OPTIMAL CONTROL OF LINEAR SYSTEMS WITH STATE EQUALITY CONSTRAINTS

Sangho Ko , Robert R. Bitmead [1]

Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, California, 92093-0411, USA.
{sko,rbitmead}@ucsd.edu

Abstract: This paper deals with the optimal control problem for systems with state linear equality constraints. For deterministic linear systems, first we find various existence conditions for constraining state feedback control and determine all constraining feedback gains, from which the optimal feedback gain is derived by using the result of singular optimal control. For systems with stochastic process noises, it is shown that the same gain used for constraining the deterministic system also optimally constrains the expectation of states inside the constraint subspace and minimizes the expectation of the squared constraint error. We compare performance between unconstrained and constrained controllers for both deterministic and stochastic systems.

Keywords: Linear optimal control, Singular control, Constraints, Kalman filters, Projection

1. INTRODUCTION

Dealing with constraints on the state and/or input variable is one of the fundamental tasks in control synthesis problems and, hence, has drawn much attention of the dynamics and control community, since it is closely connected with system performance and, thus, fulfillment of given system specifications. Recently, a number of modern model-based control design methods sought to deal with system constraints directly rather than through their implicit incorporation via penalty or barrier functions. Such is the case for Model Predictive Control, where part of the attraction of the approach is the introduction of constraints into the formulation without compromising the scalar control objective function (Maciejowski, 2002).

From the viewpoint of its origin, a state/input constraint can be a *physical constraint*, physically imposed upon the system state and/or input, or a *design constraint*, deliberately imposed to avoid undesirable states by using corrective control action. Of the various kinds of constraints, this paper focuses only on a special case: equality state constraints (which are also known as algebraic equation state constraints.) In the case of physical state equality constraint, it is always possible to reduce the system parametrization to fit in a lower dimensional state space. For robotic systems with (hard) holonomic constraints, McClamroch and Wang (1988) derived stable controllers by decomposing the constrained system into a reduced order dynamic system and a static system. But, sometimes keeping the non-reduced state space has also good reasons as described in Hemami and Wyman (1979), where a general dynamic model for biped locomotion was derived in a non-reduced state equation form and a pole-assignment algorithm was devised for a linearized state equation, which, in general, does not satisfy the hard constraint without control input. Hence, their design methodology can be applied for the system having design constraints. That is, for a system represented by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

they found, so called, $\mathcal{X}_c$-constrained linear feedback gain $\mathbf{K}$ such that

$$(\mathbf{A} - \mathbf{B}\mathbf{K})(\mathcal{X}_c) \subset \mathcal{X}_c$$

where $\mathcal{X}_c$ is a given constraint subspace. This pole-assignment problem was also studied in Yu and Müller (1994) in which a method of designing a pole-assignment controller was developed for

[1] Research supported by USA National Science Foundation Grants ECS-0200449 and ECS-0225530.

constraining the state of a system, firstly by finding a suitable form of feedback gain and then by designing a specific pole-assignment controller. Existence conditions for the pole-assignment controller were also studied. Similarly to the above-mentioned equality (or algebraic) constraint approach, the so-called *stabilized constraint* method was considered by Hahn (1992) and Yu *et al.* (1996). Here, instead of considering algebraic constraint relations, the stabilized constraints having stable stationary solutions were used, where the limiting solution is identical to the algebraic constraints. Hahn (1992) studied a pole-assignment controller and Yu *et al.* (1996) designed a linear quadratic (LQ) regulator for systems with stabilized constraints.

In the case of the LQ regulator problem, we expect that the state equality constraints cause reduction of the allowable input space which, in turn, produces a performance degradation in terms of optimal performance index, compared to that of the unconstrained LQ regulator problem. This paper studies this problem. First, we study the existence conditions for $\mathcal{X}_c$-constrained feedback input and then find all $\mathcal{X}_c$-constrained feedback gains. Then, we derive the constrained optimal performance index and then compare it with that of the unconstrained case. Also, we extend this approach to the case of systems with stochastic disturbances.

2. CONSTRAINED CONTROL

We consider the following problem: for a given discrete system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \tag{1}$$

with a design constraint

$$\mathbf{x}_k \in \mathcal{X}_c = \{\mathbf{x} : \mathbf{D}\mathbf{x} = \mathbf{0}\}, \tag{2}$$

find the optimal control law

$$\mathbf{u}_k = -\mathbf{K}_k\mathbf{x}_k$$

which minimizes

$$J = \mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N + \sum_{k=0}^{N-1}\left[\mathbf{x}_k^T\mathbf{Q}_c\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}_c\mathbf{u}_k\right] \tag{3}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ and $\mathbf{u}_k \in \mathbb{R}^m$, and it is assumed that $\mathbf{Q}_N \geq \mathbf{0}$, $\mathbf{Q}_c \geq \mathbf{0}$, $\mathbf{R}_c > \mathbf{0}$, and $\mathbf{D} \in \mathbb{R}^{c\times n}$ has full row rank. If $\mathbf{D}$ is not of full row rank, there exist redundant state constraints. In that case, we can simply remove linearly dependent rows from $\mathbf{D}$.

For this, first, we have to find the set of all $\mathbf{K} \in \mathbb{R}^{m\times n}$

$$\mathcal{K}_{\mathcal{X}_c} \triangleq \left\{\mathbf{K} : (\mathbf{A} - \mathbf{B}\mathbf{K})(\mathcal{X}_c) \subset \mathcal{X}_c\right\} \tag{4}$$

and one says that such a feedback map $\mathbf{K} : \mathcal{X} \mapsto \mathcal{U}$ is *$\mathcal{X}_c$-constrained.* It can be shown (Hemami and Wyman, 1979) that the set $\mathcal{K}_{\mathcal{X}_c}$ of all $\mathcal{X}_c$-constrained feedbacks for $(\mathbf{A}, \mathbf{B})$ is an affine subset of the set of all linear state-variable feedbacks $\mathbf{K}$. For later use, we need the following Lemma 1.

Lemma 1. (Skelton *et al.* (1998), Theorem 2.3.1). Let $\mathbf{A}$, $\mathbf{X}$, $\mathbf{B}$, and $\mathbf{Y}$ be matrices with consistent dimensions. Then the following statements are equivalent:

(i) The equation $\mathbf{AXB} = \mathbf{Y}$ has a solution $\mathbf{X}$.
(ii) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Y}$ satisfy $\mathbf{A}\mathbf{A}^\dagger\mathbf{Y}\mathbf{B}^\dagger\mathbf{B} = \mathbf{Y}$.
(iii) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Y}$ satisfy

$$(\mathbf{I} - \mathbf{A}\mathbf{A}^\dagger)\mathbf{Y} = \mathbf{0}, \quad \mathbf{Y}(\mathbf{I} - \mathbf{B}^\dagger\mathbf{B}) = \mathbf{0}.$$

In this case, all solutions are

$$\mathbf{X} = \mathbf{A}^\dagger\mathbf{Y}\mathbf{B}^\dagger + \mathbf{G} - \mathbf{A}^\dagger\mathbf{A}\mathbf{G}\mathbf{B}\mathbf{B}^\dagger \tag{5}$$

where $\mathbf{G}$ is an arbitrary matrix with consistent dimension.

