OUTPUT FEEDBACK MODEL MATCHING THROUGH SELF-BOUNDED CONTROLLED INVARIANT SUBSPACES

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Abstract: Model matching by output feedback is completely treated in the geometric approach framework. Self-bounded controlled invariant subspaces are shown to play a crucial role in the synthesis of minimal-order dynamic regulators achieving model matching by output feedback with stability. The approach provides insight into the internal eigenstructure of the minimal self-bounded controlled invariant subspace, thus paving the way to an effective treatment of nonminimum-phase systems.

Keywords: Linear systems, geometric approach, output feedback model matching.

1. INTRODUCTION

The synthesis of a minimal-order regulator achieving model matching by output feedback with stability is devised by using pure geometric arguments (Wonham, 1985; Basile and Marro, 1992). Although model matching problems has been widely discussed in the literature, after the pioneering work (Morse, 1973), which provided a state feedback solution for linear multivariable systems, just few papers were written approaching the problem by means of geometric/structural tools. The most of them, however, have addressed different classes of systems: nonlinear systems (Kotta, 1994), nonlinear recursive systems (Kotta, 1997), linear systems with delays (Picard *et al.*, 1996), 2D systems (Loiseau and Brethe, 1997), periodic systems (Colaneri and Kučera, 1997). In this paper, we consider model matching by dynamic feedforward and, since it can be reduced to a problem of measurable signal decoupling, we establish connections between structural and stabilizability conditions for measurable signal decoupling and structural and stabilizability properties of the system and the model. Theorem 2 relates the structural condition for measurable signal decoupling to a relative-degree condition on the system and the model. Theorems 3 and 4 relate the stabilizability condition for measurable signal decoupling to the invariant zero structure of the system and the eigenstructure of the model. These theorems exploit the properties of self-bounded controlled invariant subspaces, for the first time considered in the frame of model matching. Since Theorems 2 and 4 state sufficient conditions, they should also be regarded as guidelines to define an admissible model for a given system, in a nonconventional model matching problem where the designer may intervene on the model itself. Theorems 5, 6, and 7 give additional insight into the internal eigenstructure of the minimal self-bounded and suggest a straightforward procedure to deal with nonminimum-phase systems. Finally, we show how output feedback model matching can be reduced, from the structural point of view, to an equivalent feedforward problem (Theorem 8) and how the synthesis carried out with the criteria previously considered also guarantees internal stability of the closed loop (Theorems 9 and 10).

2. MODEL MATCHING BY DYNAMIC FEEDFORWARD

The original model matching problem is reduced to an equivalent signal decoupling problem where the signal to be decoupled is measurable. Hence, a feedforward solution is considered like that presented in (Zattoni, 2004). The discrete time-invariant linear system

$$x_s(t+1) = A_s\, x_s(t) + B_s\, u(t), \tag{1}$$
$$y_s(t) = C_s\, x_s(t), \tag{2}$$

is considered, where $x \in \mathcal{X}_s = \mathbb{R}^{n_s}$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$ respectively denote the state, the control input, and the controlled output. The system is assumed to be stable. The set of all admissible control input functions is defined as the set $\mathcal{U}_f$ of all bounded functions with values in $\mathbb{R}^p$. The discrete time-invariant linear model

$$x_m(t+1) = A_m\, x_m(t) + B_m\, h(t), \tag{3}$$
$$y_m(t) = C_m\, x_m(t), \tag{4}$$

is also considered, where $x \in \mathcal{X}_m = \mathbb{R}^{n_m}$, $h \in \mathbb{R}^s$, and $y \in \mathbb{R}^q$ respectively denote the state, the exogenous input, and the measurable output. Also the model is assumed to be stable. The set of all admissible exogenous input functions is defined as the set $\mathcal{H}_f$ of all bounded functions with values in $\mathbb{R}^s$. The matrices B_s, B_m, C_s, C_m are assumed to be full rank. The symbols $\mathcal{B}_s$, $\mathcal{B}_m$, $\mathcal{C}_s$, $\mathcal{C}_m$ are respectively used for $\operatorname{im} B_s$, $\operatorname{im} B_m$, $\ker C_s$, $\ker C_m$.

