ADAPTIVE MULTI-PERIODIC REPETITIVE CONTROL FOR A CLASS OF NON-LINEAR SYSTEMS

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Abstract: Recently it was shown that for the linear time-invariant plant ASPR(Almost Strictly Positive Real)/ASNR(Almost Strictly Negative Real) property can be used to design an adaptive multi-periodic repetitive control system that guarantee the Lyapunov stability of the whole system. This paper attempts to extend this concept to SP(Strictly Passive)/ASP(Almost Strictly Passive) non-linear plant. The general non-linear plant without *a priori* knowledge of parameters is considered. A Nussbaum-type adaptive gain is introduced when the control direction of non-linear plant is unknown. Lyapunov stability analysis is presented and simulation is given accordingly. *Copyright©2005 IFAC*

Keywords: iterative methods, passive, non-linear systems, Lyapunov stability, adaptive control

1. INTRODUCTION

Repetitive control has proven to be very effective for a system to track/reject periodic reference/disturbance signal in practical applications. In many cases, the reference and/or disturbance periodic signals may contain different fundamental frequencies and the ratio of these frequencies can be irrational. So the so-called multi-periodic repetitive control was analysed by several authors (Weiss and Hafele, 1999) (Owens *et al.*, 2004) (Dang and Owens, 2004).

In (Weiss and Hafele, 1999), Weiss gave a H^∞ stability condition for linear multi-periodic system based on input-output transfer function, which requires that the plant be positive real or approximately so. The Lyapunov stability analysis is given by (Owens *et al.*, 2004). He proved that asymptotic/exponential stability is guaranteed if the linear plant is PR(Positive Real)/SPR(Strictly Positive Real). When the real plant is not necessarily positive real, however minimum-phase and having sign definite high-frequency gain, Dang(Dang and Owens, 2004) designed an universal adaptive multi-periodic repetitive control system, which doesn't need any plant parameter information. Such plant is called ASPR(Almost Strictly Positive Real) or ASNR(Almost Strictly Negative Real).

It is known that SP(Strictly Passive)/ASP(Almost Strictly Passive) properties of non-linear plant are equivalent to SPR/ASPR properties in linear case. Motivated by this observation, this paper attempts to show that controlled plant structures like passivity and almost passivity render possibility of an adaptive multi-periodic repetitive control scheme for such non-linear plant. Same as (Dang and Owens, 2004), non-identifier-based universal adaptive control algorithm is designed. A rigorous Lyapunov stability analysis is presented and some simulations are given accordingly..

The paper is organized as follows. Section 2 defines the problem. In Section 3, some basic definitions and results about strictly passive and almost strictly passive system are recalled. In Section 4, an adaptive multi-periodic repetitive control scheme is designed for the SP/ASP general non-linear plant. Lyapunov second method is applied for system stability analysis. In Section 5, a Nussbaum-type adaptive gain is introduced when the control direction of non-linear plant is unknown. Simulation results are presented in Section 6. Finally in Section 7, conclusion is given.

2. PROBLEM DEFINITION

The considered general non-linear plant is described as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned} \tag{1}$$

where $x(t) \in X = R^n$, $u(t) \in U = R^m$, $y(t) \in Y = R^m$. Assume that the plant has at least one equilibrium. Without loss of generality, we assume that the plant (1) has an equilibrium at the origin, that is, $f(0,0) = 0$, $h(0,0) = 0$.

The non-linear adaptive MIMO multi-periodic repetitive control system is shown in Fig. 1. The R, D, Y, U, E are reference, disturbance, output, control input and error respectively. The plant $\sum_G$ is finite-dimensional and non-linear. Both reference $r(t)$ and disturbance $d(t)$ are multi-periodic with components of period $\tau_i, i = 1, ..., p$. These periods are assumed known. The multi-periodic repetitive controller is $M(s) = \sum_{i=1}^{p} \frac{\alpha_i I}{1 - W_i(s) e^{-s\tau_i}}$, we select $\sum_{i=1}^{p} \alpha_i = 1$ without loss of generality. $W_i(s)$ is a low-pass filter introduced to filter out noise and/or trade off tracking accuracy against closed-loop robustness. $C_1(s), C_2(s)$ are feed-forward gains given in the following sections designed to guarantee the Lyapunov stability of the whole system including the plant.