The following Lemma 2 provides equivalent existence conditions for the non-empty set $\mathcal{K}_{\mathcal{X}_c}$.

Lemma 2. The following statements are equivalent:

(i) $\mathcal{K}_{\mathcal{X}_c}$ is non-empty.
(ii) $\mathbf{A}(\mathcal{X}_c) \subset \mathcal{X}_c + \mathbf{B}(\mathcal{U})$
(iii) For any basis matrix $\mathbf{Z}$ of the subspace $\mathcal{X}_c$,

$$\mathbf{DBKZ} = \mathbf{DAZ}. \tag{6}$$

(iv) $\mathbf{P}_{\mathcal{N}(\mathbf{DB})^T}\mathbf{DAZ} = \mathbf{0}$
(v) There exists a $c \times c$ matrix $\mathbf{H}$ such that

$$\mathbf{HD} = \mathbf{D}(\mathbf{A} - \mathbf{BK}).$$

In this case, all $\mathcal{X}_c$-constrained feedback gains are given by

$$\mathbf{K} = \mathbf{G} + \left[\mathbf{G}_0 - \mathbf{P}_{\mathcal{R}(\mathbf{DB})^T}\mathbf{G}\right]\mathbf{P}_{\mathcal{N}(\mathbf{D})} \tag{7}$$

where $\mathbf{G}$ is an arbitrary matrix with consistent dimension and $\mathbf{G}_0 \triangleq (\mathbf{DB})^\dagger\mathbf{DA}$. Here $\mathbf{P}_{\mathcal{R}(\mathbf{DB})^T} = (\mathbf{DB})^\dagger(\mathbf{DB})$, $\mathbf{P}_{\mathcal{N}(\mathbf{DB})^T} = \mathbf{I} - (\mathbf{DB})(\mathbf{DB})^\dagger$, and $\mathbf{P}_{\mathcal{N}(\mathbf{D})}$ are the orthogonal projectors onto the row and the left null space of $\mathbf{DB}$, the null space of $\mathbf{D}$, respectively.

PROOF. Proofs for (i) $\leftrightarrow$ (ii) and (i) $\leftrightarrow$ (v) are given in Wonham (1979) and Hemami and Wyman (1979), and Castelan and Hennet (1992), respectively. Proof for (ii) $\leftrightarrow$ (iii) is very similar to the proof for (i) $\leftrightarrow$ (ii), which can be easily verified. The condition (iv) is obtained from (ii) or (iii) of Lemma 1 by applying it to (6) and also we obtain the all $\mathcal{X}_c$-constrained feedback (7) using (5). □

Remark 1. If $(\mathbf{DB})$ is invertible or has full row rank, $\mathbf{P}_{\mathcal{N}(\mathbf{DB})^T} = \mathbf{I} - (\mathbf{DB})(\mathbf{DB})^\dagger = \mathbf{0}$. Then, the condition (iv) of Lemma 2 is always satisfied. Therefore, $\mathcal{K}_{\mathcal{X}_c}$ is non-empty. Specially, if $(\mathbf{DB})$ is invertible, the feedback gain becomes a fixed one

$$\begin{aligned}\mathbf{u}_k &= -\mathbf{K}\mathbf{x}_k = -\mathbf{K}\mathbf{P}_{\mathcal{N}(\mathbf{D})}\mathbf{x}_k \\ &= -(\mathbf{DB})^{-1}\mathbf{DAP}_{\mathcal{N}(\mathbf{D})}\mathbf{x}_k.\end{aligned}$$

Hence, in this case, we cannot change control law design and need to change design constraints $(\mathbf{D})$ or system input matrix $(\mathbf{B})$, if this unique controller does not show desirable results.

Remark 2. The feedback gain (7) guarantees $\mathbf{x}_k \in \mathcal{X}_c$ for all k, provided that the initial state vector $\mathbf{x}_0$ is chosen from the constraint set $\mathcal{X}_c$. Now, suppose that $\mathbf{x}_k \notin \mathcal{X}_c$. Then, from (1), (7) and the condition (iv) of Lemma 2, it can be shown that

$$\mathbf{D}\big[\mathbf{x}_{k+1}^r - (\mathbf{A}-\mathbf{B}\mathbf{G})\mathbf{x}_k^r\big] = \mathbf{0}$$

where $\mathbf{x}_k^r$ is the $\mathcal{R}(\mathbf{D}^T)$-component of $\mathbf{x}_k$. Therefore, there exists a $\boldsymbol{\lambda}_k \in \mathcal{X}_c = \mathcal{N}(\mathbf{D})$ such that

$$\mathbf{x}_{k+1}^r - \mathbf{P}_{\mathcal{R}(\mathbf{D}^T)}(\mathbf{A}-\mathbf{B}\mathbf{G})\mathbf{x}_k^r = \mathbf{P}_{\mathcal{N}(\mathbf{D})}(\mathbf{A}-\mathbf{B}\mathbf{G})\mathbf{x}_k^r + \boldsymbol{\lambda}_k. \tag{8}$$

Since the left-hand side of (8) is in the row space of $\mathbf{D}$ and the right-hand side is in the null space of $\mathbf{D}$, we have

$$\mathbf{x}_{k+1}^r = \mathbf{P}_{\mathcal{R}(\mathbf{D}^T)}(\mathbf{A}-\mathbf{B}\mathbf{G})\mathbf{x}_k^r.$$

Therefore, for stable $(\mathbf{A}-\mathbf{B}\mathbf{G})$ (which also implies that $\mathbf{P}_{\mathcal{R}(\mathbf{D}^T)}(\mathbf{A}-\mathbf{B}\mathbf{G})$ is also stable), $\mathbf{x}_k^r \to \mathbf{0}$, asymptotically. This means, for an initial condition not in the constraint set, the system satisfies the constraint asymptotically.