Problem 1. (Model Matching by Minimal-Order Dynamic Feedforward) Refer to Fig. 1. Let Σ_s be ruled by (1), (2), with $x_s(0) = 0$. Let Σ_m be ruled by (3), (4), with $x_m(0) = 0$. Let $\sigma(A_s) \subset \mathbb{C}^{\odot}$ and $\sigma(A_m) \subset \mathbb{C}^{\odot}$. Design a linear dynamic feedforward compensator $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ of minimal order, such that $\sigma(A_c) \subset \mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t \geq 0)$, $y_s(t) = y_m(t)$ for all $t \geq 0$.

Theorem 1. Problem 1 is equivalent to a measurable signal decoupling problem stated for the system

$$x(t+1) = A\, x(t) + B\, u(t) + H\, h(t), \tag{5}$$
$$y(t) = C\, x(t), \tag{6}$$

where the matrices are $A = \operatorname{diag}\{A_s,\, A_m\}$, $B = \begin{bmatrix} B_s^\top & O \end{bmatrix}^\top$, $H = \begin{bmatrix} O & B_m^\top \end{bmatrix}^\top$, $C = [\, C_s \;\; -C_m\,]$.

Proof: Set $x(t) = \begin{bmatrix} x_s(t)^\top & x_m(t)^\top \end{bmatrix}^\top$ and $y(t) = y_s(t) - y_m(t)$. The statement directly follows from the comparison of (1), (2) and (3), (4) with (5), (6). ■

In view of Theorem 1, the dynamic feedforward compensator Σ_c designed according to the procedure detailed in (Zattoni, 2004) preserves the features therein illustrated: minimum number of internal unassignable dynamics, in particular. Furthermore, Theorem 1 not only provides a straightforward technique to design a feedforward compensator with the properties mentioned above, but also enables connections to be established between the necessary and sufficient condition for measurable signal decoupling with stability (Basile and Marro, 1992) and the geometric properties of the original system and model. This investigation appears to be particularly useful from a practical point of view, since it provides easy-to-check conditions to verify solvability of the considered model matching problem and suggests how to modify a problem which is originally not solvable, in order to achieve a feasible and satisfactory trade-off. The next properties and theorems show that, if the system is right-invertible and the model is reachable, a straightforward relation established between a pair of easy-to-compute vectors in the model matching problem, namely the vector relative degree of the system and the vector minimum delay of the model, implies that the structural condition of the measurable signal decoupling problem, i.e. $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$, holds. The following definitions and properties are stated for a generic discrete time-invariant linear system

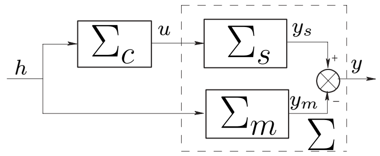

Fig. 1. Block diagram for feedforward model matching.

$$x(t+1) = A\, x(t) + B\, u(t) \tag{7}$$
$$y(t) = C\, x(t), \tag{8}$$

where $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$ respectively denote the state, the control input, and the controlled output, and where the matrices B and C are assumed to be full rank. The symbols $\mathcal{B}$ and $\mathcal{C}$ stand for $\operatorname{im} B$ and $\ker C$, respectively. The symbol $\mathcal{U}_f$ denotes the set of all admissible control input functions, defined as the set of all bounded functions with values in $\mathbb{R}^p$. The symbol $\mathcal{I}_q$ stands for the set $\{\, i \in \mathbb{Z}^+ :\;\; 1 \leq i \leq q \,\}$. For the sake of brevity, the proofs of the following properties are omitted and the reader should refer to (Marro and Zattoni, 2004).

Definition 1. Consider the system (7), (8) with $x(0) = 0$. Let (A, B, C) be right-invertible. The vector relative degree is the vector $\rho = [\, \rho_1 \ldots \rho_q \,]^\top$, where

$$\rho_i = \min_{u(\cdot)\,\in\,\mathcal{U}_f} \big\{ \bar{t} \in \mathbb{Z}^+ :\; y_i(\bar{t}) \neq 0,\;\; y_i(t) = 0,\; \forall\, t < \bar{t},$$

$$y_j(t)=0,\ \forall\, t\geq 0,\ j\in\mathcal{I}_q,\ j\neq i\}\ ,\ i\in\mathcal{I}_q.$$