The control design objective is to design adaptive feed-forward controller $C_1(s), C_2(s)$ that guarantees the BIBO stability of the whole system under the condition that only minimal plant information should be needed. The paper does not propose that the algorithm discussed is ideal in practice. Rather, it demonstrates the potential for achieving stability under conditions of extreme uncertainty. With this in mind, the community can confidently address the issue of improved general-purpose algorithms based on more complete plant information.

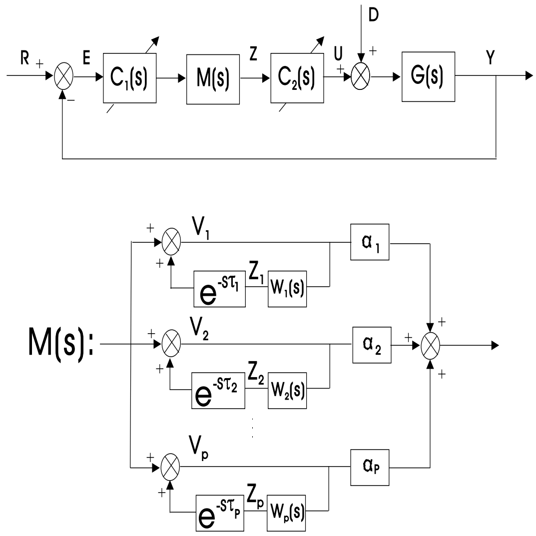

Fig. 1. Non-linear adaptive MIMO multi-periodic repetitive control system

3. PRELIMINARY DEFINITIONS AND SOME PROPOSITIONS

In this section, we review a number of basic concepts related to the notions of dissipativity and passivity. Then some propositions are given and proved, which prepares the stability proofs for adaptive stabilization and tracking in Section 4.

Definition 1. Strictly Dissipative(Sepulchre *et al.*, 1997): Assume that associated with the system (1) there is a function $w : U \times Y \to R$, called the supply rate, which is locally integrable for every $u \in U$, that is, it satisfies $\int_{t_0}^{t_1} |w(u(t), y(t))| dt < \infty$ for all $t_1 \geq t_0$. We say that the system (1) is strictly dissipative in X with the supply rate $w(u, y)$ if there exists a function $V_1(x)$, $V_1(0) = 0$ and $V_1(x) \geq 0$, such that for all $x \in X$ and a positive definite function $S(x(t))$,

$$\begin{aligned} &V_1(x(t)) - V_1(x(0)) \\ &= \int_0^t w(u(\tau), y(\tau)) d\tau - \int_0^t S(x(\tau)) d\tau \end{aligned} \tag{2}$$

for all $u \in U$ and all $t \geq 0$. The function $V_1(x)$ is called a storage function.

Definition 2. Strictly Passive(Willems, 1972)(Byrnes and Isidori, 1991): A system (1) is strictly passive if it is strictly dissipative with supply rate $w(u(t), y(t)) = y^T(t)u(t)$.

Definition 3. Output Feedback Passive(Sepulchre *et al.*, 1997): A system (1) is said to be OFP(Output Feedback Passive) if it is dissipative with supply rate $w(u(t), y(t)) = y^T(t)u(t) - \rho y^T(t)y(t)$ for some $\rho \in R$. The output feedback passive properties are quantified with the notation $OFP(\rho)$.

Remark 1. It's obvious that positive sign of ρ means that the system has an excess of passivity and conversely negative sign of ρ means that the system has a shortage of passivity.

Definition 4. Almost Strictly Passive(Kaufman *et al.*, 1997): A system (1) is almost strictly passive(ASP) if there exists a positive definite static(constant or time-varying, however bounded) feedback matrix $K_e(y)$ such that the resulting closed-loop system is strictly passive(SP).

Proposition 1. A system (1) is ASP if there exists a non-negative function $V_1(x(t))$ and a positive definite function $S(x(t))$ such that

$$\begin{aligned} V_1(x(t)) - V_1(x(0)) &= \int_0^t y^T(\tau)u(\tau)d\tau \\ +M \int_0^t y^T(\tau)y(\tau)d\tau &- \int_0^t S(x(\tau))d\tau \end{aligned} \tag{3}$$

for all $u(t)$ and all $t \geq 0$ and some positive constant M.