3. CONSTRAINED LQ OPTIMAL CONTROL

For the system (1) to be constrained in $\mathcal{X}_c$, the state feedback gain must be of the structure (7) and, for any $\mathbf{x}_k \in \mathcal{X}_c$, the input $\mathbf{u}_k$ is given by

$$\begin{aligned}\mathbf{u}_k &= -\Big[\mathbf{G}+\mathbf{G}_0-\mathbf{P}_{\mathcal{R}(\mathbf{DB})^T}\mathbf{G}\Big]\mathbf{P}_{\mathcal{N}(\mathbf{D})}\mathbf{x}_k,\\ &= -\mathbf{G}_0\ \mathbf{x}_k + \mathbf{P}_{\mathcal{N}(\mathbf{DB})}\underbrace{(-\mathbf{G}\mathbf{x}_k)}_{\triangleq \bar{\mathbf{u}}_k}\end{aligned} \tag{9}$$

where $\mathbf{P}_{\mathcal{N}(\mathbf{DB})} = \mathbf{I} - \mathbf{P}_{\mathcal{R}(\mathbf{DB})^T}$ is the orthogonal projector onto the null space of $\mathbf{DB}$. Therefore, with (9), the state equation (1) can be rewritten as

$$\mathbf{x}_{k+1} = \bar{\mathbf{A}}\mathbf{x}_k + \bar{\mathbf{B}}\bar{\mathbf{u}}_k \tag{10}$$

where $\bar{\mathbf{A}} \triangleq (\mathbf{A}-\mathbf{B}\mathbf{G}_0)$ and $\bar{\mathbf{B}} \triangleq \mathbf{B}\mathbf{P}_{\mathcal{N}(\mathbf{DB})}$. It is easily shown that $\mathcal{N}(\mathbf{D})$ is $\bar{\mathbf{A}}$-invariant (if the condition (iv) of Lemma 2 is satisfied) and also we can see that the column space of $\bar{\mathbf{B}}$ lies in $\mathcal{N}(\mathbf{D})$. Therefore, the new system representation (10) corresponds to a physically constrained system. With the new input $\bar{\mathbf{u}}_k$, we can reconstruct the performance index (3) as

$$J = \mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N + \sum_{k=0}^{N-1}\Big[\mathbf{x}_k^T\bar{\mathbf{Q}}_c\mathbf{x}_k + 2\mathbf{x}_k^T\bar{\mathbf{H}}_c\bar{\mathbf{u}}_k + \bar{\mathbf{u}}_k^T\bar{\mathbf{R}}_c\bar{\mathbf{u}}_k\Big] \tag{11}$$

where

$$\begin{aligned}\bar{\mathbf{Q}}_c &\triangleq \mathbf{Q}_c + \mathbf{G}_0^T\mathbf{R}_c\mathbf{G}_0\\ \bar{\mathbf{H}}_c &\triangleq -\mathbf{G}_0^T\mathbf{R}_c\mathbf{P}_{\mathcal{N}(\mathbf{DB})}\\ \bar{\mathbf{R}}_c &\triangleq \mathbf{P}_{\mathcal{N}(\mathbf{DB})}\mathbf{R}_c\mathbf{P}_{\mathcal{N}(\mathbf{DB})}.\end{aligned} \tag{12}$$

Note that the new weighting matrix $\bar{\mathbf{R}}_c$ becomes non-negative definite (singular) for a given positive definite original weighting matrix $\mathbf{R}_c$ due to the projector $\mathbf{P}_{\mathcal{N}(\mathbf{DB})}$. Therefore, in order to find the optimal feedback gain $\mathbf{G}^*$ such that $\bar{\mathbf{u}}_k^* = -\mathbf{G}^*\mathbf{x}_k$ minimizes J, we have to solve a singular optimal control problem. However, for any discrete-time linear-quadratic optimal control problem, regardless of the singularity or otherwise of any matrices in the cost, the associated Riccati equation is well defined and may be solved in a straightforward manner to yield a solution of the optimal control problem (Clements and Anderson, 1978), which is summarized by the following definition and lemma:

Definition 1. The set of *admissible weighting matrices*, denoted by $\mathcal{S}$, is the set of $n \times n$ symmetric matrices $\mathbf{P}$ such that $\bar{\mathbf{B}}^T\mathbf{P}\bar{\mathbf{B}} + \bar{\mathbf{R}}_c \geq \mathbf{0}$ and $\mathcal{N}(\bar{\mathbf{B}}^T\mathbf{P}\bar{\mathbf{B}} + \bar{\mathbf{R}}_c) \subset \mathcal{N}(\bar{\mathbf{A}}^T\mathbf{P}\bar{\mathbf{B}} + \bar{\mathbf{H}}_c)$.

Lemma 3. [Clements and Anderson (1978)] The optimal control problem for the system (10) with the performance index (11) has a solution on $[0, N]$ for terminal weighting matrix $\mathbf{Q}_N$ if and only if the $n \times n$ symmetric matrix $\mathbf{P}_{k+1} \in \mathcal{S}$ for each $k = 0, \cdots, N-1$, where $\mathbf{P}_k$ is defined by the recursion, with $\mathbf{P}_N = \mathbf{Q}_N$,

$$\begin{aligned}\mathbf{P}_k &= \bar{\mathbf{A}}^T\mathbf{P}_{k+1}\bar{\mathbf{A}} + \bar{\mathbf{Q}}_c\\ &\quad -\big(\bar{\mathbf{A}}^T\mathbf{P}_{k+1}\bar{\mathbf{B}}+\bar{\mathbf{H}}_c\big)\mathbf{R}_B^{\dagger}\big(\bar{\mathbf{A}}^T\mathbf{P}_{k+1}\bar{\mathbf{B}}+\bar{\mathbf{H}}_c\big)^T\end{aligned}$$

where $\mathbf{R}_B \triangleq \bar{\mathbf{B}}^T\mathbf{P}_{k+1}\bar{\mathbf{B}} + \bar{\mathbf{R}}_c$. If $\mathbf{P}_k$ is so defined, then the control sequence $\mathbf{U}_l^{*N-1}$ defined by

$$\mathbf{u}_k^* = -\mathrm{R}_B^{\dagger}\left(\bar{\mathbf{A}}^T\mathbf{P}_{k+1}\bar{\mathbf{B}} + \bar{\mathbf{H}}_c\right)^T\mathbf{x}_k$$

achieves the infimum of J for each $k = l, \cdots, N-1$. That is, with $\mathbf{U}_l^{*N-1} = \big[\mathbf{u}_l^*, \cdots, \mathbf{u}_{N-1}^*\big]$, we have

$$J_{N-l}^*(\mathbf{x}_l,\ \mathbf{S}) = J_{N-l}(\mathbf{x}_l, \mathbf{U}_l^{*N-1}, \mathbf{S}) = \mathbf{x}_l^T\mathbf{P}_l\mathbf{x}_l.$$

It can be easily verified that, for the minimization problem with the new performance index shown in (11) and the system matrices in (10), any $n \times n$ symmetric matrix $\mathbf{P}$ is in the admissible set $\mathcal{S}$ and we can apply Lemma 3, from which we obtain Theorem 1.