Property 1. Consider the system (7), (8) with (A,B,C) right-invertible. For any $i\in\mathcal{I}_q$, let C_i be i-th row of C, $\mathcal{C}_i=\ker C_i$, $\bar{\mathcal{C}}_i=\cap_{j\,\in\,\mathcal{I}_q,\,j\,\neq\,i}\,\mathcal{C}_j$, $\bar{\mathcal{V}}_i^*=\max\mathcal{V}(A,\mathcal{B},\bar{\mathcal{C}}_i)$, $\bar{\mathcal{R}}_i^{(1)}=\mathcal{B}\cap\bar{\mathcal{V}}_i^*$, and, finally, $\bar{\mathcal{R}}_i^{(\eta)}=(A\,(\bar{\mathcal{R}}_i^{(\eta-1)}\cap\bar{\mathcal{V}}_i^*)+\mathcal{B})\cap\bar{\mathcal{V}}_i^*$, $\eta=2,\ldots,k_i$, k_i $(\leq n)$ the least integer such that $\bar{\mathcal{R}}_i^{(k_i+1)}=\bar{\mathcal{R}}_i^{(k_i)}$. Then, for any $i\in\mathcal{I}_q$, ρ_i is the least integer such that $C_i\,\bar{\mathcal{R}}_i^{(\rho_i)}\neq 0$.

Definition 2. Consider the system (7), (8) with $x(0)=0$. Let (A,B) be reachable. The vector minimum delay is the vector $\delta=[\,\delta_1\ldots\delta_q\,]^\top$, where

$$\delta_i=\min_{u(\cdot)\,\in\,\mathcal{U}_f}\left\{\,\bar{t}\in\mathbb{Z}^+ : y_i(\bar{t})\neq 0,\ y_i(t)=0,\ \forall\, t<\bar{t}\,\right\},\quad i\in\mathcal{I}_q.$$

Property 2. Consider the system (7), (8). Let (A,B) be reachable. For any $i\in\mathcal{I}_q$, let C_i denote the i-th row of C. Let $\mathcal{R}^{(1)}=\mathcal{B}$, $\mathcal{R}^{(\eta)}=A\,\mathcal{R}^{(\eta-1)}+\mathcal{B}$, $\eta=2,\ldots,k$, where $k\leq n$ is the least integer such that $\mathcal{R}^{(k+1)}=\mathcal{R}^{(k)}$. Then, δ_i is the least integer such that $C_i\,\mathcal{R}^{(\delta_i)}\neq 0$.

Theorem 2. Consider the system (1), (2) and the model (3), (4). Let (A_s,B_s,C_s) be right-invertible and (A_m,B_m) be reachable. Consider the system (5), (6), defined according to Theorem 1. Let δ_m denote the vector minimum delay of the model and ρ_s the vector relative degree of the system. Then,

$$\delta_m\geq\rho_s\quad\Longrightarrow\quad\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B},$$

Proof: For any $i\in\mathcal{I}_q$, let $\delta_{m,i}\geq\rho_{s,i}$. For any $h(\cdot)\in\mathcal{H}_f$, consider the corresponding effect, with the initial condition $x_m(0)=0$, at the, generic, i-th component of the output y_m. By Definition 2, for any $i\in\mathcal{I}_q$, $\bar{t}_i\geq\delta_{m,i}$ exists, such that $y_{m,i}(\bar{t}_i)\neq 0$ and $y_{m,i}(t)=0$ for all $t<\bar{t}_i$. Due to functional controllability of (A_s,B_s,C_s), if $\delta_{m,i}\geq\rho_{m,i}$, then $u_i(\cdot)\in\mathcal{U}_f$ exists, such that, with the initial condition $x_s(0)=0$, $y_{s,i}(t)=y_{m,i}(t)$ for all $t\geq\bar{t}_i$, $y_{s,i}(t)=0$, for all $t<\bar{t}_i$, and $y_{s,j}(t)=0$, for all $t\geq 0$, with $j\in\mathcal{I}_q$, $j\neq i$. Consequently, by superposition, for any input function $h(\cdot)\in\mathcal{H}_f$, which, with $x_m(0)=0$, produces a certain output $y_m(t)$, $t\geq 0$, a control function $u(\cdot)\in\mathcal{U}_f$ exists, such that $y_s(t)=y_m(t)$, for all $t\geq 0$. In the equivalent measurable signal decoupling problem, this means that for any $h(\cdot)\in\mathcal{H}_f$, $u(\cdot)\in\mathcal{U}_f$ exists, such that $y(t)=0$, for all $t\geq 0$. In other words, for any $h(\cdot)\in\mathcal{H}_f$, $u(\cdot)\in\mathcal{U}_f$ exists, such that the corresponding state trajectory, $x(t)$, $t\geq 0$, starting from $x(0)=0$, is steered on an $(A,\mathcal{B})$-controlled invariant, say $\mathcal{V}$, such that $\mathcal{V}\subseteq\mathcal{C}$ and $\mathcal{H}\subseteq\mathcal{V}+\mathcal{B}$. Finally, since $\mathcal{V}\subseteq\mathcal{V}^*$, the latter inclusion implies $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$. ■