Proof: According to Definition 4, the closed-loop system from $u_r(t)$ to $y(t)$ is SP, where $u_r(t)$ is the new control input of the closed-loop system. From $u(t) = u_r(t) - k_e(y)y(t)$, we have $u_r(t) = u(t) + k_e(y)y(t)$. Then due to passivity, for all $u(t)$ and all $t \geq 0$ there exists a nonnegative function $V_1(x(t))$, and a positive definite function $S(x(t))$ such that

$$\begin{aligned} & V_1(x(t)) - V_1(x(0)) \\ &= \int_0^t y^T(\tau)u_r(\tau)d\tau - \int_0^t S(x(\tau))d\tau \\ &= \int_0^t y^T(\tau)u(\tau)d\tau + M\int_0^t y^T(\tau)y(\tau)d\tau \\ &- \int_0^t S(x(\tau))d\tau \end{aligned} \tag{4}$$

where M is some positive constant. □

Remark 2. It's obvious that an almost passive system is an output feedback passive with $\rho < 0$, that means the system has a shortage of passivity.

4. GENERAL PASSIVE/ALMOST PASSIVE NON-LINEAR PLANT

Based on the above definitions, our main result is stated in the following theorem.

Theorem 1. Consider the SP or ASP plant $\sum_G$ described by (1). The reference/disturbance satisfies $r \in L_2[0,\infty)$, $d \in L_2[0,\infty)$. All filters $W_i(.)$ satisfy $|W_i(.)| < 1$. If the following constant/adaptive feed-forward control laws

(α): $C_1(.) := k$ is any positive constant and $C_2(.) := I$, if $\sum_G$ is SP.

(β): $C_1(.) := k(t)$ is an adaptive gain with adaptive law $\dot{k}(t) = \|e(t)\|^2$ and $C_2(.) := I$, if $\sum_G$ is ASP.

and arbitrary $x_0 \in X$ is applied to (1), then the non-linear multi-periodic repetitive system in Fig. 1 is stable in the BIBO(bounded-input/bounded-output) sense that

$$\int_{t_1}^t \|e(\theta)\|^2 d\theta < M_1 + M_2 \int_{t_1}^t \|r(\theta)\|^2 d\theta \tag{5}$$

for some positive constants M_1, M_2, finite time t_1 and also $k(.) \in L_\infty[0,\infty)$, $\lim_{t\to\infty} k(t) = k_\infty < \infty$ for (β).

Proof: See Appendix 1.

Remark 3. Because perfect zero-tracking for periodic reference signals will be lost if the low-pass filter is not selected to be 1. we need to revise the adaptive scheme of $k(t)$ as

$$\dot{k}(t) = \begin{cases} \|e(t)\|(\|e(t)\| - \delta) & \text{if } \|e(t)\| \geq \delta \\ 0 & \text{if } \|e(t)\| < \delta \end{cases}$$

The inclusion of δ in the adaptive scheme of $k(t)$ is to prevent its divergence and also considers the possible system output measurement error.

Remark 4. Here we only assume $r \in L_2[0,\infty)$, which is not satisfied by all multi-periodic signals(*e.g.* $r = 5 + sin2\pi t + cos\sqrt{2}\pi t$). In (Owens *et al.*, 2004), Owens assumed that $\frac{1}{T}\int_0^T r^T(t)r(t)dt < +\infty$ for any value of T including $T = \infty$, which multi-periodic signals satisfy. We can't assume that here because we can't give a rigorous proof of $k(.) \in L_\infty[0,\infty)$ although the simulation in Section 6 shows that $k(.)$ is bounded.

5. NON-LINEAR PLANT WITH UNKNOWN CONTROL DIRECTIONS

In section 4, the signs of unknown parameters multiplying *control* variables, called *control direction* in (Ye and Jiang, 1998)(Jiang *et al.*, 1995)(Kaloust and Qu, 1995), are required to be known *a priori.* These signs represent motion directions of the plant under any *control* and knowledge of these signs makes adaptive control design much easier. The objective of this section is to develop control design procedure which does not require *a priori* knowledge of control directions.

Now we introduce a Nussbaum-type adaptive feed-forward gain: $N(\lambda(t))\Gamma$ where $\Gamma = I_{m\times m}, N(.): R \to R$ is any continuous function of Nussbaum-type(Nussbaum, 1983) , that is, $N(.)$ has the properties

$\sup_{k>k_0} \frac{1}{k-k_0}\int_{k_0}^{k} N(\tau)d\tau = +\infty$ and

$\inf_{k>k_0} \frac{1}{k-k_0}\int_{k_0}^{k} N(\tau)d\tau = -\infty$.