Theorem 1. For the system given in (1) with the state equality constraints (2) which satisfies one of conditions of Lemma 2, and the performance index (3), the solution to the LQ control problem is given by the optimal control law in the following state feedback form:

$$\mathbf{u}_k^c = -\mathbf{G}_0\ \mathbf{x}_k + \mathbf{P}_{\mathcal{N}(\mathbf{DB})}\bar{\mathbf{u}}_k^c$$

Here $\bar{\mathbf{u}}_k^c$ satisfies

$$\bar{\mathbf{u}}_k^c = -\big(\bar{\mathbf{B}}^T\mathbf{P}_{k+1}^c\bar{\mathbf{B}}+\bar{\mathbf{R}}_c\big)^{\dagger}\big(\bar{\mathbf{A}}^T\mathbf{P}_{k+1}^c\bar{\mathbf{B}}+\bar{\mathbf{H}}_c\big)^T\mathbf{x}_k$$

where, with $\mathbf{P}_N^c = \mathbf{Q}_N$,

$$\begin{aligned}\mathbf{P}_k^c &= \bar{\mathbf{A}}^T\mathbf{P}_{k+1}^c\bar{\mathbf{A}} + \bar{\mathbf{Q}}_c\\ &\quad -\big(\bar{\mathbf{A}}^T\mathbf{P}_{k+1}^c\bar{\mathbf{B}}+\bar{\mathbf{H}}_c\big)\big(\bar{\mathbf{B}}^T\mathbf{P}_{k+1}^c\bar{\mathbf{B}}+\bar{\mathbf{R}}_c\big)^{\dagger}\\ &\quad \times\big(\bar{\mathbf{A}}^T\mathbf{P}_{k+1}^c\bar{\mathbf{B}}+\bar{\mathbf{H}}_c\big)^T \text{ for } k=0,\cdots,N-1.\end{aligned} \tag{13}$$

The optimal value of the performance index is given by $J^c = \mathbf{x}_0^T\mathbf{P}_0^c\mathbf{x}_0$.

By using (10) and (12), the control RDE (13) can be also written as

$$\begin{aligned}\mathbf{P}_k^c &= \mathbf{A}^T\mathbf{P}_{k+1}^c\mathbf{A} + \mathbf{Q}_c + \boldsymbol{\nabla}_k^c\\ &\quad -\mathbf{A}^T\mathbf{P}_{k+1}^c\mathbf{B}\big(\mathbf{B}^T\mathbf{P}_{k+1}^c\mathbf{B}+\mathbf{R}_c\big)^{-1}\mathbf{B}^T\mathbf{P}_{k+1}^c\mathbf{A}^T\end{aligned} \tag{14}$$

where

$$\boldsymbol{\nabla}_k^c \triangleq (\mathbf{K}_k^c - \mathbf{G}_0)^T\,\boldsymbol{\Omega}_k^c\,(\mathbf{K}_k^c - \mathbf{G}_0) \tag{15}$$

with

$$\begin{aligned}\mathbf{K}_k^c &\triangleq \big(\mathbf{B}^T\mathbf{P}_{k+1}^c\mathbf{B} + \mathbf{R}_c\big)^{-1}\mathbf{B}^T\mathbf{P}_{k+1}^c\mathbf{A}\\ \mathbf{G}_0 &\triangleq (\mathbf{DB})^{\dagger}\mathbf{DA}\\ \boldsymbol{\Omega}_k^c &\triangleq \Big[\hat{\mathbf{R}}_B - \hat{\mathbf{R}}_B\hat{\mathbf{R}}_B^{(2)}\hat{\mathbf{R}}_B\Big].\end{aligned} \tag{16}$$

Here $\hat{\mathbf{R}}_B \triangleq \left(\mathbf{B}^T\mathbf{P}^c_{k+1}\mathbf{B} + \mathbf{R}_c\right)$ and $\hat{\mathbf{R}}^{(2)}_B$ represents a (2)-inverse of $\hat{\mathbf{R}}_B$ which satisfies the second condition of the Moore-Penrose inverse, viz.

$$\hat{\mathbf{R}}^{(2)}_B \triangleq \mathbf{P}_{\mathcal{N}(\mathbf{DB})}\left[\mathbf{P}_{\mathcal{N}(\mathbf{DB})}\hat{\mathbf{R}}_B\mathbf{P}_{\mathcal{N}(\mathbf{DB})}\right]^{\dagger}\mathbf{P}_{\mathcal{N}(\mathbf{DB})} = \left[\mathbf{P}_{\mathcal{N}(\mathbf{DB})}\hat{\mathbf{R}}_B\mathbf{P}_{\mathcal{N}(\mathbf{DB})}\right]^{\dagger}. \tag{17}$$

It can be shown that

$$\hat{\mathbf{R}}_B \geq \hat{\mathbf{R}}_B\hat{\mathbf{R}}^{(2)}_B\hat{\mathbf{R}}_B. \tag{18}$$

Therefore, comparing (14) with the Riccati equation that would be obtained for unconstrained LQ optimal control

$$\mathbf{P}^u_k = \mathbf{A}^T\mathbf{P}^u_{k+1}\mathbf{A} + \mathbf{Q}_c - \mathbf{A}^T\mathbf{P}^u_{k+1}\mathbf{B}\left(\mathbf{B}^T\mathbf{P}^u_{k+1}\mathbf{B} + \mathbf{R}_c\right)^{-1}\mathbf{B}^T\mathbf{P}^u_{k+1}\mathbf{A}^T \tag{19}$$

yields

$$\mathbf{P}^u_k \leq \mathbf{P}^c_k, \text{ for all } k.$$

Therefore, from the monotonicity property of the Riccati Difference Equation (RDE) (Bitmead and Gevers, 1991), we obtain the following corollary.

Corollary 1.

$$J^u = \mathbf{x}^T_0\mathbf{P}^u_0\mathbf{x}_0 \leq \mathbf{x}^T_0\mathbf{P}^c_0\mathbf{x}_0 = J^c. \tag{20}$$

Corollary 1 tells us that the optimal performance index of the constrained case is greater than that of unconstrained case, due to the design constraints on state variables which are given by (2). This is a formal derivation of a self-evident property that constraining the admissible control set worsens performance.

4. STOCHASTIC CONSTRAINED OPTIMAL CONTROL

Now we consider the optimal control of stochastic systems

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k \tag{21}$$

with state equality constraints represented (2). Here the process noise $\mathbf{w}_k$ is assumed to have a gaussian distribution of zero-mean and covariance $\mathbf{Q}_e$. Completing the squares as in Åström (1970), the following Lemma 4 is obtained which will be used for generalizing the result of deterministic LQ control of Theorem 1.

