QUANTIZED FEEDBACK CONTROL FOR SAMPLED-DATA SYSTEMS

Minyue Fu [*] **Shinji Hara** [**,1]

[*] *School of EECS, University of Newcastle, Australia*
[**] *Department of Information Physics and Computing, University of Tokyo, Japan*

Abstract: This paper studies the problem of quantized feedback control for sampled-data systems which employ a quantizer to transmit feedback signals at a given sampling rate. The so-called static (memoryless) quantizers are considered. Given a continuous-time system, the design objective is to stabilize the system or to achieve certain performance using the coarsest quantization density. We study the possible advantages of over-sampling where the input/output signals of the system are sampled at a faster rate than the quantizer. Our first result is for stabilization, and it shows that the coarsest quantization density achievable using quantized state feedback can be generically achieved using output feedback with any over-sampling ratio. Our second result provides a solution to the quantized feedback H_∞ problem for both with and without over-sampling.

Keywords: Quantized Feedback Control, Sampled-Data Systems, H_∞ Control.

1. INTRODUCTION

Control using quantized feedback has been an important research area for a long time as seen in Kalman's pioneer work (Kalman 1956), which studied the effect of quantization in a sampled data system. Recently, there is a new line of research on quantized feedback control where a quantizer is regarded as an information coder; see, e.g., (Baillieul 2001), (Brockeet and Liberzon 2000), (Elia 2000), (Elia and Mitter 2001), (Nair and Evans 2003), (Fu and Xie 2003a, 2003b) and references there in. The fundamental question of interest is how much information needs to be communicated by the quantizer in order to achieve a certain control objective including stabilization.

In (Elia and Mitter 2001), the problem of quadratic stabilization of discrete-time single-input systems using quantized feedback is studied, where the quantizer is assumed to be static and time-invariant (i.e. memoryless and with fixed quantization levels). They proved that the best quantizer is the so-called *logarithmic* and that the coarsest quantization density is given explicitly in terms of the system's unstable poles. The results have been generalized to the multi-input systems by sector bound approach in (Fu and Xie 2003a).

However, most of the results on quantized feedback control are for discrete-time systems only. In this paper, we study the problem of quantized feedback control for sampled-data systems. In such a system, a continuous-time plant is controlled by a digital compensator with AD/DA converters through a communication network channel or quantizer. There are two important issues to be investigated for the sampled-data case which do not appear in the discrete-time case. One is the inter-sample behaviors which have to be taken into account to evaluate the control performance. The other is the extra freedom of designing sample and hold schemes, which may improve the

[1] This research was partially supported by Australian Research Council, the 21st Century COE Program on Information Science and Technology Strategic Core, and CREST of JST (Japan Science and Technology Agency).

control performance significantly. Note that modern approaches to sampled-data systems such as lifting techniques are quite powerful to treat the inter-sample behaviors reasonably; see, e.g., (Chen and Francis 1995) and references there in. However, most of the results ignore the quantization effects.

The work in this paper is concerned with the use of static (memoryless) quantizers. Given a continuous-time system with a certain quantization sampling rate, the design objective is to stabilize the system or to achieve certain performance using the coarsest quantization density. The main focus of the paper is to study the possible advantages of over-sampling where the input/output signals of the system are sampled at a faster rate than the quantizer. Two main results are obtained. The first one is for quantized feedback stabilization, and it shows that the coarsest quantization density achievable using state feedback can be generically achieved using output feedback with any over-sampling ratio. The second result provides a solution to the quantized feedback H_∞ problem for both with and without over-sampling.

2. PROBLEM FORMULATION

The type of sampled-data systems we consider in this paper is depicted in Figure 1. The continuous-time plant $G(s)$ has the following realization:

$$\begin{aligned} \dot{x}_c(t) &= A_c x(t) + B_{1c} w_c(t) + B_{2c} u_c(t) \\ z_c(t) &= C_{1c} x_c(t) + D_{11c} w_c(t) + D_{12c} u_c(t) \\ y_c(t) &= C_{2c} x_c(t) + D_{21c} w_c(t) + D_{22c} u(t) \end{aligned} \quad (1)$$

where $x_c(t) \in \mathbb{R}^n$ is the state, $w_c(t) \in \mathbb{R}^{m_1}$ is the process noise, $u_c(t) \in \mathbb{R}^{m_2}$ is the control input, $z_c(t) \in \mathbb{R}^{p_1}$ is the output used for measuring the performance, $y_c(t) \in \mathbb{R}^{p_2}$ is the output used for control feedback. For simplicity, we only consider the single-input, single-output case, i.e., $m_2 = p_2 = 1$.

