CONTROL SYNTHESIS OF SYSTEMS WITH UNCERTAIN PARAMETERS BY CONVEX OPTIMIZATION [1]

Francesc Pozo * **Fayçal Ikhouane** ** **José Rodellar** **

* *Escola Universitària d'Enginyeria Tècnica Industrial de Barcelona. Universitat Politècnica de Catalunya. Comte d'Urgell, 187. 08036 Barcelona. Spain.*
** *Departament de Matemàtica Aplicada III. Universitat Politècnica de Catalunya. Jordi Girona, 1-3. 08034 Barcelona. Spain.*

Abstract: The last few years witnessed an increasing interest in the problem of control synthesis of nonlinear systems. A recently derived stability criterion for nonlinear systems –which has a remarkable convexity property– and the development of numerical methods for verification of positivity allows the computation –via semidefinite programming– of stabilizing controllers for the case of systems with polynomial or rational vector fields. Using the theory of semialgebraic sets these computational tools are extended in this paper for the case of polynomial or rational systems with uncertainty parameters.

Keywords: Uncertainty parameters, nonlinear control, semidefinite programming, sum of squares decomposition.

1. INTRODUCTION

Analysis and control of nonlinear systems are among the most challenging problems in systems and control theory. Despite many years of research, the stability and performance of nonlinear systems is an open problem and there is still no universal methodology for constructing controllers that stabilize such systems.

In this respect, Lyapunov functions have long been recognized as one of the most fundamental analytical tools for analysis and synthesis of nonlinear control systems; see, for example (Krstić *et al.*, 1995; Isidori, 1995).

[1] Supported by CICYT through grant DPI2002-04018-C02-01. The second author acknowledges the support of the Spanish Ministry of Science and Technology through the "Ramón y Cajal" program.

Thanks to a strong development of computational tools based on Lyapunov functions, there are many methods based on convex optimization, exploiting the fact that the set of Lyapunov functions for a given system is convex.

However, a serious obstacle in the problem of control synthesis of nonlinear systems is that the *joint* search for the controller $u(x)$ and a Lyapunov function $V(x)$ is not convex. Consider, for instance, the synthesis problem for the system

$$\dot{x} = f(x) + g(x)u.$$

The set of u and V satisfying the condition

$$\frac{\partial V}{\partial x}\left[f(x) + g(x)u(x)\right] < 0$$

is not convex and could be even not connected (Prieur and Praly, 1999).

The convergence criterion presented in (Rantzer, 2001) based on the density function ρ has a remarkable convexity property. Indeed, the set of $(\rho, u\rho)$ satisfying the dual criterion or divergence inequality

$$\nabla \cdot [\rho(f + gu)] > 0 \tag{1}$$

is convex. This convexity property will be exploited in the computation of stabilizing controllers. Recent numerical methods for verification of positivity of multivariate polynomials based on sum of squares decompositions are used (Prajna *et al.*, 2002; Prajna *et al.*, 2004*b*).

In this paper, these computational tools are extended, thanks to the theory of semialgebraic sets, for the case of polynomial or rational systems with uncertainty parameters. This theory allows to consider the uncertain parameters as new polynomial variables, but the dynamic of the system is not augmented. This way, our objective is to find a *worst case* controller, in the sense that the system is stabilized for all the possible values of the uncertain parameters.

The outline of the paper is as follows. The sum of squares relaxations and the dual theorem of Lyapunov that allows the computation of stabilizing controllers of polynomial or rational systems are presented in Section 2. The Putinar's theorem and the extension of the computational tools for the case of systems with uncertainty parameters are fully developed in Section 3. An example is presented in Section 4. Finally, the paper will be ended by some conclusions in Section 5.

2. COMPUTATIONAL APPROACH

In order to understand the possibilities and limitations of computational approaches to nonlinear stability, an issue that has to be addressed is how to deal numerically with functional inequalities such as the standard Lyapunov one, or the divergence inequality (1).

2.1 First relaxation: the sum of squares approach

It is well-known that the problem of checking global nonnegativity of a polynomial of quartic (or higher) degree is computationally hard, even in the restricted case of polynomial functions (Prestel and Delzell, 2001). For this reason, we need tractable sufficient conditions that guarantee nonnegativity, and that are not overly conservative. A particularly interesting sufficient condition is given by the existence of a sum of squares decomposition (Parrilo, 2000): can the polynomial $p(x)$ be written as

$$p(x) = \sum_i p_i^2(x),$$

for some polynomials $p_i(x)$? Obviously, if this is the case, then $p(x)$ takes only nonnegative values.