Lemma 4. Assume that the RDE (14) with the initial condition $\mathbf{P}^c_N = \mathbf{Q}_N$ has a solution which is non-negative definite for $k \in [0, N]$. Let $\mathbf{x}_k$ be the solution of the stochastic difference equation (21). Then,

$$\begin{aligned} J_{sto} &\triangleq \mathbf{x}^T_N\mathbf{Q}_N\mathbf{x}_N + \sum_{k=0}^{N-1}\left(\mathbf{x}^T_k\mathbf{Q}_c\mathbf{x}_k + \mathbf{u}^T_k\mathbf{R}_c\mathbf{u}_k\right) \\ &= \mathbf{x}^T_0\mathbf{P}^c_0\mathbf{x}_0 + \sum_{k=0}^{N-1}\Big[\mathbf{w}^T_k\mathbf{P}^c_{k+1}(\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k) \\ &\quad +(\mathbf{A}\mathbf{x}_k+\mathbf{B}\mathbf{u}_k)^T\mathbf{P}^c_{k+1}\mathbf{w}_k+\mathbf{w}^T_k\mathbf{P}^c_{k+1}\mathbf{w}_k+V_k\Big] \end{aligned} \tag{22}$$

where

$$\begin{aligned} V_k &= \mathbf{u}^T_k\hat{\mathbf{R}}_B\mathbf{u}_k + (\mathbf{u}_k + \mathbf{K}_k\mathbf{x}_k)^T\hat{\mathbf{R}}_B\mathbf{K}^c_k\mathbf{x}_k \\ &\quad + \mathbf{x}^T_k\mathbf{K}^{c^T}_k\hat{\mathbf{R}}_B(\mathbf{u}_k + \mathbf{K}_k\mathbf{x}_k) - \mathbf{x}^T_k\mathbf{K}^T_k\hat{\mathbf{R}}_B\mathbf{K}_k\mathbf{x}_k. \end{aligned} \tag{23}$$

Here the matrix $\mathbf{K}_k$ is defined by

$$\mathbf{K}_k = \mathbf{G}_0 + \hat{\mathbf{R}}^{(2)}_B\hat{\mathbf{R}}_B\left(\mathbf{K}^c_k - \mathbf{G}_0\right). \tag{24}$$

Although we have already analyzed the deterministic LQ case in Section 3, we derive the result again here now using Lemma 4, since this procedure will be used for the incomplete state information case in Section 4.2.

4.1 Deterministic Case

For a deterministic system, $\mathbf{w}_k \equiv \mathbf{0}$. Thus from (22) of Lemma 4 we have

$$\begin{aligned} J_{sto} = J &= \mathbf{x}^T_N\mathbf{Q}_N\mathbf{x}_N + \sum_{k=0}^{N-1}\left[\mathbf{x}^T_k\mathbf{Q}_c\mathbf{x}_k + \mathbf{u}^T_k\mathbf{R}_c\mathbf{u}_k\right] \\ &= \mathbf{x}^T_0\mathbf{P}^c_0\mathbf{x}_0 + \sum_{k=0}^{N-1}V_k. \end{aligned} \tag{25}$$

Now in order to incorporate the constraint (2) into the process of minimizing J given in (25), let us express J in terms of $\bar{\mathbf{u}}_k$ by using the $\mathcal{X}_c$-constrained feedback input form (9). Then, we obtain

$$J(\bar{\mathbf{u}}_k) = \mathbf{x}^T_0\mathbf{P}^c_0\mathbf{x}_0 + \sum_{k=0}^{N-1}\bar{V}_k(\mathbf{x}_k, \bar{\mathbf{u}}_k) \tag{26}$$

where

$$\bar{V}_k(\mathbf{x}_k, \bar{\mathbf{u}}_k) = \mathbf{x}^T_k\tilde{\mathbf{F}}\mathbf{x}_k + 2\mathbf{x}^T_k\tilde{\mathbf{H}}\bar{\mathbf{u}}_k + \bar{\mathbf{u}}^T_k\tilde{\mathbf{G}}\bar{\mathbf{u}}_k$$

with

$$\begin{aligned} \tilde{\mathbf{F}} &= \mathbf{G}^T_0\hat{\mathbf{R}}_B\mathbf{G}_0 + \mathbf{K}^T_k\hat{\mathbf{R}}_B\mathbf{K}^c_k + \mathbf{K}^{c^T}_k\hat{\mathbf{R}}_B\mathbf{K}_k \\ &\quad - \mathbf{K}^T_k\hat{\mathbf{R}}_B\mathbf{K}_k - \mathbf{G}^T_0\hat{\mathbf{R}}_B\mathbf{K}^c_k - \mathbf{K}^{c^T}_k\hat{\mathbf{R}}_B\mathbf{G}_0 \\ \tilde{\mathbf{H}} &= \mathbf{G}^T_0\hat{\mathbf{R}}_B\mathbf{P}_{\mathcal{N}(\mathbf{DB})} + \mathbf{K}^{c^T}_k\hat{\mathbf{R}}_B\mathbf{P}_{\mathcal{N}(\mathbf{DB})} \\ \tilde{\mathbf{G}} &= \mathbf{P}_{\mathcal{N}(\mathbf{DB})}\hat{\mathbf{R}}_B\mathbf{P}_{\mathcal{N}(\mathbf{DB})}. \end{aligned} \tag{27}$$

For minimizing $J(\bar{\mathbf{u}}_k)$ or $\bar{V}_k(\mathbf{x}_k, \bar{\mathbf{u}}_k)$ with respect to $\bar{\mathbf{u}}_k$, we use the following Lemma 5.

Lemma 5. [Clements and Anderson (1978)] Consider the quadratic form $q(\mathbf{z}, \mathbf{v}) = \mathbf{z}^T\mathbf{F}\mathbf{z} + 2\mathbf{z}^T\mathbf{H}\mathbf{v} + \mathbf{v}^T\mathbf{G}\mathbf{v}$ for matrices $\mathbf{F} = \mathbf{F}^T$, $\mathbf{G} = \mathbf{G}^T$ and $\mathbf{H}$ and vectors $\mathbf{z}$ and $\mathbf{v}$ of arbitrary but consistent dimensions, and define $q^*(\mathbf{z}) = \inf_{\mathbf{v}} q(\mathbf{z}, \mathbf{v})$. The following three conditions are equivalent:

(i) $q^*(\mathbf{z}) > -\infty$ for each $\mathbf{z}$
(ii) $\mathbf{G} \geq \mathbf{0}$, $\mathcal{N}(\mathbf{G}) \subset \mathcal{N}(\mathbf{H})$
(iii) there exists a symmetric matrix $\mathbf{X}$ such that

$$\begin{bmatrix} \mathbf{F} - \mathbf{X} & \mathbf{H} \\ \mathbf{H}^T & \mathbf{G} \end{bmatrix} \geq \mathbf{0}. \tag{28}$$

Moreover, if any one of the above conditions holds, then (iii) is satisfied by $\mathbf{X}^* = \mathbf{F} - \mathbf{H}\mathbf{G}^{\dagger}\mathbf{H}^T$. In addition, $\mathbf{X}^* \geq \mathbf{X}$ for any other $\mathbf{X}$ satisfying (iii). Finally if for each $\mathbf{z}$ we set

$$\mathbf{v}^* = -\mathbf{G}^\dagger \mathbf{H}^T \mathbf{z}, \tag{29}$$

then

$$q^*(\mathbf{z}) = q(\mathbf{z}, \mathbf{v}^*) = \mathbf{z}^T \mathbf{X}^* \mathbf{z}. \tag{30}$$