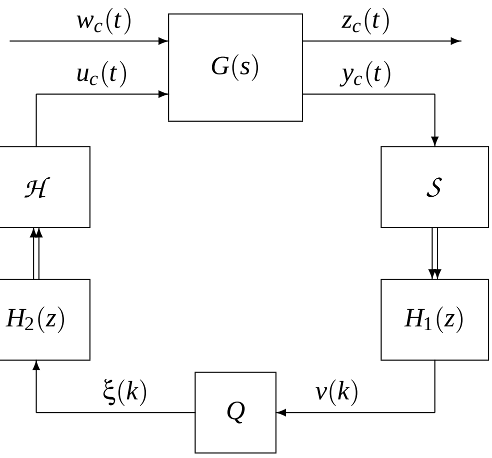

Fig. 1. Sampled-Data System

In Figure 1, Q is a quantizer, and $\mathcal{S}$ and $\mathcal{H}$ represent generalized sampling and hold functions, respectively. It is assumed that the quantizer operates with sampling period T, whereas $\mathcal{S}$ and $\mathcal{H}$ operate with sampling period $h = T/q$ for some integer $q \geq 1$. More precisely, $\mathcal{S}$ takes a sample every h time interval and every q of these samples are stacked up together as the input to $H_1(z)$ which produces an output $v(k)$ to the quantizer Q every T time interval. The quantized signal $\xi(k)$ is sent to $H_2(z)$ every T time interval to produce q outputs to $\mathcal{H}$ which is a zeroth-order hold (ZOH) with the time interval of h. It is further assumed that the process noise $w_c(t)$ is constant within each sampling period h (See Remark 4 about relaxing this assumption).

We will call $T = qh$ the *quantization sampling period*, h the *input-output sampling period* and q the *over-sampling ratio*. The transfer functions $H_1(z)$ and $H_2(z)$ will be referred to as *pre-quantizer controller* and *post-quantizer controller*, respectively.

In order to develop the sampled-data model for the system in Figure 1, we first find the sampled-data model corresponding to $T = h$ (i.e., $q = 1$). It is easy to verify that this model is given by

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k) \\ z(k) &= C_1 x(k) + D_{11} w(k) + D_{12} u(k) \\ y(k) &= C_2 x(k) + D_{21} w(k) + D_{22} u(k) \end{aligned} \quad (2)$$

where

$$\begin{aligned} x(k) &= x_c(kh);\ w(k) = w_c(kh) \\ u(k) &= u_c(kh);\ y(k) = y_c(kh) \end{aligned}$$

and $z(k)$ is related to $z_c(t)$ by

$$\|z(k)\|^2 = \int_{kh}^{(k+1)h} \|z_c(t)\|^2 dt$$

The matrices in (2) are given by

$$\begin{aligned} A &= \exp(A_c h); \\ B_i &= \int_0^h \exp(A_c(h-\tau) d\tau B_{ic},\ \ i = 1,2 \\ C_2 &= C_{2c},\ D_{21} = D_{21c},\ D_{22} = D_{22c} \end{aligned}$$

$$[C_1\ D_{11}\ D_{12}] = \left\{ \int_0^h M(\tau) M^T(\tau) d\tau \right\}^{1/2}$$

$$M(\tau) = \begin{bmatrix} C_{1c} \exp(A_c \tau) \\ D_{11c} + C_{1c} \int_0^\tau \exp(A_c(\tau-\theta)) d\theta B_{1c} \\ D_{12c} + C_{1c} \int_0^\tau \exp(A_c(\tau-\theta)) d\theta B_{2c} \end{bmatrix}$$