In this respect, it is interesting to notice that many methods used in control theory for constructing Lyapunov functions (for example, backstepping (Krstić *et al.*, 1995)) use either implicitly or explicitly a sum of squares approach.

The problem of checking if a given polynomial can be written as a sum of squares can be solved via convex optimization, in particular semidefinite programming. For our purposes, however, it will enough to know that while the standard semidefinite programming machinery can be interpreted as searching for a semidefinite element over an affine family of quadratic forms, the new tools provide a way of finding a sum of squares, over an affine family of polynomials. In particular, a freely available MATLAB toolbox for formulating and solving sum of squares programs can be used for this purpose.

2.2 Second relaxation: the dual theorem of Lyapunov

Lyapunov's second theorem has long been recognized as one of the most fundamental tools for analysis and synthesis of nonlinear systems. The importance of the criterion stems from the fact that it allows stability of a system to be verified without solving the differential equation explicitly.

Lyapunov's theorem has a close relative, Theorem 1. The relationship between the two theorems can be considered as an analogous to the duality that has been used since 1940s for closely related problems in calculus of variations.

Theorem 1. Given the equation

$$\dot{x}(t) = f(x(t)),$$

where $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f(0) = 0$, suppose there exists a non-negative $\rho \in \mathcal{C}^1(\mathbb{R}^n - \{0\}, \mathbb{R})$, called density function, such that

- ▶ $\dfrac{\rho(x)f(x)}{|x|}$ is integrable on $\{x \in \mathbb{R}^n : |x| \geq 1\}$ and (2)
- ▶ $[\nabla \cdot (f\rho)] > 0$ for almost all x. (3)

Then, for **almost all** initial states $x(0)$ the trajectory $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$.

Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

Proof. See (Rantzer, 2001).

2.3 Control synthesis of polynomial systems

Consider the system

$$\dot{x} = f(x) + g(x)u$$

where $f(x)$ and $g(x)$ are polynomial vectors. To apply the tools presented previously to the stabilization of this system, consider the following parameterized representation for ρ and $u\rho$:

$$\rho(x) = \frac{a(x)}{b(x)^\alpha}, \quad u(x)\rho(x) = \frac{c(x)}{b(x)^\alpha},$$

where $a(x), b(x), c(x)$ are polynomials, $b(x)$ is positive, and α is chosen large enough so as to satisfy the integrability condition (2) in Theorem 1. Note that by choosing this particular representation, we presuppose that we will be searching for ρ and u that are rationals. In particular, the resulting control law will be

$$u(x) = \frac{c(x)}{a(x)}.$$

In this case, the divergence criterion can be written as

$$\begin{aligned}
\nabla \cdot [\rho(f + gu)] &= \nabla \cdot \left[\frac{1}{b^\alpha}(fa + gc)\right] \\
&= -\alpha \frac{1}{b^{\alpha+1}} \nabla b \cdot (fa + gc) + \frac{1}{b^\alpha} \nabla \cdot (fa + gc) \\
&= \frac{1}{b^{\alpha+1}} \left[b \nabla \cdot (fa + gc) - \alpha \nabla b \cdot (fa + gc)\right].
\end{aligned}$$

Since b is positive, we only need to satisfy the inequality

$$b\nabla \cdot (fa + gc) - \alpha \nabla b \cdot (fa + gc) > 0. \quad (4)$$

For fixed b, α, the inequality (4) is linear in a and c. Instead of checking positivity, we check that the left-hand side is a *sum of squares*, and then the problem can be solved using semidefinite programming.

2.4 Control synthesis of rational systems

Consider the system

$$\dot{x} = f(x) + g(x)u$$

where $f(x)$ and $g(x)$ are vectors whose components are *ratios* of polynomials. Without loss of generality, we can consider that

$$f(x) = \frac{\tilde{f}(x)}{h(x)}, \quad g(x) = \frac{\tilde{g}(x)}{h(x)},$$

where $\tilde{f}(x), \tilde{g}(x)$ and $h(x)$ are polynomial expressions.