The next results show that, on the assumption that the structural condition of the equivalent measurable signal decoupling problem holds, the stabilizability condition, namely internal stabilizability of the subspace $\mathcal{V}_m$, the minimal $(A,\mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$, is implied by a straightforward condition involving the invariant zeros of the plant and the poles of the model. In Theorem 3, as well as in Theorem 2, the given system is assumed to be right-invertible. Properties are reported without proofs which can be found in (Marro and Zattoni, 2004).

Property 3. Consider the systems (1), (2), (3), (4) and (5), (6), where (5), (6) is defined according to Theorem 1. Let $\mathcal{S}_s^*=\min\mathcal{S}(A_s,\mathcal{C}_s,\mathcal{B}_s)$ and $\mathcal{S}^*=\min\mathcal{S}(A,\mathcal{C},\mathcal{B})$. Let S_s^* be a basis matrix of $\mathcal{S}_s^*$. Then, $\mathcal{S}^*=\mathrm{im}\,[S_s^{*\top}\ O]^\top$.

Property 4. Consider the systems (1), (2), (3), (4) and (5), (6), where (5), (6) is defined according to Theorem 1. Let (A_s,B_s,C_s) be right-invertible. Then, (A,B,C) is right-invertible.

Property 5. Consider the systems (1), (2), (3), (4) and (5), (6), where (5), (6) is defined according to Theorem 1. Let $\mathcal{R}_{\mathcal{V}_s^*}=\max\mathcal{V}(A_s,\mathcal{B}_s,\mathcal{C}_s)\cap\mathcal{S}_s^*$ and $\mathcal{R}_{\mathcal{V}^*}=\max\mathcal{V}(A,\mathcal{B},\mathcal{C})\cap\mathcal{S}^*$. Let $R_{\mathcal{V}_s^*}$ be a basis matrix of $\mathcal{R}_{\mathcal{V}_s^*}$. Then, $\mathcal{R}_{\mathcal{V}^*}=\mathrm{im}\,[R_{\mathcal{V}_s^*}^\top\ O]^\top$.

Property 6. Consider the systems (1), (2), (3), (4) and (5), (6), where (5), (6) is defined according to Theorem 1. Let (A_s,B_s,C_s) be right-invertible. Let $\mathcal{V}_s^*=\max\mathcal{V}(A_s,\mathcal{B}_s,\mathcal{C}_s)$ and $\mathcal{V}^*=\max\mathcal{V}(A,\mathcal{B},\mathcal{C})$. Let V_s^* be a basis matrix of $\mathcal{V}_s^*$. Then, $\mathcal{V}^*=\mathrm{im}\begin{bmatrix}V_s^* & V_1\\ O & V_2\end{bmatrix}$, where both V_1 and V_2 are non-zero matrices and $\mathrm{rank}\,[V_1^\top\ V_2^\top]^\top=n_m$.

Theorem 3. Consider the systems (1), (2), (3), (4) and (5), (6), where (5), (6) is defined according to Theorem 1. Let (A_s,B_s,C_s) be right-invertible. Then, $\mathcal{Z}(A,B,C)=\mathcal{Z}(A_s,B_s,C_s)\uplus\sigma(A_m)$.

Proof: Let V^* denote a basis matrix of $\mathcal{V}^*$ and let F be any real matrix such that $(A+BF)\,\mathcal{V}^*\subseteq\mathcal{V}^*$. Then, a matrix X of appropriate dimension exists, such that $(A+BF)\,V^*=V^*X$. According to Property 6, $V^*=\begin{bmatrix}V_s^* & V_1\\ O & V_2\end{bmatrix}$, where V_s^* is a basis matrix of $\mathcal{V}_s^*$ and $\mathrm{rank}\,[V_1^\top\ V_2^\top]^\top=n_m$. Thus, the previous equation may also be written as

$$\begin{bmatrix}A_s+B_sF_1 & B_sF_2\\ O & A_m\end{bmatrix}\begin{bmatrix}V_s & V_1\\ O & V_2\end{bmatrix}=$$

$$\begin{bmatrix} V_s & V_1 \\ O & V_2 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \qquad (9)$$

where the structures A and B have been taken into account and where F and X have been partitioned according to V^*. The upper block-triangular structure of $A+BF$ and the particular structure of V^* in (9) imply $\sigma((A+BF)|_{\mathcal{V}^*}) = \sigma((A_s+B_sF_1)|_{\mathcal{V}_s^*}) \uplus \sigma(A_m)$. Finally, the thesis follows by virtue of Property 5. ■