It is easily verified that $\tilde{\mathbf{G}} \geq \mathbf{0}$, $\mathcal{N}(\tilde{\mathbf{G}}) \subset \mathcal{N}(\tilde{\mathbf{H}})$ and hence Lemma 5 can be used for minimizing $\bar{V}_k(\mathbf{x}_k, \bar{\mathbf{u}}_k)$. From (30) together with (27) and (24), we arrive at

$$\bar{V}_k^*(\mathbf{x}_k, \bar{\mathbf{u}}_k^*) \triangleq \min_{\bar{\mathbf{u}}_k} \bar{V}_k(\mathbf{x}_k, \bar{\mathbf{u}}_k) = \mathbf{x}_k^T \mathbf{X}^* \mathbf{x}_k = 0, \tag{31}$$

since $\mathbf{X}^* = \tilde{\mathbf{F}} - \tilde{\mathbf{H}}\tilde{\mathbf{G}}^\dagger \tilde{\mathbf{H}}^T = \mathbf{0}$, for which (17) was used. Using (29), the optimal control $\bar{\mathbf{u}}_k^*$ is given by

$$\begin{aligned}\bar{\mathbf{u}}_k^* &= -\tilde{\mathbf{G}}^\dagger \tilde{\mathbf{H}}^T \mathbf{x}_k \\ &= -\hat{\mathbf{R}}_B^{(2)} \mathbf{P}_{\mathcal{N}(\mathbf{DB})} \hat{\mathbf{R}}_B (\mathbf{K}_k^c - \mathbf{G}_0) \mathbf{x}_k.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{u}_k^* &= -\mathbf{G}_0 \mathbf{x}_k + \mathbf{P}_{\mathcal{N}(\mathbf{DB})} \bar{\mathbf{u}}_k^* \\ &= -\Big[\mathbf{G}_0 + \hat{\mathbf{R}}_B^{(2)} \hat{\mathbf{R}}_B (\mathbf{K}_k^c - \mathbf{G}_0)\Big] \mathbf{x}_k \\ &= -\mathbf{K}_k \mathbf{x}_k.\end{aligned}$$

By substituting (31) into (26), we obtain the same results of Theorem 1.

4.2 Incomplete State Information

The system we now consider is driven also by the process noise $\mathbf{w}_k$ as shown in (21) and the state information is available only from a measurement given by

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k \tag{32}$$

where $\mathbf{v}_k$ is gaussian with zero-mean and covariance of $\mathbf{R}_e$. Therefore, in this case, we cannot constrain the state $\mathbf{x}_k$ in $\mathcal{X}_c$ since the exact state information is not available for the $\mathcal{X}_c$-constrained feedback such as $\mathbf{u}_k = -\mathbf{K}_k \mathbf{x}_k$. If we use the Kalman predictor $\hat{\mathbf{x}}_k \triangleq \mathcal{E}\{\mathbf{x}_k | \mathbf{y}_{k-1}\}$ for the feedback

$$\mathbf{u}_k = -\mathbf{K}_k \hat{\mathbf{x}}_k,$$

we have $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k - \mathbf{B}\mathbf{K}_k \hat{\mathbf{x}}_k + \mathbf{w}_k$ which is generally not in $\mathcal{X}_c$. However, taking expectation yields

$$\mathcal{E}\{\mathbf{x}_{k+1}\} = (\mathbf{A} - \mathbf{B}\mathbf{K}_k)\mathcal{E}\{\mathbf{x}_k\},$$

resulting in $\mathcal{E}\{\mathbf{x}_{k+1}\} \in \mathcal{X}_c$ for any $\mathcal{E}\{\hat{\mathbf{x}}_k\} = \mathcal{E}\{\mathbf{x}_k\} \in \mathcal{X}_c$. Therefore, for the case of incomplete state information, we can constrain only the expected value of the state in the constrained subspace by using the $\mathcal{X}_c$-constrained feedback obtained for the corresponding deterministic system. In addition to this, it can be shown that the $\mathcal{X}_c$-constrained feedback gain form [2]

$$\mathbf{K} = \mathbf{G} + \Big[\mathbf{G}_0 - \mathbf{P}_{\mathcal{R}(\mathbf{DB})^T} \mathbf{G}\Big] = \mathbf{G}_0 + \mathbf{P}_{\mathcal{N}(\mathbf{DB})} \mathbf{G} \tag{33}$$

minimizes the expectation of the squared constraint error, which is defined as

$$e(\mathbf{L}_k) \triangleq \mathcal{E}\big\{\mathrm{tr}\big\|\mathbf{D}\mathbf{x}_{k+1}\big\|^2 \big| \mathbf{y}_{k-1}\big\}, \tag{34}$$

[2] The only difference between (33) and (7) is the absence of the projector $\mathbf{P}_{\mathcal{N}(\mathbf{D})}$.

with the assumption that the covariance of the estimate of the Kalman predictor $\hat{\mathbf{X}}_k = \mathcal{E}\{\hat{\mathbf{x}}_k \hat{\mathbf{x}}_k^T | \mathbf{y}_{k-1}\}$ is non-singular. Here, $\mathbf{L}_k$ represents any state estimate feedback gain.

By taking the expectation of the performance index given by (22), we find, with the assumption that $\mathbf{w}_k$ is independent of $\mathbf{x}_k$ and $\mathbf{u}_k$,

$$\begin{aligned}\mathcal{E}\Big\{J_{sto}\Big\} &= \mathcal{E}\Big\{\mathbf{x}_N^T \mathbf{Q}_N \mathbf{x}_N + \sum_{k=0}^{N-1}\Big[\mathbf{x}_k^T \mathbf{Q}_c \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R}_c \mathbf{u}_k\Big]\Big\} \\ &= \mathcal{E}\Big\{\mathbf{x}_0^T \mathbf{P}_0^c \mathbf{x}_0 + \sum_{k=0}^{N-1}\Big[\mathbf{w}_k^T \mathbf{P}_{k+1}^c \mathbf{w}_k + V_k\Big]\Big\}\end{aligned} \tag{35}$$

where V_k is given in (23). For finding the minimum of the left-hand side of (35), we use following lemmas.