To apply in this case the tools presented in the previous sections to the stabilization of this system, consider also the following parameterized representation for ρ and $u\rho$:

$$\rho(x) = \frac{a(x)}{b(x)^\alpha}, \quad u(x)\rho(x) = \frac{c(x)}{b(x)^\alpha},$$

where $a(x), b(x), c(x)$ are polynomials, $b(x)$ is positive, and α is chosen large enough so as to satisfy the integrability condition (2) in Theorem 1. The resulting control law will be

$$u(x) = \frac{c(x)}{a(x)}.$$

In this case, the divergence criterion presented in Theorem 1 can be written as can be seen in Figure 1.

Since both $b(x)$ and $h(x)^2$ are positive, we only need to satisfy the inequality

$$bh\nabla \cdot (\tilde{f}a + \tilde{g}c) - (\alpha h \nabla b + b\nabla h) \cdot (\tilde{f}a + \tilde{g}c) > 0. \quad (5)$$

For fixed b, α, the inequality is linear in a and c and the problem can be solved using semidefinite programming as in the previous section.

3. UNCERTAIN PARAMETERS

A question that arises naturally at this point is: the computational tools presented in the previous sections can be employed to stabilize systems with uncertain parameters?

With the help of the theory of semialgebraic sets this can be done by considering the uncertain parameters as new polynomial variables (*pseudo-variables*) and without augmenting the dynamic of the system. This way, our objective is to find a *worst case* controller, in the sense that the system will be stabilized for all the possible values of the uncertain parameters.

In order to formalize all these questions, we must introduce some notation and a key theorem.

Definition 1. Let $\Sigma^2 \subset \mathbb{R}[x_1, \ldots, x_n] =: \mathbb{A}$ denote the set of polynomials which can be written as a sum of squares of other polynomials, that is,

$$\Sigma^2 := \{G(x) \in \mathbb{A} \ : \ \exists h_i(x) \in \mathbb{A} \text{ such that } G(x) = \sum_{i=1}^{m} h_i(x)^2\}.$$

Definition 2. A subset of $\mathbb{R}^n$ which is a finite Boolean combination of sets of the form $\{x = (x_1, \ldots, x_n) \ : \ p(x) > 0\}$ and $\{x \ : \ q(x) = 0\}$, where $p, q \in \mathbb{R}[x_1, \ldots, x_n]$ –i.e. a set that is defined by polynomial inequalities, equalities, and nonequalities– is called a semialgebraic set.

Theorem 2. (Putinar). Suppose we are given a set

$$K := \{x \in \mathbb{R}^n \ : \ c_i(x) \geq 0, \ i = 1, \ldots, m\} \quad (6)$$

that is compact, and furthermore satisfies the condition that there exists a polynomial $h(x)$ of the form

$$\begin{aligned}\nabla \cdot [\rho(f+gu)] &= \nabla \cdot \left[\frac{1}{b^\alpha h}(\tilde{f}a+\tilde{g}c)\right] \\ &= \frac{bh}{b^{\alpha+1}h^2}\nabla \cdot (\tilde{f}a+\tilde{g}c) + \nabla\frac{1}{b^\alpha h}\cdot(\tilde{f}a+\tilde{g}c) \\ &= \frac{1}{b^{\alpha+1}h^2}\left[bh\nabla\cdot(\tilde{f}a+\tilde{g}c) - (\alpha h\nabla b + b\nabla h)\cdot(\tilde{f}a+\tilde{g}c)\right].\end{aligned}$$

Fig. 1. The divergence criterion in the rational case.

$$h(x) = s_0(x) + \sum_{i=1}^{m} s_i(x)\cdot c_i(x),$$

where the $s_i \in \Sigma^2$ are sum of squares and $c_i \in \mathbb{R}[x]$, whose level set

$$\{x \in \mathbb{R}^n \;:\; h(x) \geq 0\}$$

is compact. Then, for any polynomial $G(x)$ positive on all of K, there exist $s_0, s_1, \ldots, s_m \in \Sigma^2$, such that

$$G(x) = s_0(x) + \sum_{i=1}^{m} s_i(x)\cdot c_i(x).$$

Proof. See (Putinar, 1993).

It is worth noting that for a large host of applications, the additional constraint required for Theorem 2 is easily satisfied by the corresponding sets K. For instance, the following cases fall into this category.