Now we return to the case where $T = qh, q \geq 1$. The sampled-data system (2) can be rewritten using the standard lifting technique as follows:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}_1 \mathbf{w}(k) + \mathbf{B}_2 \mathbf{u}(k) \\ \mathbf{z}(k) &= \mathbf{C}_1 \mathbf{x}(k) + \mathbf{D}_{11} \mathbf{w}(k) + \mathbf{D}_{12} \mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}_2 \mathbf{x}(k) + \mathbf{D}_{21} \mathbf{w}(k) + \mathbf{D}_{22} \mathbf{u}(k) \end{aligned} \quad (3)$$

where

$$\mathbf{w}(k) = [w(qk)\ w(qk+1)\ \ldots\ w(q(k+1)-1)]^T$$

and $\mathbf{u}(k)$, $\mathbf{z}(k)$ and $\mathbf{y}(k)$ are similarly defined but

$$\mathbf{x}(k) = x(qk)$$

The matrices in (3) are given by

$$\mathbf{A} = A^q;\ \mathbf{B}_i = [A^{q-1}B_i\ A^{q-2}B_i\ \ldots\ B_i]$$

$$\mathbf{C}_i = \begin{bmatrix} C_i \\ C_iA \\ \vdots \\ C_iA^{q-1} \end{bmatrix};\ \mathbf{D}_{ij} = \begin{bmatrix} D_{ij} & 0 & \ldots & 0 \\ C_iB_j & D_{ij} & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ C_iA^{q-2}B_j & \ldots & C_iB_j & D_{ij} \end{bmatrix}$$

for $i, j = 1, 2$.

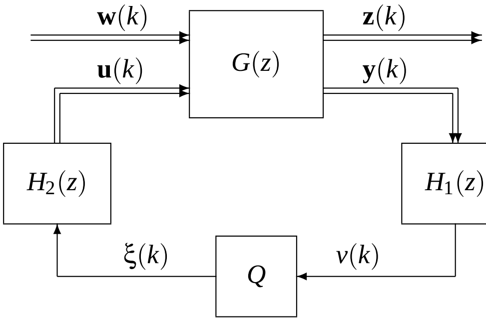

Fig. 2. Discrete-Time Model

The general setup of the sampled-data system we consider in this paper is depicted in Figure 2. In this setup, $G(z)$ is the transfer function corresponding to (3), the transfer functions, $H_1(z)$ and $H_2(z)$, are to be designed and they are of the form

$$\begin{aligned} H_1(z) &= [h_{11}(z)\ h_{12}(z)\ \ldots\ h_{1q}(z)]; \\ H_2(z) &= [h_{21}(z)\ h_{22}(z)\ \ldots\ h_{2q}(z)]^T \end{aligned}$$

Also, $H_1(z)$ and $H_2(z)$ must be chosen such that the mapping from $\mathbf{y}(k)$ to $\mathbf{u}(k)$ is causal. The block Q is a quantizer mapping from $v(k)$ to $\xi(k)$. In this paper, we only consider *static* quantizers, i.e.,

$$\xi(k) = f(v(k))$$

for a static nonlinear function $f(\cdot)$. In view of the work in (Elia and Mitter 2001) and (Fu and Xie 2003a), we consider *logarithmic quantizers*. More precisely, let

$$\begin{aligned} \mathcal{U} = &\{\pm u^{(i)} : u^{(i)} = \rho^i u^{(0)}, i = \pm 1, \pm 2, \cdots\} \\ &\cup \{\pm u^{(0)}\} \cup \{0\},\ \ 0 < \rho < 1, u^{(0)} > 0 \end{aligned} \tag{4}$$

The associated quantizer f is defined as follows:

$$f(v) = \begin{cases} u_i, & \text{if } \frac{1}{1+\delta}u_i < v \le \frac{1}{1-\delta}u_i \\ 0, & \text{if } v = 0 \\ -f(-v), & \text{if } v < 0. \end{cases} \tag{5}$$

where

$$\delta = \frac{1-\rho}{1+\rho} \tag{6}$$

The parameter ρ above can be regarded as the *quantization density*. Note that a smaller ρ corresponds to a coarser quantizer.

A quantized feedback problem can be loosely formulated as follows: *Given the continuous-time system (1), quantization sampling period and over-sampling ratio, find the coarsest quantization density such that there exists pre- and post-quantizer controllers to either stabilize the system or to meet certain performance requirement.*

The main objective of the paper is to study the possible benefits of over-sampling in quantized feedback control. We will study four scenarios:

S1: No Over-sampling. This corresponds to

$$H_1(z) = [h_{11}(z)\ 0\ \ldots\ 0]$$

$$H_2(z) = h_{21}(z)[1\ \ 1\ \ \ldots\ \ 1]^T$$

S2: Over-sampling at Input Only. This corresponds to

$$H_1(z) = [h_{11}(z)\ 0\ \ldots\ 0]$$

S3: Over-sampling at Output Only. This corresponds to

$$H_2(z) = h_{2q}(z)[z^{-1}\ \ldots z^{-1}\ 1]^T$$

(Note that the delay terms are there to ensure causality of the controller.)