Lemma 6. [Åström (1970)] Let $\mathcal{E}_{(\cdot)|\mathbf{y}}[(\cdot)|\mathbf{y}]$ denote the conditional mean given $\mathbf{y}$. Assume that the function $f(\mathbf{u}, \mathbf{y}) = \mathcal{E}_{\mathbf{x}|\mathbf{y}}[J(\mathbf{x}, \mathbf{y}, \mathbf{u})|\mathbf{y}]$ has a unique minimum with respect to $\mathbf{u} \in \mathcal{U}$ for all $\mathbf{y} \in \mathcal{Y}$. Let $\mathbf{u}^*(\mathbf{y})$ denote the value of $\mathbf{u}$ for which the minimum is achieved. Then,

$$\begin{aligned}\min_{\mathbf{u}(\mathbf{y})} \mathcal{E}_{\mathbf{x},\mathbf{y}}\Big\{J(\mathbf{x}, \mathbf{y}, \mathbf{u})\Big\} &= \mathcal{E}_{\mathbf{x},\mathbf{y}}\Big\{J(\mathbf{x}, \mathbf{y}, \mathbf{u}^*(\mathbf{y}))\Big\} \\ &= \mathcal{E}_{\mathbf{y}}\Big\{\min_{\mathbf{u}} \mathcal{E}_{\mathbf{x}|\mathbf{y}}\Big[J(\mathbf{x}, \mathbf{y}, \mathbf{u})\Big|\mathbf{y}\Big]\Big\}.\end{aligned}$$

Lemma 7. [Åström (1970)] Let $\mathbf{x}$ be normal with mean m and covariance R. Then,

$$\mathcal{E}\{\mathbf{x}^T \mathbf{S} \mathbf{x}\} = \mathbf{m}^T \mathbf{S} \mathbf{m} + \mathrm{tr}[\mathbf{S}\mathbf{R}].$$

Since in (35) only V_k depends on the input $\mathbf{u}_k$ and the state $\mathbf{x}_k$, we consider only V_k for finding the optimal feedback minimizing (35).

Let us Denote

$$\begin{aligned}&f(\mathbf{u}_k, \mathbf{y}_{k-1}) \triangleq \mathcal{E}_{\mathbf{x}|\mathbf{y}}\{V_k | \mathbf{y}_{k-1}\} \\ &= \mathcal{E}_{\mathbf{x}|\mathbf{y}}\Big\{\mathbf{u}_k^T \hat{\mathbf{R}}_B \mathbf{u}_k + (\mathbf{u}_k + \mathbf{K}_k \mathbf{x}_k)^T \hat{\mathbf{R}}_B \mathbf{K}_k^c \mathbf{x}_k \\ &\quad + \mathbf{x}_k^T \mathbf{K}_k^{c^T} \hat{\mathbf{R}}_B (\mathbf{u}_k + \mathbf{K}_k \mathbf{x}_k) - \mathbf{x}_k^T \mathbf{K}_k^T \hat{\mathbf{R}}_B \mathbf{K}_k \mathbf{x}_k \Big| \mathbf{y}_{k-1}\Big\}.\end{aligned}$$

Then, by using Lemma 7, we have

$$\begin{aligned}&f(\mathbf{u}_k, \mathbf{y}_{k-1}) = \mathbf{u}_k^T \hat{\mathbf{R}}_B \mathbf{u}_k + (\mathbf{u}_k + \mathbf{K}_k \hat{\mathbf{x}}_k)^T \hat{\mathbf{R}}_B \mathbf{K}_k^c \hat{\mathbf{x}}_k \\ &\quad + \hat{\mathbf{x}}_k^T \mathbf{K}_k^{c^T} \hat{\mathbf{R}}_B (\mathbf{u}_k + \mathbf{K}_k \hat{\mathbf{x}}_k) - \hat{\mathbf{x}}_k^T \mathbf{K}_k^T \hat{\mathbf{R}}_B \mathbf{K}_k \hat{\mathbf{x}}_k \\ &\quad + \mathrm{tr}\Big[\big(\mathbf{K}_k^T \hat{\mathbf{R}}_B \mathbf{K}_k^c + \mathbf{K}_k^{c^T} \hat{\mathbf{R}}_B \mathbf{K}_k - \mathbf{K}_k^T \hat{\mathbf{R}}_B \mathbf{K}_k\big)\boldsymbol{\Sigma}_k\Big]\end{aligned}$$

where $\boldsymbol{\Sigma}_k$ satisfies the Kalman predictor Riccati equation

$$\begin{aligned}\boldsymbol{\Sigma}_{k+1} &= \mathbf{A}\boldsymbol{\Sigma}_k \mathbf{A}^T + \mathbf{Q}_e \\ &\quad - \mathbf{A}\boldsymbol{\Sigma}_k \mathbf{C}^T (\mathbf{C}\boldsymbol{\Sigma}_k \mathbf{C}^T + \mathbf{R}_e)^{-1} \mathbf{C}\boldsymbol{\Sigma}_k \mathbf{A}^T.\end{aligned} \tag{36}$$

To incorporate the constraint $\mathcal{E}\{\mathbf{x}_k\} \in \mathcal{X}_c$, we use the input of the form

$$\mathbf{u}_k = -\mathbf{G}_0 \hat{\mathbf{x}}_k + \mathbf{P}_{\mathcal{N}(\mathbf{DB})} \bar{\mathbf{u}}_k.$$

Then, $f(\mathbf{u}_k, \mathbf{y}_{k-1})$ becomes a function of $\bar{\mathbf{u}}_k$

$$\begin{aligned}&f(\mathbf{u}_k, \mathbf{y}_{k-1}) = f(\bar{\mathbf{u}}_k, \mathbf{y}_{k-1}) = \tilde{V}(\bar{\mathbf{u}}_k, \mathbf{y}_{k-1}) \\ &\quad + \mathrm{tr}\Big[\big(\mathbf{K}_k^T \hat{\mathbf{R}}_B \mathbf{K}_k^c + \mathbf{K}_k^{c^T} \hat{\mathbf{R}}_B \mathbf{K}_k - \mathbf{K}_k^T \hat{\mathbf{R}}_B \mathbf{K}_k\big)\boldsymbol{\Sigma}_k\Big]\end{aligned}$$

where

$$\tilde{V}(\bar{\mathbf{u}}_k, \mathbf{y}_{k-1}) = \hat{\mathbf{x}}_k^T \tilde{\mathbf{F}} \hat{\mathbf{x}}_k + 2\hat{\mathbf{x}}_k^T \tilde{\mathbf{H}} \bar{\mathbf{u}}_k + \bar{\mathbf{u}}_k^T \tilde{\mathbf{G}} \bar{\mathbf{u}}_k$$

with $\tilde{\mathbf{F}}, \tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$ given by (27). Similarly to the deterministic case of Section 4.1, the minimum of $\tilde{V}(\bar{\mathbf{u}}_k, \mathbf{y}_{k-1})$ is zero, which is obtained by

$$\bar{\mathbf{u}}_k^* = -\hat{\mathbf{R}}_B^{(2)} \mathbf{P}_{\mathcal{N}(\mathbf{DB})} \hat{\mathbf{R}}_B (\mathbf{K}_k^c - \mathbf{G}_0) \hat{\mathbf{x}}_k.$$

Therefore, by applying Lemma 6 and with (35) and

$$\begin{aligned} \mathcal{E}\Big\{\mathbf{x}_0^T \mathbf{P}_0^c \mathbf{x}_0\Big\} &= \bar{\mathbf{x}}_0^T \mathbf{P}_0^c \bar{\mathbf{x}}_0 + \mathrm{tr}\Big[\mathbf{P}_0^c \bar{\boldsymbol{\Sigma}}_0\Big] \\ \mathcal{E}\Big\{\mathbf{w}_k^T \mathbf{P}_{k+1}^c \mathbf{w}_k\Big\} &= \mathrm{tr}\Big[\mathbf{P}_{k+1}^c \mathbf{Q}_e\Big], \end{aligned} \tag{37}$$

we obtain Theorem 2. Here $\mathbf{x}_0$ has a gaussian distribution with mean $\bar{\mathbf{x}}_0$ and covariance $\bar{\boldsymbol{\Sigma}}_0$.