Theorem 4. Consider the systems (1), (2), (3), (4) and (5), (6), where (5), (6) is defined according to Theorem 1. Let $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Let (A_s, B_s, C_s) be right-invertible, let $\mathcal{Z}(A_s, B_s, C_s) \subset \mathbb{C}^{\odot}$, and let $\sigma(A_m) \subset \mathbb{C}^{\odot}$. Then, $\mathcal{V}_m$, i.e. the minimal $(A, \mathcal{B}+\mathcal{H})$-controlled invariant such that $\mathcal{V}_m \subseteq \mathcal{C}$ and $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$, is internally stabilizable.

Proof: Recall that $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$ implies $\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C}) = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$ (Basile and Marro, 1992). Hence, $\mathcal{V}_m$ satisfies

$$\mathcal{V}_m = \max \mathcal{V}\,(A, \mathcal{B}, \mathcal{C}) \cap \min \mathcal{S}\,(A, \mathcal{C}, \mathcal{B}+\mathcal{H}),$$

$\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_m \subseteq \mathcal{V}^*$, and $(A+BF)\,\mathcal{V}_m \subseteq \mathcal{V}_m$ for any real matrix F such that $(A+BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. Therefore, $\sigma((A+BF)|_{\mathcal{V}_m/\mathcal{R}_{\mathcal{V}^*}}) \subseteq \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}})$. Moreover,

$$\sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \mathcal{Z}(A_s, B_s, C_s) \uplus \sigma(A_m),$$

due to Theorem 3, being (A_s, B_s, C_s) right-invertible. In conclusion, $\mathcal{Z}(A_s, B_s, C_s) \subset \mathbb{C}^{\odot}$ and $\sigma(A_m) \subset \mathbb{C}^{\odot}$ imply $\sigma((A+BF)|_{\mathcal{V}_m/\mathcal{R}_{\mathcal{V}^*}}) \subset \mathbb{C}^{\odot}$. ■

In the light of Theorems 3 and 4, a nonminimum-phase system seems to prevent the synthesis of an internally stable compensator. In fact, an invariant zero of the system outside the open unit disc results into an unstable internal unassignable eigenvalue of the subspace $\mathcal{V}_m$, thus violating the stabilizability condition of the equivalent measurable signal decoupling problem. However, also nonminimum-phase systems may be handled, at the cost of modifying the model so as to include the same unstable invariant zeros of the system, with some further constraints as specified below.

Theorem 5. Consider the systems (1), (2), (3), (4) and (5), (6), where (5), (6) is defined according to Theorem 1. Let $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Then, the invariant zero structure of $(A, [B\,H], C)$ is part of the external eigenstructure of $\mathcal{V}_m$.

Proof: The invariant zero structure of $(A, [B\,H], C)$ is the internal unassignable eigenstructure of $\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C})$. Hence, it is part of the external eigenstructure of the constrained reachability subspace

$$\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C}) \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B}+\mathcal{H}).$$

Moreover, if $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$, then

$$\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C}) = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$$

(Basile and Marro, 1992). This implies

$$\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C}) \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B}+\mathcal{H}) = \mathcal{V}_m. \; ■$$

Theorem 6. Consider the system (7), (8) and its dual, defined by the triple $(A^\top, C^\top, B^\top)$. Then, (A, B, C) and $(A^\top, C^\top, B^\top)$ have the same invariant zero structure.