S4: Over-sampling at both Input and Output. This is the general case.

3. STABILIZATION

In this section, we study the problem of quantized feedback stabilization. In this problem, the process noise $w(t)$ (or $\mathbf{w}(k)$) is set to zero, $z(t)$ (or $\mathbf{z}(k)$) is void, and $G(z)$ reduces to $G_{22}(z)$. The goal is to achieve stabilization with a minimum quantization density.

We first introduce a benchmark scenario depicted in Figure 3 so that we can compare the four scenarios discussed earlier against it. In the benchmark scenario, the full state $\tilde{\mathbf{x}}(k)$ is measured, where

$$\tilde{\mathbf{x}}(k) = [x^T(qk)\ x^T(qk+1)\ \ldots\ x^T(qk+q-1)]^T$$

$\tilde{G}_{22}(z)$ is the transfer function from $\mathbf{u}(k)$ to $\tilde{\mathbf{x}}(k)$, and $K(z)$ is a dynamic state feedback controller.

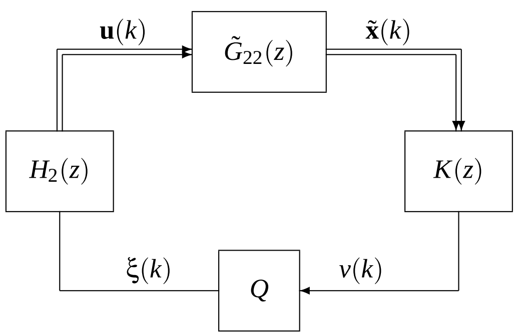

Fig. 3. State Feedback Model

For the special case where $q = 1$, $K(z)$ is static and $H_2(z) = 1$, the benchmark scenario was first studied in

(Elia and Mitter 2001) which showed that the coarsest quantization density is given by

$$\rho_{\text{inf}} = \frac{1-\delta_{\text{sup}}}{1+\delta_{\text{sup}}} \tag{7}$$

with

$$\delta_{\text{sup}}^{-1} = \prod_i |\lambda_i^u(\mathbf{A})| = \exp\left(T\sum_i \lambda_i^u(A_c)\right) \tag{8}$$

where $\lambda_i^u(\mathbf{A})$ (resp. $\lambda_i^u(A_c)$) denotes the unstable eigenvalues of $\mathbf{A}$ (resp. A_c).

The result below shows that the same result holds in general.

Theorem 1. Consider the benchmark scenario in Figure 3. The coarsest quantization density is given by (7)-(8) and this can be achieved by taking

$$K(z) = [K_1\ 0\ \dots\ 0];\ \ H_2(z) = [1\ 1\ \dots\ 1] \tag{9}$$

That is, over-sampling does not improve the coarsest quantization density.

Proof: It is obvious that the particular choice for $K(z)$ and $H_2(z)$ coincides with the special case discussed above, and hence the value of ρ_{inf} in (7)-(8) can be achieved. It remains to show that this value cannot be reduced by using other $K(z)$ and $H_2(z)$. Given $H_2(z)$, it is known (Fu and Xie 2003a) that the corresponding δ_{sup} is maximized when the full state of $\tilde{G}_{22}(z)H_2(z)$ is available for feedback and in the case,

$$\delta_{\text{sup}}^{-1} = \prod_i |\tilde{\lambda}_i^u|$$

where $\tilde{\lambda}_i^u$ are the unstable poles of $\tilde{G}_{22}(z)H_2(z)$. So δ_{sup} is maximized by taking $H_2(z) = [1\ 1\ \dots\ 1]^T$. This case is the same as $q = 1$ and thus, it suffices to choose a static $K(z)$ as in (9). ∎

Now, we consider the four scenarios in Figure 2.

Theorem 2. Consider the Scenario S1 in Figure 2. The coarsest quantization density is given by (7) with δ_{sup} given by

$$\delta_{\text{sup}} = \frac{1}{\inf_{h(z)} \|G_h(z)\|_\infty} \tag{10}$$

where

$$G_h(z) = (1-h(z)g_1(z))^{-1}h(z)g_1(z) \tag{11}$$

with

$$g_1(z) = [1\ 0\ \dots\ 0]G_{22}(z)[1\ 1 \dots\ 1]^T$$

Furthermore, if $g_1(z)$ has relative degree equal to 1 and no unstable zeros, then the coarsest quantization density for Scenario S1 matches that for the benchmark scenario.