For example, $N(.): \tau \to \tau^2\cos\tau$ suffices.

The Nussbaum-type gain here acts like a switching gain which learns the right sign(spectrum) of the plant control directions.

Theorem 2. Consider the plant $\sum_G$ described as follows:

$$\begin{aligned} &\dot{x}(t) = f(x(t), \sigma u(t)), x(0) = x_0 \\ &y(t) = h(x(t), \sigma u(t)) \end{aligned} \tag{6}$$

where $\sigma \in (1, -1)$ is unknown and the plant from $\sigma u(t)$ to $y(t)$ is ASP. Suppose that both reference $r(t)$ and disturbance $d(t)$ are identically zero. $C_1(.)$ is same as (β) case in Section 4. $C_2(.) := N(\lambda(t)))I$ is a Nussbaum-type gain with adaptive law $\dot{\lambda}(t) = e(t)^T z(t), \lambda(0) \in R$. The low-pass filter $W_i(s)$ is set to be 1 for simplicity. Then the non-linear adaptive multi-periodic repetitive system in Figure 1 is globally asymptotically stable in the sense that $y(.) \in L_2^m[0,\infty)$, $\lambda(.) \in L_\infty[0,\infty)$, $k(.) \in L_\infty[0,\infty)$ and $\lim_{t\to\infty} k(t) = k_\infty < \infty$.

Proof: See Appendix 2. It should be pointed out that we can't prove that $\lim_{t\to\infty}\lambda(t) = \lambda_\infty < \infty$ although the simulation seems to show λ converges.

6. SIMULATION

For sake of simplicity, a SISO system is examined to illustrate the control system performance. The reference is $r = r_1 + r_2$, where $r_1 = \sin\omega_1 t + 1.5\sin 5\omega_1 t$, $r_2 = \sin\omega_2 t$ and $\omega_1 = 0.2\times 2\pi$rad/sec, $\omega_2 = \sqrt{3}\times 2\pi$rad/sec. The disturbance is a square wave at a period of $\frac{1}{7}Hz$ and with peak value 7 and 3. A square wave is chosen to indicate the scheme can cope with signals with infinite frequency content. The weightings are chosen to be 0.4, 0.4, 0.2(for the disturbance rejection repetitive sub-controller).

Simulation 1. A simulation is done for ASP general passive non-linear plant described as

$$\begin{aligned} &\dot{x}_1 = -x_1u^2 - 3x_3x_1^2, \quad x_1(0) = 1 \\ &\dot{x}_2 = -2x_3 - 7x_2^3, \quad x_2(0) = -7 \\ &\dot{x}_3 = 3x_1^3 + 2x_2 + 8x_3 + u, \quad x_3(0) = 5 \\ &y = x_3 \end{aligned} \tag{7}$$

The control scheme of (β) in Section 4 is applied and we select $k(0) = 1$, $W_i(.) = 0.99$ and $\delta = 0.01$. The simulation result is given in Fig. 2.

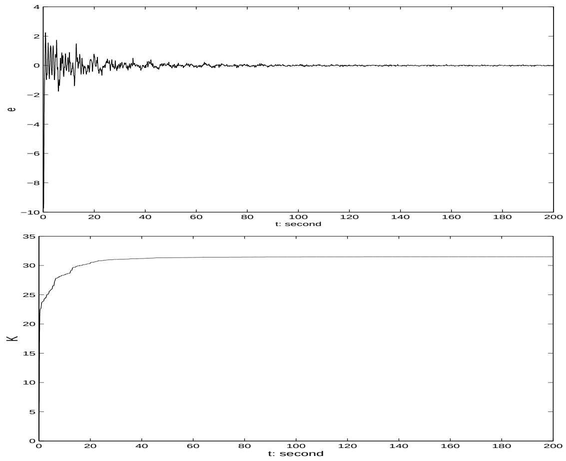

Fig. 2. Error $e(t)$ and adaptive gain $k(t)$ for ASP plant

The simulation result shows that the output converges to a bound of the reference input and the adaptive gain converges to a positive constant.