Theorem 2. Consider the state (21) and measurement equation (32). Let the admissible control strategies be such that $\mathbf{u}_k$ is a function of $\mathbf{y}_k$. Assume that $\mathbf{P}_k^c$-Riccati equation (14) with initial condition $\mathbf{P}_N^c = \mathbf{Q}_N$ has a solution $\mathbf{P}_k^c$ such that $\mathbf{P}_k^c$ is symmetric with $\mathbf{P}_k^c \in \mathcal{S}$. Then there exists a unique admissible control strategy

$$\mathbf{u}_k = -\mathbf{K}_k \hat{\mathbf{x}}_k$$

which minimizes the expected performance index (35), satisfying equality constraints $\mathcal{E}\{\mathbf{D}\mathbf{x}_k\} = \mathbf{0}$ and also minimizing the expectation of the squared constraint error given by (34). Here, $\mathbf{K}_k$ is given by (24). The minimal value of expected performance index is given by

$$\begin{aligned} &\mathcal{E}\{J_{sto}^c\} \triangleq \min_{\mathcal{E}\{\mathbf{x}_k\} \in \mathcal{N}} \mathcal{E}\{J\} = \\ &\bar{\mathbf{x}}_0^T \mathbf{P}_0^c \bar{\mathbf{x}}_0 + \mathrm{tr}\Big[\mathbf{P}_0^c \bar{\boldsymbol{\Sigma}}_0\Big] + \sum_{k=0}^{N-1} \mathrm{tr}\Big[\mathbf{P}_{k+1}^c \mathbf{Q}_e\Big] \\ &+ \mathrm{tr}\Big[\big(\mathbf{K}_k^T \hat{\mathbf{R}}_B \mathbf{K}_k^c + \mathbf{K}_k^{c^T} \hat{\mathbf{R}}_B \mathbf{K}_k - \mathbf{K}_k^T \hat{\mathbf{R}}_B \mathbf{K}_k\big) \boldsymbol{\Sigma}_k\Big] \end{aligned} \tag{38}$$

where $\boldsymbol{\Sigma}_k$ satisfies (36).

By following the similar approach used for deriving (38), we can obtain the optimal performance index for the unconstrained case

$$\begin{aligned} \mathcal{E}\{J_{sto}^u\} &= \bar{\mathbf{x}}_0^T \mathbf{P}_0^u \bar{\mathbf{x}}_0 + \mathrm{tr}\Big[\mathbf{P}_0^u \bar{\boldsymbol{\Sigma}}_0\Big] \\ &+ \sum_{k=0}^{N-1} \mathrm{tr}\Big[\mathbf{P}_{k+1}^u \mathbf{Q}_e + \big(\mathbf{K}_k^{u^T} \hat{\mathbf{R}}_k^u \mathbf{K}_k^u\big) \boldsymbol{\Sigma}_k\Big] \end{aligned} \tag{39}$$

where $\hat{\mathbf{R}}_k^u \triangleq \mathbf{B}^T \mathbf{P}_{k+1}^u \mathbf{B} + \mathbf{R}_c$. Using optimality yields the following stochastic version of Corollary 1.

Corollary 2. For the stochastic system represented by (21) and (32) with the same state and control weighting matrices, the following performance ordering holds.

$$\mathcal{E}\{J_{sto}^u\} \leq \mathcal{E}\{J_{sto}^c\} \tag{40}$$

5. CONCLUDING REMARKS

In this paper, the control problem with state linear equality constraints was considered, first by finding the existence conditions for linear feedback gains and then determining all such gains. By using the results of discrete time singular optimal control, the optimal constrained feedback gain was determined, which is also shown to constrain optimally the expected values of state variable of the corresponding stochastic system. It is also confirmed that the constrained optimal cost function is increased, due to the constraint. The procedures used for discrete-time systems here can be similarly extended to the continuous-time case, which can be found in Ko (2005).

REFERENCES

Åström, Karl J. (1970). *Introduction to Stochastic Control Theory.* Academic Press. New York, NY.

Bitmead, Robert R. and Michel Gevers (1991). Riccati difference and differential equations: Convergence, monotonicity and stability. In: *The Riccati Equation* (Sergio Bittanti, Alan J. Laub and Jan C. Willems, Eds.). pp. 263–291. Springer-Verlag. Berlin, Germany.

Castelan, Eugênio B. and Jean-Claude Hennet (1992). Eigenstructure assignment for state constrained linear continuous time systems. *Automatica* **28**(3), 605–611.

Clements, David J. and Brian D.O. Anderson (1978). *Singular Optimal Control: The Linear-Quadratic Problem.* Springer-Verlag. Berlin, Germany.

Hahn, Hubert (1992). Linear systems controlled by stabilized constraint relations. In: *Proceedings of the 31st Conference on Decision and Control.* Tucson, Arizona. pp. 840–848.

Hemami, Hooshang and Bostwick F. Wyman (1979). Modeling and control of constrained dynamic systems with application to biped locomotion in the frontal plane. *IEEE Trans. Automat. Contr.* **24**(4), 526–535.

Ko, Sangho (2005). Performance Limitations in Linear Estimation and Control: Quantization and Constraint Effects. PhD thesis. Univ. of California, San Diego. La Jolla, CA.

Maciejowski, Jan Marian (2002). *Predictive Control with Constraints.* Pearson Education Ltd. Essex, England.

McClamroch, N. Harris and Danwei Wang (1988). Feedback stabilization and tracking of constrained robots. *IEEE Trans. Automat. Contr.* **33**(5), 419–426.

Skelton, R. E., T. Iwasaki and K. Grigoriadis (1998). *A Unified Algebraic Approach to Linear Control Design.* Taylor and Francis. London, UK.

Wonham, W. Murray (1979). *Linear multivariable control: a Geometric Approach.* Springer-Verlag. New York, NY.

Yu, Tie-Jun and P. C. Müller (1994). Design of pole-assignment controller for systems with algebraic-equation constraint. *Systems & Control Letters* **23**, 463–471.

Yu, Tie-Jun, Ching-Fang Lin and P. C. Müller (1996). Design of LQ regulator for linear systems with algebraic-equation constraints. In: *Proceedings of the 35th Conference on Decision and Control.* Kobe, Japan. pp. 4146–4151.