Proof: Consider the system (7), (8) and perform the similarity transformations $T = [\,T_1\; T_2\; T_3\; T_4\,]$, where T_1, T_2, and T_3 are s. t. $\operatorname{im} T_1 = \mathcal{R}_{\mathcal{V}^*}$, $\operatorname{im}\,[\,T_1\; T_2\,] = \mathcal{V}^*$, $\operatorname{im}\,[\,T_1\; T_3\,] = \mathcal{S}^*$, and $U = [\,U_1\; U_2\,]$, where U_1 and U_2 are s. t. $\operatorname{im} U_1 = B^{-1}\mathcal{V}^*$, $\operatorname{im} U_2 = \left(B^{-1}\mathcal{V}^*\right)^\perp$. The matrices A', B', C', respectively corresponding to A, B, C in the new bases, partitioned according to T and U, have the structures

$$A' = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} & A'_{14} \\ O & A'_{22} & A'_{23} & A'_{24} \\ A'_{31} & A'_{32} & A'_{33} & A'_{34} \\ O & O & A'_{43} & A'_{44} \end{bmatrix}, \; B' = \begin{bmatrix} B'_{11} & B'_{12} \\ O & O \\ O & B'_{32} \\ O & O \end{bmatrix},$$
$$C' = \begin{bmatrix} O & O & C'_{13} & C'_{14} \end{bmatrix}.$$

Consider the dual triple in the new bases, i.e. $(A'^\top, C'^\top, B'^\top)$. By simple inspection one gets

$$\mathcal{V}_d^* = \max \mathcal{V}(A^\top, \mathcal{C}^\perp, \mathcal{B}^\perp) = \operatorname{im} V_d'^* = \operatorname{im} \begin{bmatrix} O & O \\ I & O \\ O & O \\ O & I \end{bmatrix}.$$

Let $G^\top$ be any real matrix s. t.

$$(A^\top + C^\top G^\top)\mathcal{V}_d^* \subseteq \mathcal{V}_d^*.$$

In the new bases, let $G'^\top = [G_{11}^\top \; G_{21}^\top \; G_{31}^\top \; G_{41}^\top]$. Then, $A_G'^\top = A'^\top + C'^\top G'^\top$ has the structure

$$A_G'^\top = \begin{bmatrix} A_{11}'^\top & O & A_{31}'^\top & O \\ A_{12}'^\top & A_{22}'^\top & A_{32}'^\top & O \\ A_{G13}'^\top & O & A_{G33}'^\top & O \\ A_{G14}'^\top & A_{G24}'^\top & A_{G34}'^\top & A_{G44}'^\top \end{bmatrix},$$

where $A_{Gj4}'^\top = A_{j4}'^\top + C_{14}'^\top G_{j1}'^\top$, with $j = 1, 2, 3, 4$, $A_{Gj3}'^\top = A_{j3}'^\top + C_{13}'^\top G_{j1}'^\top$, with $j = 1, 3$, and where $A_{Gj3}'^\top = A_{j3}'^\top + C_{13}'^\top G_{j1}'^\top$, with $j = 2, 4$, are set to zero by imposing $G_{j1}'^\top = -(C_{13}'^\top)^+ A_{j3}'^\top$, with $j = 2, 4$, respectively. Then, it is trivial to verify that $A_G'^\top V_d'^* = V_d'^* X$ holds, with

$$X = \begin{bmatrix} A_{22}'^\top & O \\ A_{G24}'^\top & A_{G44}'^\top \end{bmatrix}.$$

Since $X=(A^\top+C^\top G^\top)|_{\mathcal{V}_d^*}$ is lower block-triangular,

$$\sigma((A^\top+C^\top G^\top)|_{\mathcal{V}_d^*})=\sigma(A'^{\top}_{22})\uplus\sigma(A'^{\top}_{G44}).$$

Hence, the set of the internal unassignable eigenvalues of $\mathcal{V}_d^*$, i.e. $\sigma(A'^{\top}_{22})$, matches that of $\mathcal{V}^*$. ■

Theorem 7. Consider the systems (1), (2), (3), (4) and (5), (6), where (5), (6) is defined according to Theorem 1. Let $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$. Let (A_s,B_s,C_s) and (A_m,B_m,C_m) be right- and left-invertible. Let X be a real Jordan block, part of the invariant zero structure of both (A_s,B_s,C_s) and (A_m,B_m,C_m). If matrices V_s, V_m, and L of appropriate dimensions exist, such that $A_s^\top V_s-V_sX=-C_s^\top L$, $B_s^\top V_s=O$, $A_m^\top V_m-V_mX=C_m^\top L$, and $B_m^\top V_m=O$, then X is part of the eigenstructure external to $\mathcal{V}_m$.