Simulation 2. A simulation is also done for non-linear plant with unknown control directions described as

$$\begin{aligned} &\dot{x}_1 = -x_1 - 3x_3x_1^2, \quad x_1(0) = 1 \\ &\dot{x}_2 = -2x_3 - 7x_2^3, \quad x_2(0) = 5 \\ &\dot{x}_3 = 3x_1^3 + 2x_2 + 8x_3 + \sigma u, \quad x_3(0) = 8 \\ &y = x_3 \end{aligned} \tag{8}$$

Here we set $\sigma = -4$. The control scheme in Section 5 is applied and we select $k(0) = 1$, $W_i(.) = 1$ and $\lambda(0) = 0$. The simulation result is given in Fig. 3.

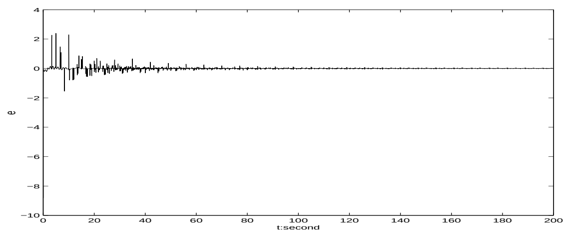

Fig. 3. Error $e(t)$ for non-linear plant with unknown control direction

The simulation result shows that the Nussbaum-type control scheme is capable for the plant to track/reject a multi-periodic reference/disturbance signal.

7. CONCLUSION

A kind of non-linear adaptive MIMO multi-periodic repetitive control system is studied. The system is proved to be stable in BIBO sense and the stability is analysed by Lyapunov second method. The adapting gains are proved to be bounded and converge. A Nussbaum-type adaptive gain is introduced when control direction of non-linear plant is unknown. Simulations show the effectiveness of the proposed adaptive scheme.

REFERENCES

Byrnes, C. I. and A. Isidori (1991). Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans. on Automatic Control* **36**, 1228–1240.

Dang, H. and D.H. Owens (2004). Mimo multi-periodic repetitive control system: adaptive control schemes. *Proceedings of 43rd IEEE Conference on Decision and Control, Bahamas.*

Jiang, Y.A., D.J. Clements and T. Hesketh (1995). Adaptive repetitive control of nonlinear systems. *Proceedings of the 34th Conference on Decidion and Control, New Orleans, U.S.A.*

Kaloust, J. and Z. Qu (1995). Continuous robust control design for nonlinear uncertain systems without a priori knowledge of control direction. *IEEE Trans. on Automatic Control* **40**, 276–282.

Kaufman, Howard, Itzhak Barkana and Kenneth Sobel (1997). *Direct adaptive control algorithms: theory and applications. second edition.* Springer. New York.

Nussbaum, R.D. (1983). Some remarks on a conjecture in parameter adaptive control. *Systems Control Letters* **3**, 243–246.

Owens, D.H., L.M. Li and S.P. Banks (2004). Multi-periodic repetitive control system: a liapunov analysis for mimo systems. *INT. J. Control* **77**, 504–515.

Sepulchre, R., M. Jankovic and P. Kokotovic (1997). *Constructive nonlinear control.* Springer. London.

Weiss, G. and M. Hafele (1999). Repetitive control of mimo systems using h^∞ design. *Automatica* **35**, 1185–1199.

Willems, J.C. (1972). Dissipative dynamical systems, part i: general theory. *Archive for Rational Mechanics and Analysis* **45**, 321–351.

Ye, X.D. and J.P. Jiang (1998). Adaptive nonlinear design without a priori knowledge of control directions. *IEEE Trans. on Automatic Control* **43**, 1617–1621.

8. APPENDIX

Appendix 1: Proof of Theorem 1

Assume

$$\begin{aligned} \dot{x}_{W_i}(t) &= A_{W_i}x_{W_i}(t) + B_{W_i}v_i(t) \\ z_i(t) &= C_{W_i}x_{W_i}(t) \end{aligned} \tag{9}$$

is a minimal realization of strictly bounded real $W_i(s)$. Then according to Corollary 1 and the inequality (10) in (Owens *et al.*, 2004), we have $(x_{W_i}^T P_{W_i} x_{W_i})' \leq \mu_i^2 v_i^T v_i - z_i^T z_i$, where $0 < \mu_i < 1$ is a constant.