1. Suppose some c_i in the definition of K satisfies, on its own, the condition $\{c_i(x) \geq 0\}$ compact. Then Theorem 2 applies. This includes any instance where we are taking intersections with ellipses, or circles, among others.
2. If K is compact, and is defined only by linear functions, then we can directly apply Theorem 2. Note that this includes all polytopes.
3. If we know that the compact set K lies inside some ball of radius R, we can simply add the interior of the ball, $\sum_i x_i^2 \leq R^2$ as a redundant constraint, thus not changing K, but automatically satisfying Theorem 2, without appreciably changing the size of definition of the problem (especially if we already have a large number of functions defining K).

In what follows we assume that the set K defined as in equation (6) satisfies the hypotheses of the Theorem 2. Using this theorem, we translate the pointwise property

$$G(x) > 0, \quad \forall x \in K \tag{7}$$

to the algebraic property

$$\exists s_1, \ldots, s_m \in \Sigma^2 \text{ such that}$$
$$\left(G(x) - \sum_{i=1}^{m} s_i(x)c_i(x)\right) \in \Sigma^2. \tag{8}$$

The membership test in equation (8) can be performed in time polynomial in the size of the polynomial $G(x)$ using the computational tools presented in (Prajna *et al.*, 2002).

3.1 Synthesis procedure

Consider the nonlinear affine system

$$\dot{x} = f(x,p) + g(x,p)u, \tag{9}$$

where $x \in \mathbb{R}^n$ is the state vector, $p = (p_1, \ldots, p_m) \in \mathbb{R}^m$ is the uncertainty parameter vector and $f, g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are polynomial functions describing the system dynamics. We are searching a controller u that stabilizes the system for all the possible values of the parameter p.

We assume that the following intervals are known:

$$\underline{p}_i \leq p_i \leq \overline{p}_i, \quad i = 1, \ldots, m. \tag{10}$$

Equation (10) can be expressed as

$$\begin{aligned} c_1 &= p_1 - \underline{p}_1 \geq 0, \\ c_2 &= \overline{p}_1 - p_1 \geq 0, \\ &\vdots \\ c_{2m-1} &= p_m - \underline{p}_m \geq 0, \\ c_{2m} &= \overline{p}_m - p_m \geq 0. \end{aligned}$$

For compactness, we also assume that the state vector x lies inside some ball of radius M:

$$c_{2m+1} = M - \sum_{i=1}^{n} x_i^2 \leq 0.$$

This way, the set K can be defined as

$$K = \{(x,p) \in \mathbb{R}^{n+m} \;:\; c_1 \geq 0, \ldots, c_{2m+1} \geq 0\}.$$

Let us define $\tilde{x} = [x,p]^{\mathrm{T}}$. The system (9) is now

$$\dot{x} = f(\tilde{x}) + g(\tilde{x})u. \tag{11}$$

To apply the tools presented in Section 2 to the stabilization of the system (11), consider the parameterized representation for ρ and $u\rho$:

$$\rho(x) = \frac{a(x)}{b(x)^\alpha}, \quad u(x)\rho(x) = \frac{c(x)}{b(x)^\alpha},$$

where $a(x), b(x), c(x)$ are polynomials, $b(x)$ is positive, and α is chosen large enough so as to satisfy the integrability condition in Theorem 1.

Remark 1. $a(x), b(x)$ and $c(x)$ are chosen as polynomials only in the variable x. This way, the controller of the system does not depend on p.

In this case, the divergence criterion can be written as

$$\begin{aligned}
&\nabla_n \cdot [\rho(f+gu)] = \\
&= \nabla_n \cdot \left[\frac{1}{b^\alpha}(fa+gc)\right] \\
&= -\alpha\frac{1}{b^{\alpha+1}}\nabla_n b \cdot (fa+gc) + \frac{1}{b^\alpha}\nabla_n \cdot (fa+gc) \\
&= \frac{1}{b^{\alpha+1}}\left[b\nabla_n \cdot (fa+gc) - \alpha\nabla_n b \cdot (fa+gc)\right].
\end{aligned}$$

where the following modified versions of the operators gradient and divergence have been considered:

$$\nabla_n V = \left[\frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_n}\right], \quad V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$
$$\nabla_n \cdot f = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}, \quad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n.$$

Since b is positive, we only need to satisfy the inequality

$$\begin{aligned}
G(\tilde{x}) :=& b\nabla_n \cdot (fa+gc) - \alpha\nabla_n b \cdot (fa+gc) > 0, \\
& \forall (x,p) \in K. \qquad (12)
\end{aligned}$$

Although the system contains uncertain parameters, the explicit expressions of the functions f and g are known and so equation (12) can be computationally treated.