Proof: Let X be part of the invariant zero structure of both (A_s,B_s,C_s) and (A_m,B_m,C_m). Then, by virtue of Theorem 6, it is also part of the invariant zero structure of $(A_s^\top,C_s^\top,B_s^\top)$ and $(A_m^\top,C_m^\top,B_m^\top)$. Since (A_s,B_s,C_s) and (A_m,B_m,C_m) are right- and left-invertible by assumption, $(A_s^\top,C_s^\top,B_s^\top)$ and $(A_m^\top,C_m^\top,B_m^\top)$ are right- and left-invertible, too. Hence, matrices V_s, V_m, L_s, and L_m of appropriate dimensions exist, s.t. $A_s^\top V_s-V_sX=-C_s^\top L_s$, $B_s^\top V_s=O$, $A_m^\top V_m-V_mX=C_m^\top L_m$, and, finally, $B_m^\top V_m=O$. In particular, if $L_s=L_m=L$, then the above equations may also be written in compact form as

$$\begin{bmatrix}A_s^\top & O\\ O & A_m^\top\end{bmatrix}\begin{bmatrix}V_s\\ V_m\end{bmatrix}-\begin{bmatrix}V_s\\ V_m\end{bmatrix}X=-\begin{bmatrix}C_s^\top\\ -C_m^\top\end{bmatrix}L, \qquad (10)$$

$$\begin{bmatrix}B_s^\top & O\\ O & B_m^\top\end{bmatrix}\begin{bmatrix}V_s\\ V_m\end{bmatrix}=\begin{bmatrix}O\\ O\end{bmatrix}. \qquad (11)$$

Since the triple $(A^\top,C^\top,[B\,H]^\top)$ is left-invertible (as a consequence of Property 3 and duality), equations (10),(11) imply that X is part of the invariant zero structure of $(A^\top,C^\top,[B\,H]^\top)$. Hence, by virtue of Theorem 6, X is part of the invariant zero structure of $(A,[B\,H],C)$, which implies that it is part of the eigenstructure external to $\mathcal{V}_m$, due to Theorem 5. ■

In view of the previous results, a real Jordan block X corresponding to an unstable invariant zero of (A_s,B_s,C_s) does not necessarily imply violation of the stabilizability condition. In fact, it may be removed from the eigenstructure internal to $\mathcal{V}_m$, by replicating it as part of the invariant zero structure of the model, with a further constraint on the so-called input distribution matrix L according to Theorem 7. Also non-left-invertible systems may be handled, by resorting to the techniques detailed in (Zattoni, 2004).

3. OUTPUT FEEDBACK MODEL MATCHING

Throughout this section, the system (1), (2) and the model (3), (4) are considered, with the assumptions made in Section 2 and the further assumption that the model is square.

Problem 2. (Model Matching by Minimal-Order Dynamic Output Feedback) Refer to Fig. 2. Let Σ_s be ruled by (1), (2), with $x_s(0)=0$. Let Σ_m be ruled by (3), (4), with $x_m(0)=0$. Let $\sigma(A_s)\subset\mathbb{C}^{\odot}$ and $\sigma(A_m)\subset\mathbb{C}^{\odot}$. Design a linear dynamic regulator $\Sigma_c\equiv(A_c,B_c,C_c,D_c)$ of minimal order, such that the loop is internally and externally stable and, for all admissible $h(t)$ $(t\geq 0)$, $y_s(t)=y_m(t)$ for all $t\geq 0$.

The next Theorem 8 shows that, from the structural point of view, the output feedback model matching problem is equivalent to a feedforward model matching problem which refers to a suitably modified model.

Theorem 8. Refer to Fig. 2. Let Σ_s be ruled by (1), (2), with $x_s(0)=0$. Let Σ_m be ruled by (3), (4), with $x_m(0)=0$. Then, $\Sigma_c\equiv(A_c,B_c,C_c,D_c)$ is a minimal-order regulator solving the structural output feedback model matching problem, i.e. such that for all admissible $h(t)$ $(t\geq 0)$, $y_s(t)=y_m(t)$ for all $t\geq 0$, if and only if Σ_c is a minimal-order compensator solving the structural feedforward model matching problem for the modified model $\Sigma'_m\equiv(A_m+B_mC_m,B_m,C_m)$.