Firstly, we consider the case that (β) is applied for a ASP plant and assume that $d = 0, r \neq 0$. Introduce a positive definitive Lyapunov function V of the form(time subscripts dropped in the following for notational convenience):

$$\begin{aligned} &V = \frac{1}{k}\widetilde{V}, \quad \widetilde{V} = V_1 + \frac{1}{2k}\sum_{i=1}^{p}\alpha_i M(t) \\ &M(t) := (\int\limits_{t-\tau_i}^{t} \|z_i(\theta)\|^2 \, d\theta + x_{W_i}^T P_{W_i} x_{W_i}) \end{aligned} \tag{10}$$

where V_1 is defined in Definition 2. By differentiating V, we have

$$\frac{dV}{dt} = \frac{1}{k}\frac{d\widetilde{V}}{dt} - \frac{1}{k^2}\frac{dk}{dt}\widetilde{V} \leq \frac{1}{k}\frac{d\widetilde{V}}{dt} \tag{11}$$

And also we have

$$\begin{aligned} &\frac{d\widetilde{V}}{dt} \\ &\leq z^T y + My^2 - S(x) - \frac{1}{2k^2}\frac{dk}{dt}\sum_{i=1}^{p}\alpha_i M(t) \\ &-\frac{1}{2k}\sum_{i=1}^{p}\alpha_i(\mu_i^2 \|v_i(t)\|^2 - \|z_i(t-\tau_i)\|^2) \\ &< z^T(r-e) + M(r-e)^2 \\ &-\frac{1}{2k}\sum_{i=1}^{p}\alpha_i(\mu_i^2 \|v_i\|^2 - \|v_i - ke\|^2) \\ &\leq z^T r + 2Mr^T r + 2Me^T e \\ &-\frac{1}{2k}\sum_{i=1}^{p}\alpha_i(\mu_i^2 - 1)\|v_i\|^2 - \frac{k}{2}e^T e \\ &\leq (2M + \frac{k}{2(1-\mu_i^2)})r^T r + (2M - \frac{k}{2})e^T e \\ &-\sum_{i=1}^{p}\alpha_i(\sqrt{\frac{1-\mu_i^2}{2k}}v_i - \frac{1}{2}\sqrt{\frac{2k}{1-\mu_i^2}}r)^2 \\ &< (2M + \frac{k}{2(1-\mu_i^2)})r^T r + (2M - \frac{k}{2})e^T e \end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned} &\frac{dV}{dt} \\ &< (\frac{2M}{k} + \frac{1}{2(1-\mu^2)})r^T r + (\frac{2M}{k} - \frac{1}{2})e^T e \\ &< \bar{M}r^T r + (\frac{2M}{k} - \frac{1}{2})e^T e \\ &\quad \bar{M} := (\frac{2M}{k(0)} + \frac{1}{2(1-\mu^2)}) \end{aligned} \tag{13}$$

Integrating (13) yields

$$V(t^{'}) < V(0) + \int_0^{t^{'}} \bar{M} r^T r dt + \int_0^{t^{'}} (\frac{2M}{k} - \frac{1}{2}) e^T e dt \quad (14)$$

We will establish $k(t) \in L_\infty[0, t^{'})$ by contradiction. Suppose $k(t) \notin L_\infty[0, t^{'})$, the term $\int_0^{t^{'}} (\frac{2M}{k} - \frac{1}{2}) e^T e dt$ will be negative infinity. $\int_0^{t^{'}} (\frac{2M}{k(0)} + \frac{1}{2(1-\mu^2)}) r^T r dt$ is positive finity. So the right part of (14) is negative, hence contradicting the non-negativity of the left hand side of (14). Therefore, we have $k(t) \in L_\infty[0, t^{'})$. When $t^{'} = \infty$, we have $k(t) \in L_\infty[0, \infty)$. Due to the monotonic increase of $k(t)$, we have $\lim_{t\to\infty} k(t) = k_\infty < \infty$.

