Automat. Contr.*, **47**(9), pp.1586.

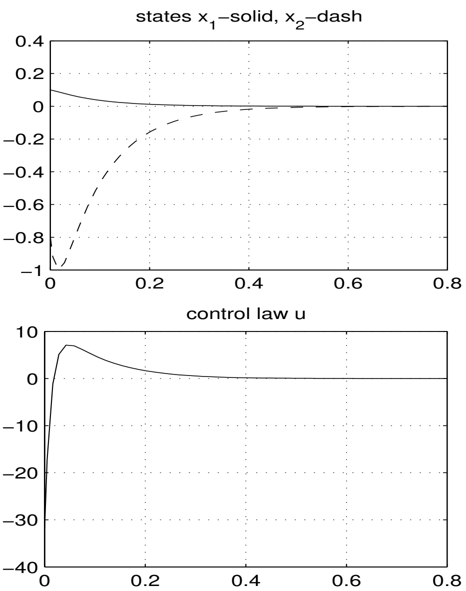

Fig. 1. $\tau \in [-0.2, 0]$, $e_1(x_1, x_{1t}) = x_{1t} \sin x_1$, $e_2(x, x_t) = x_{1t} \cos x_1 + x_{2t} \sin x_1x_2$

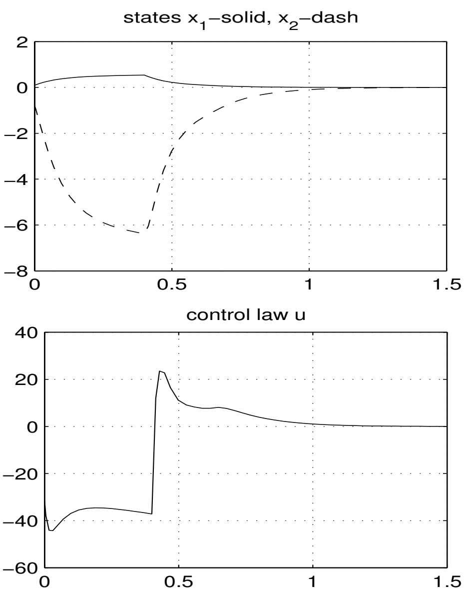

Fig. 2. $\tau \in [-7, 0]$, $e_1(x_1, x_{1t}) = x_{1t} \cos 0.6x_1$, $e_2(x, x_t) = x_{1t} \sin 0.5x_{2t} + x_{2t} \cos(x_1 + x_2)$