Proof: From the structural point of view, the block diagram in Fig. 3 is equivalent to that shown in Fig. 2. In fact, it is obtained by adding the same signal $y_m(t)$ both to the input of the loop and to the input of the model and taking into account that, on the assumption that Σ_c guarantees that, for all admissible $h(t)$, $(t\geq 0)$, $y(t)=0$ for all $t\geq 0$, it is $y_m(t)=y_s(t)$ for all $t\geq 0$. ■

Thus, the dynamic output feedback model matching problem is reduced to an equivalent feedforward model matching problem, as far as the structural aspects are concerned. The next Theorems 9 and 10 concern internal and external stability of the loop, when the plant is minimum-phase and nonminimum-phase, respectively. The

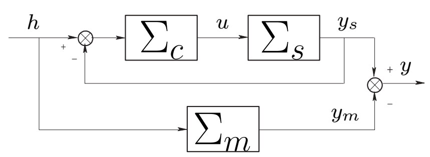

Fig. 2. Block diagram for dynamic output feedback model matching.

minimal-order regulator Σ_c is designed in order to solve the feedforward model matching problem for the modified plant from the structural point of view. This is achieved by following the procedure detailed in (Zattoni, 2004), but leaving apart the question of internal stabilizability of $\mathcal{V}_m$.

Theorem 9. Consider the system (1), (2) and the model (3), (4). Let (A_s, B_s, C_s) be right-invertible, $\sigma(A_s) \subset \mathbb{C}^{\odot}$, $\sigma(A_m) \subset \mathbb{C}^{\odot}$, and $\mathcal{Z}(A_s, B_s, C_s) \subset \mathbb{C}^{\odot}$. Let $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ be a minimal-order regulator solving the structural output feedback model matching problem according to Theorem 8. Then, the loop is internally and externally stable.

Proof: Since the structural property, namely $y(t)=0$ for all $t \geq 0$, for any admissible $h(t)$ $(t \geq 0)$, is preserved in the equivalence between the block diagrams shown in Fig. 2 and in Fig. 3, stability of the original model implies external stability of the loop. As to internal stability, note that, according to Theorem 3, the poles of Σ_c are a subset of $\mathcal{Z}(A_s, B_s, C_s) \uplus \sigma(A_m + B_m C_m)$, where $\sigma(A_m + B_m C_m)$ is not necessarily contained in the open unit disc. This implies that Σ_c is not necessarily stable. Nevertheless, the loop is internally stable since cancellations outside the open unit disc are prevented by the assumption that Σ_s is minimum-phase. ■

Theorem 10. Consider the system (1), (2) and the model (3), (4). Let (A_s, B_s, C_s) and (A_m, B_m, C_m) be right- and left-invertible. Let $\sigma(A_s) \subset \mathbb{C}^{\odot}$, $\sigma(A_m) \subset \mathbb{C}^{\odot}$, and

$$\mathcal{Z}(A_s, B_s, C_s) \cap \sigma(A_m + B_m C_m) = \emptyset.$$

Let the unstable part of the invariant zero structure of (A_s, B_s, C_s) be replicated as part of the invariant zero structure of (A_m, B_m, C_m) according to Theorem 7. Let $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ be a minimal-order regulator solving the structural output feedback model matching problem according to Theorem 8. Then, the loop is internally and externally stable.

Proof: External stability is guaranteed by stability of the model and preservation of the structural property ($y(t)=0$ for all $t \geq 0$, for any admissible $h(t)$, $t \geq 0$) in the equivalence between the block diagrams in Figs. 2 and 3. As to internal stability, since output feedback does not modify the invariant zero structure of the model, the unstable part of the invariant zero structure of (A_s, B_s, C_s), reproduced in (A_m, B_m, C_m) according to Theorem 7, is also part of the invariant zero structure of $(A_m + B_m C_m, B_m, C_m)$. Hence, due to Theorem 7, it is not part of the internal unassignable eigenstructure of $\mathcal{V}_m$, or, equivalently, it is not part of the eigenstructure of Σ_c.

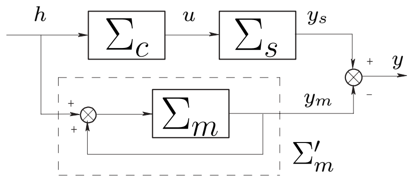

Fig. 3. Block diagram for equivalent feedforward model matching.

Thus, cancellations outside the open unit disc are avoided for nonminimum-phase plants. ■

4. CONCLUSIONS

The design of a dynamic regulator of minimal order which achieves model matching by output feedback has been thoroughly accomplished in the geometric context. The structural properties of self-bounded controlled invariant subspaces have been shown to be fundamental to both the minimization of the regulator complexity and the stabilization of the closed loop, particularly in the presence of nonminimum-phase systems.

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