We rewrite (13) as

$$\frac{dV}{dt} < \bar{M} r^T r + (\frac{2M}{k} - \frac{1}{4}) e^T e - \frac{1}{4} e^T e \quad (15)$$

Let t_1 such that $\frac{2M}{k(t_1)} = \frac{1}{4}$, then integrating (15) from t_1 to t yields

$$\begin{aligned} 0 \le V(t) &< V(t_1) + \int_{t_1}^{t} \bar{M} r^T r dt \\ &+ \int_{t_1}^{t} (\frac{2M}{k} - \frac{1}{4}) e^T e dt - \int_{t_1}^{t} \frac{1}{4} e^T e dt \\ &< V(t_1) + \int_{t_1}^{t} \bar{M} r^T r dt - \int_{t_1}^{t} \frac{1}{4} e^T e dt \end{aligned} \quad (16)$$

Therefore we have

$$\int_{t_1}^{t} \|e(\theta)\|^2 d\theta < M_1 + M_2 \int_{t_1}^{t} \|r(\theta)\|^2 d\theta \quad (17)$$

for some positive constants M_1, M_2 and some finite time t_1.

The proof for the case that $r = 0, d \neq 0$ is similar as above. Also if we set the low-pass filter $W_i(s)$ to be 1, full disturbance rejection will be achieved. We outline this analysis as follows:

$$V = V_1 + \frac{1}{2k} \sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta) + d_i(\theta)\|^2 d\theta \quad (18)$$

By differentiating V, we have

$$\begin{aligned} \frac{dV}{dt} &< -S - (\frac{k}{2} - M) y^T y \\ &- \frac{1}{2k^2} \frac{dk}{dt} \sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta) + d_i(\theta)\|^2 d\theta \\ &< -(\frac{k}{2} - M) y^T y \end{aligned} \quad (19)$$

Integrating (19) yields

$$V(t^{'}) < V(0) - \int_0^{t^{'}} (\frac{k}{2} - M) y^T y dt \quad (20)$$

Similar as before, we have $k(t) \in L_\infty[0, \infty)$, $\lim_{t\to\infty} k(t) = k_\infty < \infty$, and also $y(.) \in L_2[0, \infty)$.

In the case of (α), the proof is much simpler, therefore we omit it for limited space. □

Appendix 2: Proof of Theorem 2

Introduce a positive definitive Lyapunov function V of the form

$$V = V_1 + \frac{1}{2k} \sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \quad (21)$$

By differentiating V, we have

$$\begin{aligned} \frac{dV}{dt} &< -S + M y^T y + \sigma N(\lambda) z^T y \\ &- \frac{1}{2k} \sum_{i=1}^{p} \alpha_i (\|z_i(t)\|^2 - \|z_i(t-\tau_i)\|^2) \\ &- \frac{1}{2k^2} \frac{dk}{dt} \sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \\ &< -(\frac{k}{2} - M) y^T y + (\sigma N(\lambda) - 1) z^T y \end{aligned} \quad (22)$$

Integrating (22) yields

$$\begin{aligned} V(t^{'}) &< V(0) - \int_0^{t^{'}} (\frac{k}{2} - M) y^T y d\tau \\ &+ \int_{\lambda(0)}^{\lambda(t^{'})} (\sigma N(\lambda) - 1) d\tau \end{aligned} \quad (23)$$

Suppose $k(t) \notin L_\infty[0, t^{'})$ and $\lambda(t) \notin L_\infty[0, t^{'})$, the term $-\int_0^{t^{'}} (\frac{k}{2} - M) y^T y d\tau$ will be negative infinity. The term $\int_{\lambda(0)}^{\lambda(t^{'})} (\sigma N(\tau) - 1) d\tau$ will take arbitrary large negative or positive value when $\lambda(t^{'}) = \infty$ according to Lemma 1 in (Ye and Jiang, 1998). For example, if we select $N(\lambda) = \lambda^2 \cos\lambda$ and $\lambda(0) = 0$ without loss of generality, then we have $\int_0^{\lambda(t^{'})} (\sigma N(\tau) - 1) d\tau = \sigma[\lambda(t^{'})^2 \sin\lambda(t^{'}) - 2\lambda(t^{'}) \cos\lambda(t^{'}) + 2\sin\lambda(t^{'})] - \lambda(t^{'})$ and it will take arbitrary large negative or positive value when $\lambda(t^{'}) = \infty$. So when it takes arbitrary large negative, the right hand side of (23) will be negative, hence contradicting the non-negativity of the left hand side of (23). Therefore, we have $\lambda(t) \in L_\infty[0, t^{'})$, $k(t) \in L_\infty[0, t^{'})$. Similar as before, we have $\lim_{t\to\infty} k(t) = k_\infty < \infty$ and $y(.) \in L_2^m[0, \infty)$, which proves the result. □