Proof: Note that in Scenario S1,

$$H_1(z)G_{22}(z)H_2(z) = h_{21}(z)h_{11}(z)g_1(z)$$

Therefore, it is without loss of generality to take $h_{11}(z) = 1$. Denoting $h(z) = h_{21}(z)$, then this revised quantized feedback stabilization problem has been studied in (Fu and Xie 2003a) and the results in the theorem are directly cited from (Fu and Xie 2003a). ∎

Remark 1. What is implied in Theorem 2 is that for any given $\rho > \rho_{\text{inf}}$, a stabilizing controller can be constructed by taking $h_{11}(z) = 1$ and solving $h_{21}(z)$ such that

$$\|G_h(z)\|_\infty < \delta^{-1} \tag{12}$$

where δ and ρ are related as in (7).

Remark 2. There are two special cases where we can obtain the analytical expressions for $\inf_{h(z)} \|G_h(z)\|_\infty$ by applying the solution of the Navanlinna-Pick interpolation problem.

- Suppose $g_1(z)$ is minimum phase and its relative degree is one. Then we have

  $$\inf_{h(z)} \|G_h(z)\|_\infty = \prod_{i=1}^{N_p} |\lambda_i^u|$$

  where $\lambda_i^u, 1 \le i \le N_p$, are unstable poles of $g_1(z)$. This corresponds to the state feedback case.
- Suppose $g_1(z)$ has only one unstable pole λ and non-minimum phase zeros $z_j, 1 \le j \le N_z$, and its relative degree is $\nu \ge 1$. Then we have

  $$\inf_{h(z)} \|G_h(z)\|_\infty = |\lambda|^\nu \prod_{j=1}^{N_z} |\frac{\zeta_j\lambda - 1}{\lambda - \zeta_j}|$$

Remark 3. It is well-known that $g_1(z)$ has relative degree equal to 1 generically. However, $g_1(z)$ is not minimum phase in general. Therefore, it is seen from Remark 2 that Theorem 1 also implies that Scenario S1 is in general inferior to the benchmark scenario. See Section 5 for an example.

Theorem 3. The coarsest quantization density achievable by Scenario S2 in Figure 2 is generically identical to that of the benchmark scenario for any $q > 1$. In particular, this can be achieved by choosing

$$H_1(z) = [1\ 0\ \dots\ 0]$$

and $H_2(z)$ such that $H_1(z)G_{22}(z)H_2(z)$ is minimum phase and having relative degree equal to 1, which is generically possible.

Proof: It is well-known that $H_1(z)G_{22}(z)$ has relative degree 1 with co-prime elements generically. Therefore, it is generically possible to choose $H_2^0(z)$ such that $g_1(z) = H_1(z)G_{22}(z)H_2^0(z)$ is minimum phase and has relative degree equal to 1. Hence, by taking $H_2(z) = h(z)H_z^0(z)$, the desired result is obtained by applying Theorem 2 (Part 2) to $g_1(z)$ and $h(z)$. ∎

Theorem 4. The coarsest quantization density achievable by Scenario S3 in Figure 2 is generically identical to that of the benchmark scenario for any $q > 1$. In particular, this can be achieved by choosing

$$H_2(z) = [z^{-1} \ \dots \ z^{-1} \ 1]^T$$

and $H_1(z)$ such that $H_2(z)G_{22}(z)H_1(z)$ is minimum phase and having relative degree equal to 1, which is generically possible.

Proof. The proof is similar to that of Theorem 3.

Since both Scenarios S2 and S3 can do as well as the benchmark scenario, there is no need to consider Scenario S4 for quantized feedback stabilization.

4. H_∞ CONTROL

In this section, we study a quantized feedback H_∞ control problem for the system in Figure 2. More precisely, given an H_∞ performance bound $\gamma > 0$ and a quantization density ρ, find, if it exists, a stabilizing controller pair, $(H_1(z), H_2(z))$, such that the closed-loop system in Figure 2 has $\|\mathcal{G}_{zw}\|_\infty < \gamma$. When a solution to this problem is available, the coarsest quantization density can be found by using a simple bisection algorithm.