For fixed b, α, the inequality is linear in a, c. Instead of checking positivity, we check that the left-hand side is a *sum of squares*. Using Theorem 2 and, more precisely, equation (8), the pointwise property (12) is translated to the algebraic property

$$\exists s_1, \ldots, s_{2m+1} \in \Sigma^2 \text{ such that}$$
$$\left(G(\tilde{x}) - \sum_{i=1}^{2m} s_i(x)c_i\right) \in \Sigma^2,$$

that can be performed in polynomial time using the computational tools presented in (Prajna *et al.*, 2002).

4. A SIMPLE EXAMPLE

4.1 Example 1

In order to show the applicability of the computational approach described in Section 3, let us consider the following nonlinear system with an uncertain parameter as the coefficient of the linear term in the first equation:

$$\begin{aligned}
\dot{x}_1 &= px_2 - x_1^3 + x_1^2, \quad p \in [0.7, 1.3] \\
\dot{x}_2 &= u
\end{aligned}$$

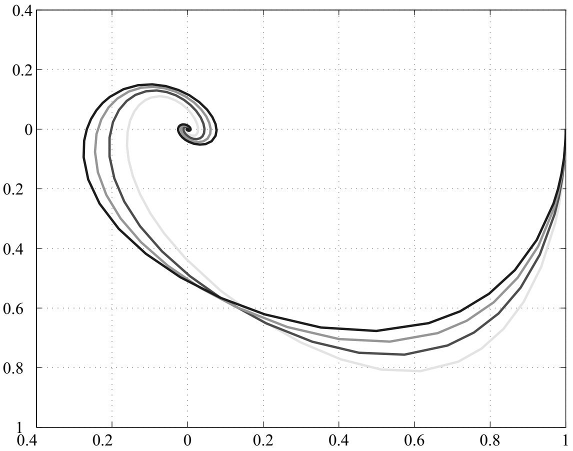

Fig. 2. Phase plot of the closed-loop system in Section 4.1. Solid curves are trajectories with initial conditions $(x_1, x_2) = (1, 0)$ and for four different values for the parameter p, $0.7, 0.9, 1.1$ and 1.3.

The system is defined by polynomial expressions in the variables x_1, x_2 and, for computational purposes, p can be treated as a third variable.

Our objective is to find a control function u that stabilizes the system for every $p \in [0.7, 1.3]$. Such a control law u for this system can be found using the techniques described in Section 2. We only need to satisfy the inequality (12) in the case $n = 2$, that is

$$\begin{aligned}
G(x,p) =& b\nabla_2 \cdot (fa+gc) - \alpha\nabla_2 b \cdot (fa+gc) > 0, \\
& \forall (x,p) \in K \qquad (13)
\end{aligned}$$

where

$$f(x_1, x_2, p) = \begin{bmatrix} px_2 - x_1^3 + x_1^2 \\ 0 \end{bmatrix},$$
$$g(x_1, x_2, p) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and b can be chosen as it is described in (Prajna *et al.*, 2004*c*)

$$\begin{aligned}
b(x_1, x_2) &= 3x_1^2 + 2x_1x_2 + 2x_2^2 \\
&= (x_1 + x_2)^2 + 2x_1^2 + x_2^2.
\end{aligned}$$

Since we will be using a cubic polynomial for $c(x_1, x_2)$, and $a(x_1, x_2)$ is taken to be a constant, we choose $\alpha = 5$ to satisfy the integrability condition. We note that $G(x, p)$ can be considered as a polynomial expression in $\mathbb{R}[x_1, x_2, p]$.

K is defined as follows

$$K = \{(x,p) \in \mathbb{R}^2 \times \mathbb{R} \; : \; c_1 \geq 0, \; c_2 \geq 0, \; c_3 \geq 0\},$$

where

$$\begin{aligned}
c_1 &= 1.3 - p, \\
c_2 &= p - 0.7, \\
c_3 &= M - \|x\|_2^2, \; M > 0.
\end{aligned}$$

We translate the equation (13) to the algebraic property

$\exists s_1, s_2, s_3 \in \Sigma^2$ such that
$(G(x,p) - s_1(x)c_1 - s_2(x)c_2 - s_3(x)c_3) \in \Sigma^2.$

After solving the sum of squares problem, the results are

$$\begin{aligned} u(x) &= -1.3446x_1 - 0.9005x_2 - 0.0902x_2^3, \\ s_1(x) &= 6.2496x_1^2 + 14.695x_2^2 + 19.1664x_1x_2 \\ &= (1.37x_1 + 2.09x_2)^2 + (2.09x_1 + 3.21x_2)^2, \\ s_2(x) &= 6.2496x_1^2 + 4.695x_2^2 - 10.8336x_1x_2 \\ &= (1.89x_1 - 1.64x_2)^2 + (1.42x_2 - 1.64x_1)^2. \end{aligned}$$

See Figure 2 for a phase plot of the closed-loop system.

5. CONCLUDING REMARKS

A recently derived stability criterion for nonlinear systems (Rantzer, 2001) and the development of numerical methods for verification of positivity (Prajna *et al.*, 2002) has made possible to state the synthesis problem in terms of convex optimization. Using the compactness of semialgebraic sets (Putinar, 1993) these tools have been extended for the case of polynomial or rational systems with uncertainty parameters. The size of the semidefinite programs makes it possible to handle problems that are otherwise too large to solve using state-of-the-art semidefinite programming solvers.

ACKNOWLEDGEMENTS

The first author is grateful with Prof. Jaume Peraire for his suggestions in the work developed at the Massachusetts Institute of Technology.

REFERENCES

Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics.

Gahinet, P., A. Nemirovski, A.J. Laub and M. Chilali (1995). *LMI Control Toolbox*. The Mathworks Incorporated.

Isidori, A. (1995). *Nonlinear Control Systems*. third ed.. Springer-Verlag.

Khalil, H.K. (1996). *Nonlinear systems*. Upper Saddle River, NJ. second ed.. Prentice-Hall, Inc.

Krstić, M. and P. Tsiotras (1999). Inverse optimal stabilization of a rigid spacecraft. *IEEE Transactions on Automatic Control* **42**, 1042–1049.

Krstić, M., I. Kanellakopoulos and P. Kokotović (1995). *Nonlinear and Adaptive Control Design*. John Wiley & Sons, Inc.

Papachristodoulou, A. and S. Prajna (2002). On the construction of lyapunov functions using the sum of squares decomposition. In: *Proceedings of the Conference on Decision and Control*.

Parrilo, P.A. (2000). Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. PhD thesis. California Institute of Technology.

Prajna, S., A. Papachristodoulou and F. Wu (2004*a*). Nonlinear control synthesis by sum of squares optimization: a lyapunov-based approach. In: *Proceedings of the Asian Control Conference*.

Prajna, S., A. Papachristodoulou and P.A. Parrilo (2002). Introducing sostools: a general purpose sum of squares programming solver. In: *Proceedings of the Conference on Decision and Control*.

Prajna, S., A. Papachristodoulou, P. Seiler and P.A. Parrilo (2004*b*). New developments in sum of squares optimization and sostools. In: *Proceedings of the American Control Conference*.

Prajna, S., P.A. Parrilo and A. Rantzer (2004*c*). Nonlinear control synthesis by convex optimization. *IEEE Transactions on Automatic Control* **49**(2), 310–314.

Prestel, A. and C.N. Delzell (2001). *Positive polynomials. From Hilbert 17th's problem to real algebra*. Springer.

Prieur, C. and L. Praly (1999). Uniting local and global controllers. In: *Proceedings of the Conference on Decision and Control*.

Putinar, M. (1993). Positive polynomials on compact semi-algebraic sets. *Indiana University Mathematical Journal* **42**(3), 969–984.

Rantzer, A. (2001). A dual to lyapunov's stability theorem. *Systems and Control Letters* **42**, 161–168.

Scherer, C. and S. Weiland (2002). Linear matrix inequalities in control. Web page `http://www.cs.ele.tue.nl/sweiland/lmi.htm`.

Tütüncü, R.H., K.C. Toh and M.J. Todd (2001). SDPT3 – a MATLAB software package for semidefinite-quadratic-linear programming, version 3.0. Web page `http://www.math.nus.edu.sg/~mattohkc/sdpt3.html`.