Following the sector bound technique in (Fu and Xie 2003a), we can write

$$\xi(k) = (1 + \Delta(v(k)))v(k) \tag{13}$$

where

$$|\Delta(v(k))| \le \delta, \ \ \delta = (1-\rho)/(1+\rho) \tag{14}$$

That is, the quantization error lies in a sector of size δ.

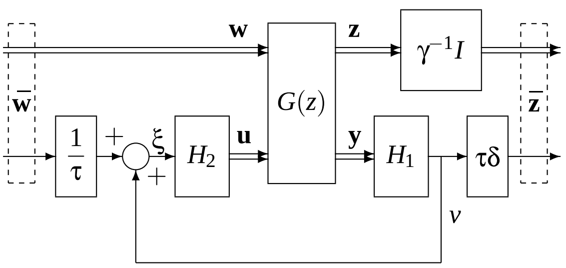

Fig. 4. Auxiliary System for H_∞ Control

We introduce an auxiliary system in Figure 4, where $\tau > 0$ is a scaling parameter to be searched. If we write $G(z) = \{G_{ij}(z)\}, \ i, j = 1, 2$, then it is easy to verify that the transfer function from $\bar{\mathbf{w}}$ to $\bar{\mathbf{z}}$ is given by

$$\bar{G}(z) = \begin{bmatrix} \frac{1}{\gamma}(G_{11} + G_{12}H_2HH_1G_{21}) & \frac{1}{\tau\gamma}G_{12}H_2H \\ (\tau\delta)HH_1G_{21} & \delta H_1G_{22}H_2H \end{bmatrix} \tag{15}$$

where

$$H(z) = (1 - H_1(z)G_{22}(z)H_2(z))^{-1} \tag{16}$$

We have the following result:

Theorem 5. Given $\gamma > 0$ and $\rho > 0$, the quantized feedback H_∞ control problem for the system in Figure 2 is solvable if there exists a controller pair $(H_1(z), H_2(z))$ and a scaling parameter $\tau > 0$ for the system in Figure 4 such that

$$\|\bar{G}(z)\|_\infty < 1 \tag{17}$$

Proof: It is straightforward to see that (17) implies that the transfer function from **w** to **z** in Figure 5 has an induced L_2-norm less than γ for any $|\Delta(k)| \le \delta$. It then follows from the sector bound approach in (Fu and Xie 2003a) that the latter is equivalent to that the quantized feedback H_∞ control problem for the system in Figure 2 is solvable for the given γ and ρ. ■

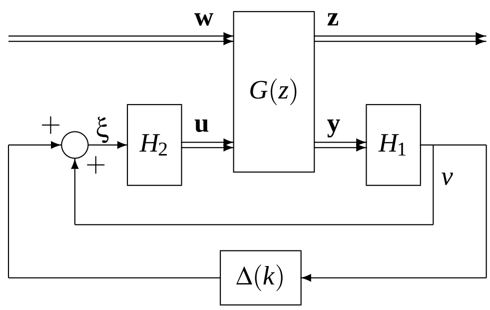

Fig. 5. Auxiliary System for H_∞ Control

Now we remark on the design of $H_1(z), H_2(z)$ and τ. From (15)-(16), it is clear that for a fixed τ, if $H_1(z)$ (resp. $H_2(z)$) is given, the design of $H_2(z)$ (resp. $H_1(z)$) is a standard H_∞ optimization problem. However, unlike the quantized feedback stabilization problems, $H_1(z)$ and $H_2(z)$ can not be designed jointly, even for scenarios S1, S2 and S3. Moreover, it is even not clear whether the optimal $H_1(z)$ and $H_2(z)$ has the same order as $\bar{G}(z)$.

To get around this difficulty, we propose to take either $H_1(z)$ or $H_2(z)$ as a constant vector. The idea is this vector is used to assign the zeros of $\bar{G}_{22}(z)$. We assume below that $H_1(z) = H_1$ is constant below although the proposed algorithm below works either way. Once H_1 is given, $H_2(z)$ and τ can be optimized. For a fixed τ, solving $H_2(z)$ is a standard H_∞ optimization problem. The parameter τ is then searched numerically. We caution that the minimum $\|\bar{G}(z)\|_\infty$ is not necessarily a convex function of τ. In addition, because the over-sampling ratio is typically small, H_1 can be found through a numerical search. To summarize, we use the following iterative algorithm.

Iterative Design Algorithm for H_∞ Control:

(1) For fixed H_1 and τ, use any H_∞ optimization algorithm on $H_1(z)$ to minimize $\|\bar{G}(z)\|_\infty$.
(2) Search numerically for an optimal τ.
(3) Search numerically for an optimal H_1.

Remark 4. We now comment on the assumption of $w_c(t)$ being constant for each input-output sampling period. This assumption can be easily relaxed. If $w_c(t)$ is relaxed to be constant for each sub-sampling period $h_0 = h/q_0$ for some integer $q_0 > 1$, then a discrete-time model similar to Figure 2 can be developed using the lifting technique. The resulting model has the same state dimension, but the dimension for $\mathbf{w}(k)$ will be increased by q_0 times. Other than this change, the result in Theorem 5 still applies. Furthermore, assumptions on $w_c(t)$ can be totally avoided by taking $q_0 \to \infty$ in which case matrices related to $\mathbf{w}(k)$ become infinite-dimensional linear operators.

5. SIMULATION EXAMPLE

To demonstrate the results in previous sections, we consider an example of system (1) with

$$A_c = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \; B_{1c} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \; B_{2c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C_{1c} = \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix}; \; D_{11c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \; D_{12c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_{2c} = [1 \;\; 1]; \; D_{21c} = 0; \; D_{22c} = 0$$

We take $T = 0.4$ and $q = 2$.

Firstly, we consider is quantized feedback stabilization in Scenario S1. For this case, we have

$$g_1(z) = [1 \;\; 0] G_{22}(z) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1.105(z-1.8181)}{(z-2.2246)(z-1.4924)}$$

Using Theorem 2, we know that $\delta_{\sup}$ is given by (10), which is computed to be

$$\delta_{\sup} = 0.0132$$

In comparison, the value of $\delta_{\sup}$ for the quantized state feedback is given by (8) which equals to

$$\delta_{\sup} = \frac{1}{2.2246 \times 1.4924} = 0.312$$

Secondly, we consider quantization feedback stabilization with Scenario S2 and take

$$H_1(z) = [1 \;\; 0]; \; H_2(z) = h(z)[1 \;\; -1.32]^T$$

which yields

$$[1 \;\; 0] G_{22}(z) \begin{bmatrix} 1 \\ -1.32 \end{bmatrix} = \frac{0.02042(z-0.7076)}{(z-2.2246)(z-1.4924)}$$

which is minimum phase and has relative degree 1. The corresponding $\delta_{\sup}$ coincides with that in the state feedback case above, as predicted by Theorem 3.

Finally, we consider the H_∞ control problem for both Scenarios 1 and 2. The result is shown in Figure 6. The two vertical lines correspond to the two values of $\delta_{\sup}$ as mentioned above. It is clear that the design corresponds to S2 can tolerate a much larger δ.

Remark 5. The simulation example above confirms that there is significant improvement by using over-sampling. However, we point out that the improvement given by over-sampling is not simply due to fast sampling. In fact, if we take $q = 1$ and $T = 0.2$ for stabilization, it gives $\delta_{\sup} = 0.2$ only.

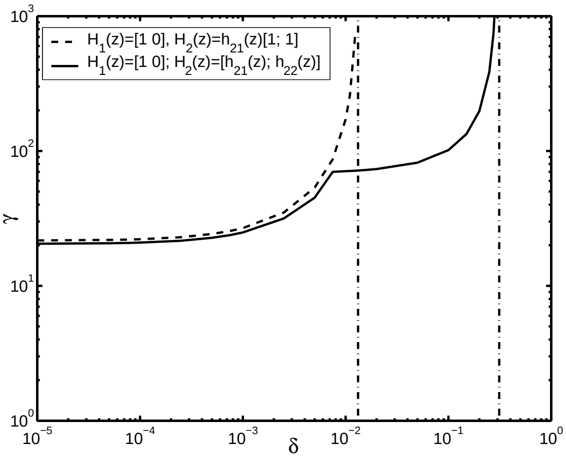

Fig. 6. Simulation for δ vs. γ

6. CONCLUSION

In this paper, we have studied two quantized feedback control problems for sampled-data systems: stabilization and H_∞ control. We have shown that the use of over-sampling can provide a significant improvement in achieving the coarsest quantization density. This is shown in Theorems 1-4 and a simulation example.

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